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Linear independence of continued fractions

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Résumé. Nous donnons un critère d'indépendance linéaire sur le corps des rationnels qui s'applique à une famille donnée de nombres réels dont les développements en fractions continues satisfont certaines conditions.

Abstract. The main result of this paper is a criterion for linear independence of continued fractions over the rational numbers. The proof is based on their special properties.

1. Introduction

Forty years ago Davenport and Roth in [2] proved that the continued fraction $[a_1, a_2, \dots]$, where $a_1, a_2, \dots$ are positive integers satisfying

$$\limsup_{n\to\infty}((\log\log a_n)\frac{\sqrt{\log n}}{n}) = \infty,$$

is a transcendental number. The generalization of transcendence is algebraic independence and there are several results concerning the algebraic independence of continued fractions. See, for instance, Bundschuh [1] or Hančl [5]. On the other hand it is a well known fact that if a positive real number has a finite continued fractional expansion then it is a rational number, and if not it is an irrational number. Irrationality is a special case of linear independence and this paper deals with such a theory. By the way, as to linear independence of series, one can find the criterion in [4], for instance.

2. Linear independence

Theorem 2.1. *Let $\epsilon > 1$ be a real number, K be a natural number and $\{a_{j,n}\}_{n=1}^{\infty}$ $(j = 1, 2, \dots, K)$ be K sequences of positive integers such that*

$$a_{j+1,n} > a_{j,n}(1 + \frac{\epsilon}{n \log n}) \tag{1}$$

and

$$a_{1,n+1} > a_{K,n}^{K-1}(1 + \frac{1}{n}) \tag{2}$$

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hold for every sufficiently large positive integer n *and* $j = 1, 2, 3, \dots, K-1$. *Then the continued fractions* $\alpha_j = [a_{j,1}, a_{j,2}, \dots]$ $(j = 1, 2, \dots, K)$ *and the number* 1 *are linearly independent over the rational numbers.*

Lemma 2.1. *Let* $a_{j,n}$, $j = 1, 2, \dots, K$, $n = 1, 2, ..$ *and* $K > 2$ *satisfy all conditions stated in Theorem 2.1. Then*

$$\prod_{n=1}^{\infty}(1 + \frac{1}{a_{j,n}}) = C_j < \infty.$$

Proof of Lemma 2.1. From (1) and (2) we obtain

$$\begin{aligned} a_{j,n} &\geq a_{1,n}(1 + \frac{\epsilon}{n \log n})^{j-1} > a_{K,n-1}^{K-1}(1 + \frac{1}{n-1})(1 + \frac{\epsilon}{n \log n})^{j-1} \\ &> a_{j,n-1}^{K-1}(1 + \frac{\epsilon}{(n-1)\log(n-1)})^{(K-1)(K-j)}(1 + \frac{1}{n-1})(1 + \frac{\epsilon}{n \log n})^{j-1} \\ &\geq a_{j,n-1}(1 + \frac{1}{n-1})(1 + \frac{\epsilon}{n \log n}) > (1 + \frac{1}{n})(1 + \frac{\epsilon}{n \log n})a_{j,n-1} \end{aligned}$$

for every sufficiently large positive integer n and $j = 1, 2, \dots, K$. By mathematical induction we get

$$a_{j,n} \geq Y \prod_{j=2}^{n}(1 + \frac{1}{j})(1 + \frac{\epsilon}{j \log j})$$

for every $n = 2, 3, \dots$ and $j = 1, 2, \dots, K$, where Y is a positive real constant which does not depend on n. It follows that

$$\prod_{n=2}^{\infty}(1 + \frac{1}{a_{j,n}}) \leq \prod_{n=2}^{\infty}(1 + \frac{1}{Y \prod_{i=2}^{n}(1 + \frac{1}{i})(1 + \frac{\epsilon}{i \log i})}) = C_j < \infty$$

because the series

$$\sum_{n=2}^{\infty} \frac{1}{\prod_{j=2}^{n}(1 + \frac{1}{j})(1 + \frac{\epsilon}{j \log j})}$$

is convergent. (To prove this last fact one can use Bertrand's criterion for convergent series, for instance. See [3] for example.) □

Proof of Theorem 2.1. If $K = 1$, then α_1 has an infinite continued fraction expansion. In this case α_1 is irrational and Theorem 2.1 holds. Now we will consider the case in which $K \geq 2$ and n is a sufficiently large positive integer. Let us assume that there exist $K+1$ integers $A_1, A_2, \dots, A_K, A_{K+1}$ (not all of which equal zero) such that

$$A_{K+1} = \sum_{j=1}^{K} A_j \alpha_j. \tag{3}$$

We can write each continued fraction α_j $(j = 1, 2, \dots, K)$ in the form

$$\alpha_j = \frac{p_{j,n}}{q_{j,n}} + R_{j,n} \tag{4}$$

where $\frac{p_{j,n}}{q_{j,n}} = [a_{j,1}, a_{j,2}, \dots, a_{j,n}]$ is the n-th partial fraction of α_j and $R_{j,n}$ is the remainder. For $R_{j,n}$ we have the estimation

$$|R_{j,n}| = |\alpha_j - \frac{p_{j,n}}{q_{j,n}}| < \frac{1}{a_{j,n+1}q_{j,n}^2} \tag{5}$$

and

$$|R_{j,n}| > \frac{c}{a_{j,n+1}q_{j,n}^2} \tag{6}$$

where $c > 0$ is a constant which depends only on $\alpha_1, \alpha_2, \dots, \alpha_K$. (For the proof see, for instance, [6].) Substituting (4) into (3) we obtain

$$A_{K+1} = \sum_{j=1}^{K} A_j(\frac{p_{j,n}}{q_{j,n}} + R_{j,n}).$$

Multiplying both sides of the last equation by $\prod_{j=1}^{K} q_{j,n}$ we obtain

$$A_{K+1}\prod_{j=1}^{K} q_{j,n} = \prod_{j=1}^{K} q_{j,n} \sum_{j=1}^{K} A_j(\frac{p_{j,n}}{q_{j,n}} + R_{j,n}).$$

This implies

$$M_n = (A_{K+1} - \sum_{j=1}^{K} A_j \frac{p_{j,n}}{q_{j,n}}) \prod_{j=1}^{K} q_{j,n} = \prod_{j=1}^{K} q_{j,n} \sum_{j=1}^{K} A_j R_{j,n} \tag{7}$$

where M_n is an integer.

First we will prove that $|M_n| > 0$. Let P be the least positive integer such that $A_P \neq 0$. (Such a P must exist because not every A_j is equal to zero.) Then we have

$$\begin{aligned}|M_n| &= |\prod_{j=1}^{K} q_{j,n} \sum_{j=1}^{K} A_j R_{j,n}| = |\prod_{j=1}^{K} q_{j,n} \sum_{j=P}^{K} A_j R_{j,n}| \\ &\geq \prod_{j=1}^{K} q_{j,n}(|A_P||R_{P,n}| - \sum_{j=P+1}^{K} |A_j||R_{j,n}|).\end{