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Multiplicative Dedekind η-function and representations of finite groups

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Résumé. Dans cet article, nous étudions le problème de trouver des groupes finis tels que les formes modulaires associées aux éléments de ces groupes au moyen de certaines représentations fidèles appartiennent à des classes particulières de formes modulaires (appelées produits η multiplicatifs). Ce problème est ouvert.

Nous trouvons des groupes métacycliques ayant cette propriété et décrivons les p-sous-groupes de Sylow, $p \neq 2$, de tels groupes. Nous donnons également un aperçu des résulats reliant les produits η multiplicatifs et les éléments d'ordre fini de $SL(5, \mathbb{C})$.

Abstract. In this article we study the problem of finding such finite groups that the modular forms associated with all elements of these groups by means of a certain faithful representation belong to a special class of modular forms (so-called multiplicative $\eta-$products). This problem is open.

We find metacyclic groups with such property and describe the Sylow p-subgroups, $p \neq 2$, for such groups. We also give a review of the results about the connection between multiplicative η-products and elements of finite orders in $SL(5, \mathbb{C})$.

1. Introduction.

The investigation of connections between modular forms and representations of finite groups is an interesting modern aspect of the theory of modular forms.

In this article we study the problem of finding such finite groups that the modular forms associated with all elements of these groups by means of a certain faithful representation belong to a special class of modular forms (so-called multiplicative η-products). This problem is open: all such groups have not been found. It will be interesting to find the complete classification. G.Mason gave an example of such a group: the group M_{24}. This group is unsolvable and large but it is possible to find groups associated with multiplicative η-products which are not subgroups in M_{24}. So we have a non-trivial problem of classification. Also sometimes for the same group we can find several faithful representations with our property. And it is the second aspect of classification. The author has shown that all groups of order 24 have this property. The representation in this case is the regular one [17].

The present paper continues the author's investigations of metacyclic groups which has been started in the articles [19],[22]. In particular we have studied completely the case of dihedral groups. We find all metacyclic groups with such property with genetic code $< a, b : a^m = e, b^s = e, b^{-1}ab = a^r >$ in the case when the cyclic groups $< a >$ and $< b >$ have no nontrivial intersection.

We consider the groups and their representations in detail and we write explicitly such representations that the associated modular forms are multiplicative η-products. Sometimes for the same group we find distinct such faithful representations.

We use our theorem about such abelian groups which is also formulated. It was proved in [21]. Also we describe the Sylow p-subgroups, $p \neq 2$, for such groups. In conclusion we give a review of the results about the connection between multiplicative η-products and elements of finite orders in $SL(5, \mathbb{C})$.

2. Multiplicative η-products.

The Dedekind η-function $\eta(z)$ is defined by the formula

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \; q = e^{2\pi i z},$$

z belongs to the upper complex half-plane.

In this article we consider the modular forms which can be completely described by the following conditions:

1. They are cusp forms of integral weight (with characters);
2. they are eigenforms with respect to all Hecke operators;
3. they have zeroes only in the cusps and every zero has multiplicity 1.

A priori we don't suppose that these functions are modified products of Dedekind η-functions. But in fact it is so, there are 28 such functions. It was proved in [18].

Let us give the complete list.

Forms of the weight 1:

$$\eta(23z)\eta(z), \; \eta(22z)\eta(2z), \; \eta(21z)\eta(3z), \; \eta(20z)\eta(4z),$$
$$\eta(18z)\eta(6z), \; \eta(16z)\eta(8z), \; \eta^2(12z).$$

Forms of the weight 2:

$$\eta(15z)\eta(5z)\eta(3z)\eta(z), \; \eta(14z)\eta(7z)\eta(2z)\eta(z), \; \eta(12z)\eta(6z)\eta(4z)\eta(2z),$$
$$\eta^2(11z)\eta^2(z), \; \eta^2(10z)\eta^2(2z), \; \eta^2(9z)\eta^2(3z), \; \eta^2(8z)\eta^2(4z), \; \eta^4(6z).$$

Forms of the weight 3:

$$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),\ \eta^3(7z)\eta^3(z),\ \eta^3(6z)\eta^3(2z),\ \eta^6(4z).$$

Forms of the weight 4:

$$\eta^4(5z)\eta^4(z),\ \eta^4(4z)\eta^4(2z),\ \eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z),\ \eta^8(3z).$$

Form of the weight 5: $\eta^4(4z)\eta^2(2z)\eta^4(z)$.

Forms of the weight 6: $\eta^6(3z)\eta^6(z),\ \eta^{12}(2z)$.

Form of the weight 8: $\eta^8(2z)\eta^8(z)$.

Form of the weight 12: $\eta^{24}(z)$.

We add to this list two cusp forms of half-integral weight, $\eta(24z), \eta^3(8z)$.

These functions we shall call *multiplicative η-products* because they have multiplicative Fourier coefficients.

D.Dummit, H.Kisilevskii and J. McKay obtained the same list of cusp forms from another point of view: they showed that among functions of the kind

$$f(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z),$$

where a_k and $t_k \in \mathbb{N}$, only these 30 functions had multiplicative coefficients. They checked it by computer calculations [3]. Yves Martin found all η-quotients. But the quotients cannot be used in our case. From various points of view these functions have been studied in recent works of American and Japanese mathematicians [1], [3], [4], [6] to [14].

3. Representations of finite groups and modular forms.

We assign modular forms to elements of finite groups by the following rule. Let Φ be such a representation of a finite group G by unimodular matrices in a space V whose dimension is divisible by 24 that for any element $g \in G$ the characteristic polynomial of the operator $\Phi(g)$ is of the form:

$$P_g(x) = \prod_{k=1}^{s} (x^{a_k} - 1)^{t_k},\ a_k \in \mathbb{N},\ t_k \in \mathbb{Z}.$$

With each $g \in G$ we can associate the function

$$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z).$$

The function $\eta_g(z)$ is a cusp form of a certain level $N(g)$ and of the weight $k(g) = \frac{1}{2}\sum_{k=1}^{s} t_k$, with the character equal to the character of the quadratic field $\mathbb{Q}(\sqrt{\prod_{k=1}^{s}(ia_k)^{t_k}})$.

We shall consider an interesting problem:

the problem of finding such finite groups that all modular forms assigned to all elements of the group by means of a faithful representation are multiplicative η-products.

Now this problem is open: all such groups have not been found.

G.Mason has shown that all functions associated with elements of the Mathieu group M_{24} by means of the representation on the Leech lattice are multiplicative η-products. There are 21 functions of this kind.

It is possible to find groups associated with multiplicative η-products which are not subgroups in M_{24}.

Theorem 3.1. *Let g be such an element in G that the function $\eta_g(z)$ associated with g by a representation (as described above) is a multiplicative η-product then the functions $\eta_h(z)$, $h = g^k$, are also multiplicative η-products.*

In Table 1 we list all the possible assignments of multiplicative η-products to elements of cyclic groups. It is very useful in our proofs.

Theorem 3.2. *For any cyclic group $Z_n, 1 \le n \le 23$, there are such representations T_1 and T_2, dim $T_1 = n_1$, dim $T_2 = n_2$, $n_1 \cdot n_2 = 24$, that the cusp forms $\eta_g(z)$ associated with all elements $\in Z_n$ by means of representations n_2T_1, n_1T_2, $T_1 \otimes T_2$ are also multiplicative η-products.*

These theorems are proved in [19], [20].

A representation of a group will be called *desired* or *of permissible type* if, by means of this representation, the multiplicative η-products are associated with all elements of this group. We shall not call these groups and representations "admissible" because the word "admissible" has another sense in the theory of automorphic representations.

In this paper we shall list the metacyclic groups with faithful representations of this kind. Permissible groups are listed up to isomorphism.

The desired groups can contain only elements whose orders do not exceed 24 and are not equal to 13, 17, 19. Due to Theorem 3.1. it is sufficient to consider the representations only for elements that do not belong to the same cyclic group. The identity element of the group corresponds to the cusp form $\eta^{24}(z)$.

If a permissible group is a subgroup of another permissible group then it is sufficient to study in detail the larger group only. The result for the subgroup immediately follows.

We shall use the following theorem which is proved in [21].

Theorem 3.3. *Let G be an abelian group and T a faithful representation such that for every $g \in G$ the characteristic polynomial of the operator $T(g)$ is of the form $P_g(x) = \prod_{k=1}^{s}(x^{a_k} - 1)^{t_k}$, the corresponding cusp form*

$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$ *being a multiplicative η-product. Then G is a subgroup of one (or several) of the following groups:*

$\mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_{14},\ \mathbb{Z}_{15},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_4 \times \mathbb{Z}_4,\ \mathbb{Z}_8 \times \mathbb{Z}_2,$

$\mathbb{Z}_{16},\ \mathbb{Z}_6 \times \mathbb{Z}_3,\ \mathbb{Z}_{18},\ \mathbb{Z}_{10} \times \mathbb{Z}_2,\ \mathbb{Z}_{20},\ \mathbb{Z}_{22}, \mathbb{Z}_{23},\ \mathbb{Z}_{12} \times \mathbb{Z}_2,\ \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_{24}.$

We include only the maximal abelian groups.

Table 1.

Group	Modular forms
$\mathbb{Z}_{24}$	$\eta(24z),\ \eta^2(12z),\ \eta^3(8z),\ \eta^4(6z),\ \eta^6(4z),\ \eta^8(3z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_{23}$	$\eta(23z)\eta(z),\ \eta^{24}(z)$
$\mathbb{Z}_{22}$	$\eta(22z)\eta(2z),\ \eta^2(11z)\eta^2(z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_{21}$	$\eta(21z)\eta(3z),\ \eta^3(7z)\eta^3(z),\ \eta^8(3z),\ \eta^{24}(z)$
$\mathbb{Z}_{20}$	$\eta(20z)\eta(4z),\ \eta^2(10z)\eta^2(2z),\ \eta^4(5z)\eta^4(z),$ $\eta^6(4z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_{18}$	$\eta(18z)\eta(6z),\ \eta^2(9z)\eta^2(3z),\ \eta^3(6z)\eta^3(2z),$ $\eta^6(3z)\eta^6(z),\ \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{Z}_{16}$	$\eta(16z)\eta(8z),\ \eta^2(8z)\eta^2(4z),\ \eta^4(4z)\eta^4(2z),\ \eta^8(2z)\eta^8(z), \eta^{24}(z)$
$\mathbb{Z}_{15}$	$\eta(15z)\eta(5z)\eta(3z)\eta(z),\ \eta^4(5z)\eta^4(z),\ \eta^6(3z)\eta^6(z),\ \eta^{24}(z)$
$\mathbb{Z}_{14}$	$\eta(14z)\eta(7z)\eta(2z)\eta(z),\ \eta^3(7z)\eta^3(z),\ \eta^8(2z)\eta^8(z),\ \eta^{24}(z)$
$\mathbb{Z}_{12}$	$\eta(12z)\eta(6z)\eta(4z)\eta(2z),\ \eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z),\ \eta^6(3z)\eta^6(z),$ $\eta^4(4z)\eta^2(2z)\eta^4(z),\ \eta^8(2z)\eta^8(z),\ \eta^{24}(z)$
$\mathbb{Z}_{12}$	$\eta^2(12z),\ \eta^4(6z),\ \eta^6(4z),\ \eta^8(3z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_{11}$	$\eta^2(11z)\eta^2(z),\ \eta^{24}(z)$
$\mathbb{Z}_{10}$	$\eta^2(10z)\eta^2(2z),\ \eta^4(5z)\eta^4(z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_9$	$\eta(18z)\eta(6z),\ \eta^2(9z)\eta^2(3z),\ \eta^6(3z)\eta^6(z),\ \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{Z}_8$	$\eta^2(8z)\eta^2(4z),\ \eta^4(4z)\eta^4(2z),\ \eta^8(2z)\eta^8(z), \eta^{24}(z)$
$\mathbb{Z}_8$	$\eta^3(8z),\ \eta^6(4z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_8$	$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),\ \eta^4(4z)\eta^2(2z)\eta^4(z),\ \eta^8(2z)\eta^8(z),\ \eta^{24}(z)$
$\mathbb{Z}_7$	$\eta^3(7z)\eta^3(z),\ \eta^{24}(z)$
$\mathbb{Z}_6$	$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z),\ \eta^6(3z)\eta^6(z),\ \eta^8(2z)\eta^8(z),\ \eta^{24}(z)$
$\mathbb{Z}_6$	$\eta^4(6z),\ \eta^8(3z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_6$	$\eta^3(6z)\eta^3(2z),\ \eta^6(3z)\eta^6(z),\ \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{Z}_5$	$\eta^4(5z)\eta^4(z),\ \eta^{24}(z)$
$\mathbb{Z}_4$	$\eta^4(4z)\eta^4(2z),\ \eta^8(2z)\eta^8(z), \eta^{24}(z)$
$\mathbb{Z}_4$	$\eta^6(4z),\ \eta^{12}(2z),\ \eta^{24}(z)$
$\mathbb{Z}_4$	$\eta^4(4z)\eta^2(2z)\eta^4(z),\ \eta^8(2z)\eta^8(z),\ \eta^{24}(z)$
$\mathbb{Z}_3$	$\eta^8(3z), \eta^{24}(z)$
$\mathbb{Z}_3$	$\eta^6(3z)\eta^6(z),\ \eta^{24}(z)$
$\mathbb{Z}_2$	$\eta^8(2z)\eta^8(z), \eta^{24}(z)$
$\mathbb{Z}_2$	$\eta^{12}(2z),\ \eta^{24}(z)$

4. Metacyclic groups and modular forms.

Metacyclic group is, by definition, a finite group with a cyclic normal subgroup such that the corresponding factor-group is also cyclic.

The genetic code of a metacyclic group is $< a, b : a^m = e, b^s = e, b^{-1}ab = a^r >$.

In this paper we completely analyze the cases when $m = 6$, 8, 9, 12, 16, 18, 24 and the cyclic groups $< a >$ and $< b >$ have no nontrivial intersection. The cases $m = 3$, 4, 5, 7, 11, 23 were considered in the article [19], the cases $m = 10$, 14, 15, 20, 21, 22 can be found in the article [22]. The result can be stated in the form of the following theorem.

Theorem 4.1. *Let G be a metacyclic group with the following genetic code*

$$< a, b : a^m = e, b^s = e, b^{-1}ab = a^r >, \ m = 6, 8, 9, 12, 16, 18, 24,$$

such that the modular form associated with each element of this group by means of a faithful representation is a multiplicative η-product and the cyclic groups $< a >$ and $< b >$ have the trivial intersection. Then the possible values of m, s, r (up to isomorphism) are listed in the Table 2.

Table 2.

m	6	6	6	8	8	8	8	8	8	9	9	12	12	12	16	16	16	18	24
s	2	4	6	2	2	2	4	4	4	2	4	2	2	2	2	2	2	2	2
r	5	5	5	3	5	7	3	5	7	8	8	5	7	11	7	9	15	17	17

The dihedral groups of permissible types were studied in detail by the author in the previous paper [15].

4.1. Metacyclic groups of the kind $< a, b : a^6 = e, b^s = e, b^{-1}ab = a^5 >$.

4.1.1. *The group $< a, b : a^6 = e, b^6 = e, b^{-1}ab = a^5 >$.*

The desired representation is the direct sum of all the irreducible representations with the multiplicity one.

The cusp form $\eta^3(6z)\eta^3(2z)$ corresponds to the elements a, a^5, the cusp form $\eta^4(6z)$ corresponds to the other elements of order 6. The cusp form $\eta^6(3z)\eta^6(z)$ corresponds to the elements a^2, a^4, the cusp form $\eta^8(3z)$ corresponds to all the other elements of order 3. The cusp form $\eta^{12}(2z)$ corresponds to the elements of order 2.

This group contains D_6 as a subgroup. Therefore the group D_6 is also permissible.

4.1.2. *The group $< a, b : a^6 = e, b^4 = e, b^{-1}ab = a^5 >$.*

This group is permissible. The desired representation is the regular representation. The cusp forms $\eta^4(6z)$, $\eta^6(4z)$, $\eta^8(3z)$,$\eta^{12}(2z)$,, $\eta^{24}(z)$ correspond to elements of the group.

Let us consider the groups whose elements have admissible orders but the groups have no representations of the desired type. The genetic code of such a group is $< a, b : a^6 = e, b^s = e, b^{-1}ab = a^5 >, s = 8, 12, 16, 24$. If $s = 12$, then the group contains $\mathbb{Z}_6 \times \mathbb{Z}_6$. If $s = 16$, then the group contains $\mathbb{Z}_{16} \times \mathbb{Z}_2$. If $s = 18$, then the group contains $\mathbb{Z}_{18} \times \mathbb{Z}_2$. If $s = 24$, then the group contains $\mathbb{Z}_{24} \times \mathbb{Z}_2$. But these abelian subgroups are not permissible.

The case $s = 8$ is more difficult.

Let T be a desired representation, T_1 be a trivial representation ($T_1(g) = 1, \forall g \in G$). Let denote as χ_T and χ_1 their characters. It has been proved in [21] that if the group $\mathbb{Z}_{12} \times \mathbb{Z}_2$ is of a desired type, then all elements of the order 12 correspond to the cusp form $\eta^2(12z)$, all elements of order 6 correspond to $\eta^4(6z)$, all elements of order 4 correspond to $\eta^6(4z)$, all elements of order 3 correspond to $\eta^8(3z)$, all elements of order 2 correspond to $\eta^{12}(2z)$. Therefore $\chi_T(g) = 0, g \neq e,\ \chi_T(e) = 24$.

Using this fact it is easy to show that in our metacyclic group the cusp forms

$\eta^2(12z)$, $\eta^3(8z)$, $\eta^4(6z)$, $\eta^6(4z)$, $\eta^8(3z)$, $\eta^{12}(2z)$, $\eta^{24}(z)$ correspond to elements of the group.

Consider the scalar product $< \chi_T, \chi_1 >= \frac{1}{48} \cdot 24 = \frac{1}{2}$. Since this number must be an integer, we obtain a contradiction and the desired representation T cannot be constructed.

4.2. Metacyclic groups of the kind $< a, b : a^8 = e, b^s = e, b^{-1}ab = a^r >$.

4.2.1. *The group* $< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^3 >$.

This group has the following irreducible representations:

$T_k(a) = 1, T_k(b) = (-1)^k, k = 1, 2,\ T_k(a) = -1, T_k(b) = (-1)^k, k = 3, 4,$

$$T_5(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^3 \end{pmatrix}, \qquad T_6(a) = \begin{pmatrix} \zeta_8^5 & 0 \\ 0 & \zeta_8^7 \end{pmatrix},$$

$$T_7(a) = \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & \zeta_8^6 \end{pmatrix}, \qquad T_5(b) = T_6(b) = T_7(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The desired representation is the direct sum

$$2(T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6) \oplus 4T_7.$$

All elements of order 8 correspond to the cusp form $\eta^2(8z)\eta^2(4z)$, all elements of order 4 correspond to the cusp form $\eta^4(4z)\eta^4(2z)$, the element a^4 corresponds to $\eta^8(2z)\eta^8(z)$, all the other elements of order 2 correspond to the cusp form $\eta^{12}(2z)$.

4.2.2. *The group* $< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^5 >$.

The desired representation is the direct sum of all irreducible representations with the multiplicity 2. The correspondence between modular forms and elements of the group is as in 4.2.1.

4.2.3. *The group* $< a, b : a^8 = e, b^2 = e, b^{-1}ab = a^7 >$.

This group has the following irreducible representations:
$T_k(a) = 1, T_k(b) = (-1)^k, k = 1, 2$; $T_k(a) = -1, T_k(b) = (-1)^k, k = 3, 4$,

$$T_5(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{pmatrix}, \qquad T_6(a) = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix},$$

$$T_7(a) = \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & \zeta_8^6 \end{pmatrix}, \qquad T_5(b) = T_6(b) = T_7(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The desired representation is the direct sum

$$2(T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6) \oplus 4T_7.$$

The correspondence between modular forms and elements of the group is as in 4.2.1.

We can consider another desired representation:

$$3(T_1 \oplus T_2 \oplus T_7) \oplus 2(T_3 \oplus T_4 \oplus T_5 \oplus T_6).$$

All elements of order 8 correspond to the cusp form $\eta^2(8z)\eta(4z) \times \eta(2z)\eta^2(z)$, the elements of order 4 correspond to the cusp form $\eta^4(4z) \times \eta^2(2z)\eta^4(z)$, the element a^4 corresponds to $\eta^8(2z)\eta^8(z)$, all other elements of order 2 correspond to the cusp form $\eta^{12}(2z)$.

4.2.4. *The group* $< a, b : a^8 = e, b^4 = e, b^{-1}ab = a^3 >$.

This group has the following irreducible representations:
$T_k(a) = 1, T_k(b) = i^k, k = 1, 2, 3, 4$; $T_k(a) = -1, T_k(b) = i^k, k = 5, 6, 7, 8$,

$$T_9(a) = T_{10}(a) = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^3 \end{pmatrix}, \quad T_{11}(a) = T_{12}(a) = \begin{pmatrix} \zeta_8^5 & 0 \\ 0 & \zeta_8^7 \end{pmatrix},$$

$$T_{13}(a) = T_{14}(a) = \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & \zeta_8^6 \end{pmatrix}, \quad T_9(b) = T_{11}(b) = T_{13}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_{10}(b) = T_{12}(b) = T_{14}(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The desired representation is the direct sum that contains T_{13}, T_{14} with the multiplicity 2, all other representations with the multiplicity 1.

All elements of order 8 correspond to the cusp form $\eta^2(8z)\eta^2(4z)$, the elements a^2, a^6, a^2b^2, a^6b^2 correspond to the cusp form $\eta^4(4z)\eta^4(2z)$, all other elements of order 4 correspond to the cusp form $\eta^6(4z)$, the element a^4 corresponds to $\eta^8(2z)\eta^8(z)$, elements b^2 and a^4b^2 correspond to the cusp form $\eta^{12}(2z)$.

4.2.5. *The group* $< a, b : a^8 = e, b^4 = e, b^{-1}ab = a^5 >$.

The desired representation is the direct sum of all the irreducible representations with the multiplicity one. The correspondence between modular forms and elements of the group is as in 4.2.4.

4.3. Metacyclic groups of the kind $< a, b : a^m = e, b^s = e, b^{-1}ab = a^r >,\ m = 9, 18$.

The group $\mathbb{Z}_{18} \times \mathbb{Z}_k$ is not permissible if $k > 1$. The group $\mathbb{Z}_9 \times \mathbb{Z}_k$ is not permissible if $k > 2$.

So we must consider for $m = 9$ the variants:

$$s = 2, r = 8;\ s = 3, r = 4, 7;\ s = 4, r = 8;\ s = 6, r = 2, 5$$

and for $m = 18$

$$s = 2, r = 17;\ s = 3, r = 7, 13;\ s = 6, r = 5, 11.$$

The groups D_9 and D_{18} are permissible [15].

4.3.1. *The group* $< a, b : a^9 = e, b^3 = e, b^{-1}ab = a^4 >$.

This group is of the order 27 and has 8 elements of order 3.

Let T be a desired representation, u elements correspond to $\eta^6(3z)\eta^6(z)$. The number u can be equal to 2,4,6,8. The scalar product

$$< \chi_T, \chi_1 >= \frac{1}{27} \cdot (24 + 6u) = \frac{1}{9} \cdot (8 + 2u),$$

where χ_1 is the character of the trivial representation.

This number must be an integer, but it is not an integer if $u = 2, 4, 6, 8$. We obtain a contradiction and the desired representation T cannot be constructed.

This group is isomorphic to the group $G_1 \cong< a, b : a^9 = e, b^3 = e, b^{-1}ab = a^7 >$. So the group G_1 is not permissible. G_1 is a subgroup in the group

$$G_2 \cong< a, b : a^{18} = e, b^3 = e, b^{-1}ab = a^7 > .$$

$$G_2 \cong G_3 \cong< a, b : a^{18} = e, b^3 = e, b^{-1}ab = a^{13} > .$$

So the groups G_2 and G_3 are not permissible.

4.3.2. *The group* $< a, b : a^9 = e, b^6 = e, b^{-1}ab = a^2 >$.

This group contains a subgroup $< a, c : a^9 = e, c^3 = e, b^{-1}ab = a^4 >$ $(c = b^2)$ which is not permissible.

Our group is isomorphic to the group $G_1 \cong< a, b : a^9 = e, b^6 = e, b^{-1}ab = a^5 >$. So the group G_1 is not permissible. G_1 is a subgroup in the group

$$G_2 \cong< a, b : a^{18} = e, b^6 = e, b^{-1}ab = a^5 > .$$

$$G_2 \cong G_3 \cong< a, b : a^{18} = e, b^3 = e, b^{-1}ab = a^{11} > .$$

So the groups G_2 and G_3 are not permissible.

4.3.3. *The group* $< a, b : a^9 = e, b^4 = e, b^{-1}ab = a^8 >$.

This group is of order 36 and has 12 conjugacy classes. This group is permissible. The desired representation contains two irreducible 2-dimensional representations which send the element a to the matrix of order 3 with multiplicity 2, other irreducible representations are included with multiplicity 2. The cusp forms $\eta(18z)\eta(6z)$, $\eta^2(9z)\eta^2(3z)$, $\eta^3(6z)\eta^3(2z), \eta^6(4z)$, $\eta^6(3z)\eta^6(z)$, $\eta^{12}(2z)$, $\eta^{24}(z)$ correspond to elements of the group.

4.4. Metacyclic groups of the kind $< a, b : a^{12} = e, b^s = e, b^{-1}ab = a^r >$.

This group contains a subgroup which is generated by the elements a^2 and b. This subgroup has a genetic code: $< c, d : c^6 = e, d^s = e, d^{-1}cd = c^r >$. Taking into account the results of 4.1., we see that our group is not permissible if $s \neq 2, 4, 6$. The group $< a, b : a^{12} = e, b^6 = e, b^{-1}ab = a^r >$ is not permissible because it contains the subgroup $\mathbb{Z}_{12} \times \mathbb{Z}_3$ which is not permissible.

4.4.1. *The groups* $< a, b : a^{12} = e, b^2 = e, b^{-1}ab = a^r >$.

The number r can be equal to 5, 7, 11. In all these cases the regular representation is the desired one. It has been proved in [17] that the regular representation for any group of order 24 is permissible. The cusp forms $\eta^2(12z)$, $\eta^4(6z)$, $\eta^6(4z)$, $\eta^8(3z)$, $\eta^{12}(2z)$, $\eta^{24}(z)$ correspond to elements of the group.

4.4.2. *The groups* $< a, b : a^{12} = e, b^4 = e, b^{-1}ab = a^r >$.

In this case the number r is equal to one of the values 5,7 or 11. The subgroup $H \cong < a > \times < b >$ is isomorphic to $\mathbb{Z}_{12} \times \mathbb{Z}_2$. It has been proved in [21] that if the group $\mathbb{Z}_{12} \times \mathbb{Z}_2$ is permissible then there is the single possibility for the correspondence between elements of the group and cusp forms: $\eta^2(12z)$, $\eta^4(6z)$, $\eta^6(4z)$, $\eta^8(3z)$, $\eta^{12}(2z)$, $\eta^{24}(z)$ correspond to the elements of the group $\mathbb{Z}_{12} \times \mathbb{Z}_2$. In our group of order 48 each element conjugate to an element in H. So if T is a permissible representation of our group then $\chi_T(g) = 0$, if $g \neq e$. Consider the scalar product $< \chi_T, \chi_{tr} >= \frac{24}{48} = \frac{1}{2}$. But this number must be an integer. We obtain a contradiction.

4.5. Metacyclic groups of the kind $< a, b : a^{16} = e, b^s = e, b^{-1}ab = a^r >$.

Because the group $\mathbb{Z}_{16} \times \mathbb{Z}_m$ is not permissible if $m > 1$ we must consider only the cases $s = 2, r = 7, 9, 15; s = 4, r = 3, 5, 11, 13$.

The dihedral group D_{16} is permissible. It was considered in [15].

4.5.1. *The group* $< a, b : a^{16} = e, b^2 = e, b^{-1}ab = a^7 >$.

This group has the following irreducible representations:

$$T_k(a) = \begin{pmatrix} \zeta_{16}^k & 0 \\ 0 & \zeta_{16}^{7k} \end{pmatrix}, \ k = 1,2,3,4, \qquad T_5(a) = \begin{pmatrix} \zeta_{16}^6 & 0 \\ 0 & \zeta_{16}^{10} \end{pmatrix},$$

$$T_6(a) = \begin{pmatrix} \zeta_{16}^9 & 0 \\ 0 & \zeta_{16}^{15} \end{pmatrix}, \qquad T_7(a) = \begin{pmatrix} \zeta_{16}^{11} & 0 \\ 0 & \zeta_{16}^{13} \end{pmatrix},$$

$$T_k(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ k = 1, \dots 7,$$

$$T_k(a) = 1, \ T_k(b) = (-1)^k, \ k = 8, 9,$$

$$T_k(a) = -1, \ T_k(b) = (-1)^k, \ k = 10, 11.$$

The desired representation is the direct sum

$$T_1 \oplus 2T_2 \oplus T_3 \oplus 2T_4 \oplus 2T_5 \oplus T_6 \oplus T_7 \oplus T_8 \oplus T_9 \oplus T_{10} \oplus T_{11}.$$

All elements of order 16 correspond to the cusp form $\eta(16z)\eta(8z)$, all elements of order 8 correspond to the cusp form $\eta^2(8z)\eta^2(4z)$, the elements a^4, a^{12} correspond to the cusp form $\eta^4(4z)\eta^4(2z)$, the element a^8 corresponds to $\eta^8(2z)\eta^8(z)$, all other elements of order 2 correspond to the cusp form $\eta^{12}(2z)$.

4.5.2. *The group* $< a, b : a^{16} = e, b^2 = e, b^{-1}ab = a^9 >$.

The desired representation is the direct sum of all the irreducible representations with multiplicity 1. The correspondence between modular forms and elements of the group is as in 4.5.1.

4.5.3. *The groups* $< a, b : a^{16} = e, b^4 = e, b^{-1}ab = a^r, r = 3, 5, 11, 13 >$.

The subgroup $H \cong < a^2 > \times < b^2 > \cong \mathbb{Z}_8 \times \mathbb{Z}_2$. It has been proved in [21] that the representation of the group $\mathbb{Z}_8 \times \mathbb{Z}_2 \cong < f > \times < h >$ is permissible only in the following three cases:

(1) all elements of order 8 correspond to the cusp form $\eta^2(8z)\eta^2(4z)$, all elements of order 4 correspond to the cusp form $\eta^4(4z)\eta^4(2z)$, the element f^4 corresponds to $\eta^8(2z)\eta^8(z)$, two other elements of order 2 correspond to the cusp form $\eta^{12}(2z)$.

(2) all elements of order 8 correspond to the cusp form $\eta^2(8z)\eta^2(4z)$, all elements of the order 4 correspond to the cusp form $\eta^4(4z)\eta^4(2z)$, all elements of order 2 correspond to $\eta^8(2z)\eta^8(z)$.

(3) all elements of order 8 correspond to the cusp form $\eta^3(8z)$, all elements of order 4 correspond to the cusp form $\eta^6(4z)$, the element h corresponds to $\eta^8(2z)\eta^8(z)$, two other elements of order 2 correspond to the cusp form $\eta^{12}(2z)$.

Let us consider the case $r = 3$.

Let T be a desired representation, Φ be such one-dimensional representation that $\Phi(a) = 1, \Phi(b) = i$. Let denote as χ_T and χ_Φ their characters. Because the element a corresponds to $\eta(16z)\eta(8z)$ the element a^2 corresponds to the cusp form $\eta^2(8z)\eta^2(4z)$, the element a^4 corresponds to the cusp form $\eta^4(4z)\eta^4(2z)$, the element a^8 corresponds to the cusp form $\eta^8(2z)\eta^8(z)$. So for the group H the third variant is excluded. If an element $g \neq e,\ b^2,\ a^8,\ a^8b^2$ then we have $\chi_T(g) = 0$ or $\chi_\Phi(g) + \chi_\Phi(g^3) = 0$. The elements $b^2,\ a^8b^2$ are conjugate and correspond to the same cusp form. So we see that the scalar product

$$< \chi_T, \chi_\Phi >= \frac{1}{64} \cdot (\chi_T(e)\bar{\chi}_\Phi(e) + \chi_T(a^8)\bar{\chi}_\Phi(a^8) + 2\chi_T(b^2)\bar{\chi}_\Phi(b^2)).$$

If the elements b^2, a^8b^2 correspond to $\eta^{12}(2z)$ then $\chi_T(b^2) = 0$ and the scalar product is equal to $< \chi_T, \chi_\Phi > = \frac{1}{64} \cdot (24+8) = \frac{1}{2}$. But this number must be an integer. We obtain a contradiction and the desired representation T cannot be constructed.

If the elements $b^2,\ a^8b^2$ correspond to $\eta^8(2z)\eta^8(z)$ then $\chi_T(b^2) = 8$ and the scalar product is equal to $< \chi_T, \chi_\Phi > = \frac{1}{64} \cdot (24+8-2\cdot 8) = \frac{1}{4}$. Since this number must be an integer, we obtain a contradiction.

The group $< a, b : a^{16} = e, b^4 = e, b^{-1}ab = a^3 >$ is isomorphic to $< a, b : a^{16} = e, b^4 = e, b^{-1}ab = a^{11} >$.

The group $< a, b : a^{16} = e, b^4 = e, b^{-1}ab = a^5$ is not permissible. It can be proved similarly. This group is isomorphic to $< a, b : a^{16} = e, b^4 = e,$ $b^{-1}ab = a^{13} >$.

4.6. Metacyclic groups of the kind $< a, b : a^{24} = e, b^s = e,$ $b^{-1}ab = a^r >$.

The group $\mathbb{Z}_{24} \times \mathbb{Z}_m$ is not permissible if $m > 1$. So we have the single possibility:$s = 2$.

The group $< a, b : a^{24} = e, b^2 = e, b^{-1}ab = a^{17} >$ is permissible.

The desired representation contains all irreducible representations with multiplicity 1 except such one-dimensional representations Φ that $\Phi(b) = -1$. The cusp form $\eta(24z)$ corresponds to all elements of order 24, the cusp form $\eta^2(12z)$ corresponds to all elements of order 12, the cusp form $\eta^3(8z)$ corresponds to all elements of order 8, the cusp form $\eta^4(6z)$ corresponds to all elements of order 6, the cusp form $\eta^6(4z)$ corresponds to all elements of order 4, the cusp form $\eta^8(3z)$ corresponds to all elements of order 3, the cusp form $\eta^{12}(2z)$ corresponds to a^{12}, the cusp form $\eta^8(2z)\eta^8(z)$ corresponds to $b, a^{12}b$.

All other groups $< a, b : a^{24} = e, b^2 = e, b^{-1}ab = a^r >$ are not permissible. Let us prove it. If T is a permissible representation of such metacyclic group then $\chi_T(g) = 0$, if $g \neq e$. Then the scalar product

$< \chi_T, \chi_{tr} >= \frac{24}{48} = \frac{1}{2}$. But this number must be an integer. We obtain a contradiction. The theorem is proved.

4.7. Metacyclic groups and multiplicative η-products. Main theorem.

Now we can formulate the following result.

Theorem 4.2. *Let G be such a metacyclic group with the following genetic code*

$$< a, b : a^m = e, b^s = e, b^{-1}ab = a^r >,$$

such that a modular form associated with each element of this group by means of a faithful representation is a multiplicative η-product and the cyclic groups $< a >$ and $< b >$ have no nontrivial intersection. Then for m, s, r (up to an isomorphism) there are only the following possibilities:

$\mathbf{m = 3}$, $s = 2,\ 4,\ 6,\ 8,\ 12,\ 18,\ r = 2.$
$\mathbf{m = 4}$, $s = 2,\ 4,\ 6,\ 8,\ 10,\ 24,\ r = 3.$
$\mathbf{m = 5}$, $s = 4,\ 8,\ 12,\ r = 2;\ s = 2,\ 4,\ 6,\ 8,\ r = 4.$
$\mathbf{m = 6}$, $s = 2,\ 4,\ 6,\ r = 5.$
$\mathbf{m = 7}$, $s = 3,\ 6,\ r = 2;\ s = 6,\ 12,\ r = 3;\ s = 2,\ 4,\ 6,\ r = 6.$
$\mathbf{m = 8}$, $s = 2,\ 4,\ r = 3;\ s = 2,\ 4,\ r = 5;\ s = 2,\ 4,\ r = 7.$
$\mathbf{m = 9}$, $s = 2,\ r = 8;\ s = 4,\ r = 8.$
$\mathbf{m = 10}$, $s = 4,\ 8,\ r = 3;\ s = 2,\ 4,\ r = 9.$
$\mathbf{m = 11}$, $s = 2,\ 4,\ r = 10;\ s = 10,\ 20,\ r = 2;\ s = 10,\ r = 4;\ s = 5,\ r = 5.$
$\mathbf{m = 12}$, $s = 2,\ r = 5,\ 7,\ 11.$
$\mathbf{m = 14}$, $s = 4,\ r = 3;\ s = 6,\ r = 3;\ s = 3,\ r = 9;\ s = 2,\ r = 13.$
$\mathbf{m = 15}$, $s = 4,\ r = 2;\ s = 2,\ r = 4,\ 14.$
$\mathbf{m = 16}$, $s = 2,\ r = 7,\ 9,\ 15.$
$\mathbf{m = 18}$, $s = 2,\ r = 17.$
$\mathbf{m = 20}$, $s = 2,\ r = 9,\ 19;\ s = 4,\ r = 17.$
$\mathbf{m = 21}$, $s = 2,\ r = 8,\ 20;\ s = 3,\ r = 4;\ s = 6,\ r = 2.$
$\mathbf{m = 22}$, $s = 2,\ r = 21;\ s = 5,\ r = 3;\ s = 10,\ r = 7.$
$\mathbf{m = 23}$, $s = 2,\ r = 22;\ s = 11,\ r = 10;\ s = 22,\ r = 5.$
$\mathbf{m = 24}$, $s = 2,\ r = 17.$

It has been proved in this article and in the works [19], [22].

5. Sylow subgroups of permissible groups

Theorem 5.1. *Let G be a finite group such that there is a faithful representation T with the following property. For every $g \in G$ the characteristic polynomial of the operator $T(g)$ is of the form $P_g(x) = \prod_{k=1}^{s}(x^{a_k} - 1)^{t_k}$, and the corresponding cusp form $\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$ is a multiplicative η-product.*

Then for the Sylow p-subgroups, $p \neq 2$, there are only the following possibilities:

$$S(3) \cong \mathbb{Z}_3, \quad S(3) \cong \mathbb{Z}_3 \times \mathbb{Z}_3, \quad S(3) \cong \mathbb{Z}_9,$$

$$S(3) \cong < a, b, c : a^3 = e, b^3 = 3, c^3 = e, ab = bac, ac = ca, bc = cb >,$$

$$S(5) \cong \mathbb{Z}_5, \quad S(7) \cong \mathbb{Z}_7, \quad S(11) \cong \mathbb{Z}_{11}.$$

Proof.

The Sylow 3-subgroups.

We shall describe all the cases in detail.

A permissible 3-group can contain only elements of order 1, 3 and 9.

Let T be a desired representation, T_1 be the trivial representation $(T_1(g) = 1, \forall g \in G.)$

The group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

We must consider three cases.

(1) All elements of order 3 correspond to the cusp form $\eta^8(3z)$. Then $\chi_T(e) = 24, \chi_T(g) = 0, ord(g) \neq 3$. The scalar product

$$< \chi_T, \chi_1 >= \frac{24}{9} = \frac{8}{3}.$$

But this number must be an integer. We obtain a contradiction and the desired representation T cannot be constructed.

(2) All elements of order 3 correspond to the cusp form $\eta^6(3z)\eta^6(z)$. In this case the group is permissible. The desired representation contains T_1 with multiplicity 8, all other irreducible representations with multiplicity 2.

(3) In this case u elements correspond to the cusp form $\eta^6(3z)\eta^6(z)$, and v elements correspond to the cusp form $\eta^8(3z)$, where u and v are positive integers. The group is permissible. Since g and g^2 correspond to the same modular form, then u and v are even. The scalar product

$$< \chi_T, \chi_1 >= \frac{1}{9} \cdot (24 + 6u) = \frac{1}{3} \cdot (8 + 2u).$$

Since this number must be an integer we have $u = 2$. This variant is suitable. Let f and f^2 be the elements which correspond to $\eta^6(3z)\eta^6(z)$. Let T_k be an one-dimensional representation of our group and m_k be its multiplicity in T. If $T_k(f) = 1$ then $m_k = 4$. For other one-dimensional representation $m_k = 2$. The representation T is permissible.

The group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Let us prove that this group is not permissible. We must consider two cases.

(1) All elements of order 3 correspond to the cusp form $\eta^6(3z)\eta^6(z)$. The scalar product

$$< \chi_T, \chi_1 >= \frac{1}{27} \cdot (24 + 26 \cdot 6) = \frac{20}{3}.$$

But this number must be an integer. We obtain a contradiction.

(2) In this case u elements correspond to the cusp form $\eta^6(3z)\eta^6(z)$, and v elements correspond to the cusp form $\eta^8(3z)$, where u and v are positive integers. The numbers u and v are even. The scalar product

$$< \chi_T, \chi_1 >= \frac{1}{27} \cdot (24 + 6u) = \frac{1}{9} \cdot (8 + 2u).$$

Since this number must be an integer we have $u = 14$, $m_1 = 4$. Let T_k be an one-dimensional representation of our group and m_k be its multiplicity in T.

Let u_1 be the number of elements g which correspond to the cusp form $\eta^6(3z)\eta^6(z)$ and $T_k(g) = 1$; let u_2 be the number of elements g which correspond to the cusp form $\eta^6(3z)\eta^6(z)$ and $T_k(g) = \zeta_3$; let u_3 be the number of elements g which correspond to the cusp form $\eta^6(3z)\eta^6(z)$ and $T_k(g) = \zeta_3^2$. Then $u_2 = u_3, u_1 + 2u_2 = 14$.

$$\begin{aligned} < \chi_T, \chi_k > &= \frac{1}{27} \cdot (24 + 6(u_1 + \zeta_3 \cdot u_2 + \zeta_3^2 \cdot u_3)) \\ &= \frac{1}{27} \cdot (24 + 6u_1 - 6u_2) \\ &= \frac{1}{9} \cdot (8 + 2(u_1 - u_2)). \end{aligned}$$

If T_k is not trivial then $u_1 \neq 14$, and $u_1 - u_2 = 5$. We obtain $u_1 = 8$, $u_2 = 3$, and $Ker(T_k)$ does not contain elements corresponding to the cusp form $\eta^8(3z)$. It contains 8 elements which correspond to the cusp form $\eta^6(3z)\eta^6(z)$ and the element e. Its order is equal to 9. If an element h corresponds to the cusp form $\eta^8(3z)$ then $T_k(h) \neq 1$ for any $k \neq 1$. Then among eigenvalues of the operator $T(h)$ there are only 4 eigenvalues equal to 1 and the characteristic polynomial of the operator $T(h)$ cannot be $(x^3 - 1)^8$. We obtain a contradiction.

The group $\mathbb{Z}_9 \times \mathbb{Z}_3$.

Let us prove that this group is not permissible.

In this group there are 8 elements of the order 3, 18 elements of the order 9 and the element e.

$$\chi_T(e) = 24;\ \chi_T(g) = 0, ord(g) = 9;\ \chi_T(g) = 6, ord(g) = 3.$$

The number $< \chi_T, \chi_1 >= \frac{8}{3}$. But this number must be an integer. We obtain a contradiction.

The group $S(3) \cong < a, b, c : a^3 = e, b^3 = e, c^3 = e, ab = bac, ac = ca, bc = cb >$.

This group is of order 27 and it has 11 conjugacy classes. They are:

$$1.e\ 2.c\ 3.c^2\ 4.a, ac, ac^2\ 5.b, bc, bc^2\ 6.ab, abc, abc^2\ 7.a^2b, a^2bc, a^2bc^2$$

$$8.a^2, a^2c, a^2c^2\ 9.b^2, b^2c, b^2c^2\ 10.ab^2, ab^2c, ab^2c^2\ 11.a^2b^2, a^2b^2c, a^2b^2c^2.$$

The commutant is generated by the element c.

$$G/G' \cong \mathbb{Z}_3 \times \mathbb{Z}_3.$$

This group has the following irreducible representations:

$$\begin{array}{lll} T_k(a) = {\zeta_3}^k, & T_k(b) = \zeta_3, & T_k(c) = 1,\ k = \overline{1,3}, \\ T_k(a) = {\zeta_3}^k, & T_k(b) = {\zeta_3}^2, & T_k(c) = 1,\ k = \overline{4,6}, \\ T_k(a) = {\zeta_3}^k, & T_k(b) = 1, & T_k(c) = 1,\ k = \overline{7,9}, \end{array}$$

$$T_{10}(a) = T_{11}(b) = \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{10}(b) = T_{11}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$T_{10}(c) = \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \quad T_{11}(c) = T_{10}(c^2).$$

The desired representation is the direct sum

$$2(T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus T_7 \oplus T_8 \oplus T_9) \oplus T_{10} \oplus T_{11}.$$

The elements $a, b, a^2, b^2, ac, bc, a^2c, b^2c, ac^2, bc^2, a^2c^2, b^2c^2$ correspond to the cusp form $\eta^8(3z)$, all other elements of the order 3 correspond to the cusp form $\eta^6(3z)\eta^6(z)$.

The group $< a, b : a^9 = e, b^3 = e, b^{-1}ab = a^4 >$ is not permissible. This has been proved in 4.3.1.

The Sylow p-subgroups, $p = 5, 7, 11$.

Let us prove that the group $\mathbb{Z}_5 \times \mathbb{Z}_5$ is not permissible. In this group there are 24 elements of the order 5.

$$\chi_T(e) = 24;\ \chi_T(g) = 4,\ ord(g) = 5.$$

The number $< \chi_T, \chi_1 >= \frac{120}{25} = \frac{24}{5}$. But this number must be an integer. We obtain a contradiction. The elements of order 25 do not correspond to the multiplicative η-products. So we have the only possibility: $S(5) \cong \mathbb{Z}_5$.

We consider the cases $p = 7, 11$ by an analogous way. Now we can prove the following result.

Theorem 5.2. *There is no finite solvable group G such that a faithful representation assigns to every element of G a multiplicative η-product, every product being assigned to some element.*

Proof.

The order of this group will be equal to $2^k \cdot 3^m \cdot 5 \cdot 7 \cdot 11$. According to the theorem of Ph.Hall [5] there is a subgroup of order 35 in this group. It is known that there is only one group of order 35. It is $\mathbb{Z}_{35}$. But elements of order 35 do not correspond to the multiplicative η-products. The theorem is proved.

In conclusion we shall formulate an open problem. It will be very interesting

to find an algebraic structure such that all multiplicative η-products, and only them, are associated with its elements in a natural way.

6. Multiplicative η-products and the adjoint representations $SL(5, \mathbb{C})$.

In this section we give some results about the connection between multiplicative η-products and elements of finite order in $SL(5, \mathbb{C})$ by means of the adjoint representation. They were proved in [16] and [22].

Theorem 6.1. *All multiplicative η-products of weight greater than 1 can be associated, by the adjoint representation, with elements of finite order in the group $SL(5, \mathbb{C})$. The eigenvalues of the element $g \in SL(5, \mathbb{C})$ that corresponds to a given cusp form can be found uniquely, up to a permutation of the values, up to raising eigenvalues to a power coprime with the order of the element g, and up to the multiplying each eigenvalue by the same fifth root of unity.*

Theorem 6.2. *The maximal finite subgroups of $SL(5, \mathbb{C})$, whose elements g have such characteristic polynomials $P_g(x) = \prod_{k=1}^{s}(x^{a_k} - 1)^{t_k}$ that the corresponding cusp forms $\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$ are multiplicative η-products, are the direct products of the group $\mathbb{Z}_5$ (which is generated by the scalar matrix) and one of the following groups: $S_4, A_4 \times \mathbb{Z}_2, Q_8 \times \mathbb{Z}_3, D_4 \times \mathbb{Z}_3, D_6$, the binary tetrahedral group, the metacyclic group of order 21, the group of order 12 : $< a, b : a^3 = b^2 = (ab)^2 >$, all groups of order 16, $\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9, \mathbb{Z}_{10}, \mathbb{Z}_{11}, \mathbb{Z}_{14}, \mathbb{Z}_{15}$.*

Theorem 6.3. *Let Ad be the adjoint representation of the group $SL(5,\mathbb{C})$ and $g \in SL(5,\mathbb{C})$, $ord(g) \neq 3,6,9,21$, is such that the characteristic polynomial of the operator $Ad(g)$ is of the form*

$$P_g(x) = \prod_{k=1}^{s} (x^{a_k} - 1)^{t_k}, \ a_k \in \mathbb{N}, \ t_k \in \mathbb{N}.$$

Then the corresponding cusp form $\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$ is a multiplicative η-product of the weight $k(g) > 1$, and all multiplicative η-products of the weight $k(g) > 1$ can be obtained by this way.

If $ord(g) = 3,6,9,21$ then by this way we can obtain all multiplicative η-products of the weight $k(g) > 1$. Moreover in this correspondence there are five modular forms which are not multiplicative η-products:

$$\eta^4(3z)\eta^{12}(z), \ \eta^7(3z)\eta^3(z), \ \eta^2(6z)\eta^6(2z), \ \eta^2(9z)\eta(3z)\eta^3(z), \ \eta(21z)\eta^3(z).$$

Sketch of the proof.

We shall express in this sketch all main ideas of the proof.

Let $T : SL(5,\mathbb{C}) \to GL(V)$ be a natural representation of $SL(5,\mathbb{C})$ in a five-dimensional vector space V; $T^* : SL(5,\mathbb{C}) \to GL(V^*)$ is the conjugate representation.

Let us consider the representation $T \otimes T^* : SL(5,\mathbb{C}) \to GL(V \otimes V^*)$. This representation can be decomposed into the direct sum $T_1 \oplus T_2$, where T_1 is the adjoint representation Ad of the group $SL(5,\mathbb{C})$ in 24-dimensional space, T_2 is one-dimensional identity representation. Let $\lambda_1, \ \lambda_2, \ \lambda_3, \ \lambda_4, \ \lambda_5$ be the eigenvalues of the operator $T(g)$.The elements $\lambda_k/\lambda_m, \ 1 \leq l, \ m \leq 5$, are the eigenvalues of the operator $T \otimes T^*(g)$. Excluding one eigenvalue equal to 1, we obtain the set of eigenvalues of the operator $Ad(g)$.

Since there are four or more values equal to 1 among the eigenvalues of the operator $Ad(g)$, the weight of the modular form $\eta_g(z)$ associated with g is greater than 1.

The number equal to 1 eigenvalues of the operator $Ad(g)$ is determined by the number of equal eigenvalues of the operator $T(g)$. This correspondence is described in the table next page. We denote identical values by identical symbols and different values by different symbols.

Let us consider the problem for each order from 1 to 24. The identity element corresponds to $\eta^{24}(z)$. Let us denote the characteristic polynomial of the operator $Ad(g)$ by $P_g(x)$, the primitive root of unity of degree m by ζ_m, the m-th cyclotomic polynomial by Φ_m. We shall denote the number of units among eigenvalues of the operator $Ad(g)$ by s.

Table 3.

Eigenvalues of $T(g)$	Number of units among the eigenvalues of the operator $Ad(g)$
(a, b, c, d, e)	4
(a, a, b, c, d)	6
(a, a, b, b, c)	8
(a, a, a, b, c)	10
(a, a, a, b, b)	12
(a, a, a, a, b)	16
(a, a, a, a, a)	24

The case ord(g) = 2.

We have

$$P_g(x) = (x^2 - 1)^k (x - 1)^m, \ 2k + m = 24, \ s = k + m, \ 0 < k, \ 0 \leq m.$$

The values s from the table which satisfy these conditions are $s = 12, 16$. So $k = 12, m = 0$; $k = 8, m = 8$. The modular form $\eta^{12}(2z)$ corresponds to the set $(1, 1, 1, -1, -1)$ of eigenvalues of the operator $T(g)$, the form $\eta^8(2z)\eta^8(z)$ corresponds to the set $(1, -1, -1, -1, -1)$ of eigenvalues of the operator $T(g)$.

The case ord(g) = 3.

We have

$$P_g(x) = (x^3 - 1)^k (x - 1)^m, \ 3k + m = 24, \ s = k + m, 0 < k, 0 \leq m.$$

The values from the table which satisfy these conditions are $s = 8, 10, 12, 16$. We have $k = 8, m = 0$; $k = 7, m = 3$; $k = 6, m = 6$; $k = 4, m = 12$. The modular form $f_1 = \eta^8(3z)$ corresponds to the set of eigenvalues of the operator $T(g)$: $(\zeta_3, \zeta_3, \zeta_3^2, \zeta_3^2, 1)$, the form $f_2 = \eta^6(3z)\eta^6(z)$ corresponds to $(\zeta_3, \zeta_3, \zeta_3, 1, 1)$. These two functions are multiplicative η-products.

The functions $f_3 = \eta^7(3z)\eta^3(z)$ and $f_4 = \eta^4(3z)\eta^{12}(z)$ are not multiplicative η-products. The function f_3 corresponds to the set of eigenvalues $(\zeta_3, \zeta_3^2, 1, 1, 1)$, the function f_4 corresponds to the set of eigenvalues $(\zeta_3^2, \zeta_3, \zeta_3, \zeta_3, \zeta_3)$. We note the interesting relations:

$$f_3^2 = f_1 f_2, \ f_2^2 = f_1 f_4.$$

The case ord(g) = 4.

We have

$$P_g(x) = (x^4-1)^k(x^2-1)^m(x-1)^l,\ 4k+2m+l = 24,\ s = k+m+l,\ 2|(k+m).$$

The characteristic polynomial can be written as the product of cyclotomic polynomials,namely, $P_g = \Phi_4^k\Phi_2^{k+m}\Phi_2^{k+m+l}$.

Let us consider the characteristic polynomial of the operator $Ad(g^2)$: $P_{g^2} = \Phi_2^{2k}\Phi_1^{2k+2m+l}$. There are two possibilities : $P_{g^2}(x) = (x^2-1)^{12}$ or $P_{g^2}(x) = (x^2-1)^8(x-1)^8$. In this case $2k = 12,\ 2k+2m++l = 12$, in the second case $2k = 8,\ 2k+2m++l = 16$. The following values satisfy to the conditions:

$$k = 6,\ m = 0,\ l = 0;\ k = 4,\ m = 4,\ l = 0;$$

$$k = 4,\ m = 2,\ l = 4;\ k = 4,\ m = 0,\ l = 8.$$

In the latter case the set of eigenvalues must be of type (a, a, a, b, b). But in this case there are at most two distinct values not equal to 1 among the quotients $\frac{\lambda_k}{\lambda_m}$. Since the set of eigenvalues of the operator $Ad(g)$ must contain $i, -i$ and -1,we obtain a contradiction. The three other possibilities correspond to multiplicative η-products. The modular form $\eta^6(4z)$ corresponds to the set $(i, -i, -1, -1, 1)$ of eigenvalues of the operator $T(g)$, the form $\eta^4(4z)\eta^4(2z)$ corresponds to the set $(i, -i, 1, 1, 1)$ of eigenvalues of the operator $T(g)$, the form $\eta^4(4z)\eta^2(2z)\eta^4(z)$ corresponds to the set $(i, -i, i, -i, 1)$ of eigenvalues of the operator $T(g)$.

The case ord(g) = 5.

We have

$$P_g(x) = (x^5-1)^k(x-1)^m,\ 5k+m = 24,\ s = k+m,\ 0 < k \le 4,\ 0 \le m.$$

We obtain $s = 24 - 4k$; therefore, $4|s$. Since there are four or more distinct values not equal to 1 among the the quotients $\frac{\lambda_k}{\lambda_m}$, we have $s \le 10$. Taking these conditions into account, we obtain the unique possibility $k = 4, m = 4$. The form $\eta^4(5z)\eta^4(z)$ corresponds to $(\zeta_5^4, \zeta_5^2, \zeta_5^2, \zeta_5, \zeta_5)$.

We consider the other cases by the analogous way.

In conclusion we write the table of the eigenvalues of the elements in $SL(5, \mathbb{C})$ which correspond to multiplicative η-products by the described way.

Table 4.

Eigenvalues	Cusp forms
1,1,1,1,1	$\eta^{24}(z)$
-1,-1,-1,-1,1	$\eta^8(2z)\eta^8(z)$
-1,-1,1,1,1	$\eta^{12}(2z)$
$\zeta_3, \zeta_3, \zeta_3$, 1,1	$\eta^6(3z)\eta^6(z)$
$\zeta_3^2, \zeta_3^2, \zeta_3, \zeta_3$,1	$\eta^8(3z)$
$\zeta_4^3, \zeta_4^2, \zeta_4, \zeta_4, \zeta_4$	$\eta^4(4z)\eta^2(2z)\eta^4(z)$
$\zeta_4^3, \zeta_4^3, \zeta_4, \zeta_4$,1	$\eta^4(4z)\eta^4(2z)$
$\zeta_4^3, \zeta_4^2, \zeta_4^2, \zeta_4$,1	$\eta^6(4z)$
$\zeta_5^3, \zeta_5, \zeta_5$, 1,1	$\eta^4(5z)\eta^4(z)$
$\zeta_6^4, \zeta_6^3, \zeta_6^3, \zeta_6, \zeta_6$	$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$
$\zeta_6^5, \zeta_6^3, \zeta_6^2, \zeta_6^2$,1	$\eta^3(6z)\eta^3(2z)$
$\zeta_6^5, \zeta_6^4, \zeta_6^2, \zeta_6$,1	$\eta^4(6z)$
$\zeta_8^7, \zeta_8^5, \zeta_8^3, \zeta_8$,1	$\eta^2(8z)\eta^2(4z)$
$\zeta_8^5, \zeta_8^5, \zeta_8^3, \zeta_8^2, \zeta_8$,	$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$
$\zeta_9^8, \zeta_9^5, \zeta_9^3, \zeta_9^2$,1	$\eta^2(9z)\eta^2(3z)$
$\zeta_{10}^8, \zeta_{10}^6, \zeta_{10}^5, \zeta_{10}$,1	$\eta^2(10z)\eta^2(2z)$
$\zeta_{11}^9, \zeta_{11}^5, \zeta_{11}^4, \zeta_{11}^3, \zeta_{11}$	$\eta^2(11z)\eta^2(z)$
$\zeta_{12}^9, \zeta_{12}^7, \zeta_{12}^4, \zeta_{12}^3, \zeta_{12}$,	$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$
$\zeta_{14}^{11}, \zeta_{14}^9, \zeta_{14}^7, \zeta_{14}, \zeta_{14}$,	$\eta(14z)\eta(7z)\eta(2z)\eta(z)$
$\zeta_{15}^{12}, \zeta_{15}^{10}, \zeta_{15}^7, \zeta_{15}$,1	$\eta(15z)\eta(5z)\eta(3z)\eta(z)$

References

[1] A.J.F. Biagioli, *The construction of modular forms as products of transforms of the Dedekind eta-function.* Acta Arith. **LIV** (1990), 274–300.

[2] H.S.M. Coxeter, W.O.J. Moser, *Generators and relations for discrete groups.* Springer-Verlag (1965), 161 pp.

[3] D. Dummit, H. Kisilevsky, J. McKay, *Multiplicative products of η-functions.* Contemp. Math. **45** (1985), 89–98.

[4] B. Gordon, S. Sinor, *Multiplicative properties of η-products.* Lecture Notes in Math. **1395** (1989), 173–200. (Springer-Verlag)

[5] M. Hall,jr, *The theory of groups.* The Macmillan Company. New York (1959).

[6] K. Harada, *Another look at the Frame shapes of finite groups.* J. Fac. Sci. Univ. Tokyo. Sect. IA. Math. **34** (1987), 491–512.

[7] T. Hiramatsu *Theory of automorphic forms of weight 1.* Advanced Studies in Pure Math. **13** (1988), 503–584.

[8] N. Ishii, *Cusp forms of weight one, quartic reciprocity and elliptic curves.* Nagoya Math. J. **98** (1985), 117–137.

[9] M. Koike, *On McKay's conjecture.* Nagoya Math. J. **95** (1984), 85–89.

[10] T. Kondo, *Examples of multiplicative η-products.* Sci. Pap. Coll. Arts and Sci. Univ. Tokyo **35** (1986), 133–149 .

[11] Y. Martin, K. Ono, *Eta-quotients and elliptic curves.* Proc. Amer. Math. Soc. **125** (1997), 3169–3176.

[12] G. Mason, *Finite groups and Hecke operators.* Math. Ann. **282** (1989), 381–409.

[13] G. Mason, *M_{24} and certain automorphic forms.* Contemp. Math. **45** (1985), 223–244.

[14] K. Ono, *Shimura sums related to imaginary quadratic fields*. Proc. Japan Acad. **70 (A)** (1994), 146–151.
[15] G.V. Voskresenskaya, *Modular forms and representations of the dihedral group*. Math. Notes. **63** (1998), 130–133.
[16] G.V. Voskresenskaya, *Cusp forms and finite subgroups in* $SL(5, \mathbb{C})$. Functional Anal. Appl. **29** (1995), 71–73.
[17] G.V. Voskresenskaya, *Modular forms and regular representations of groups of order 24*. Math. Notes **60** (1996), 292–294.
[18] G.V. Voskresenskaya, *One special class of modular forms and group representations*. J. Th. Nombres Bordeaux **11**(1999), 247–262.
[19] G.V. Voskresenskaya, *Metacyclic groups and modular forms*. Math. Notes. **67** (2000), 18–25.
[20] G.V. Voskresenskaya, *Finite groups and multiplicative* η*-products*. Vestnik SamGU **16** (2000), 18–25.
[21] G.V. Voskresenskaya, *Abelian groups and modular forms* Vestnik SamGU **28** (2003), 21–34.
[22] G.V. Voskresenskaya, *Multiplicative products of Dedekind* η*-functions and group representations*. Math. Notes. **73** (2003), 482–495.

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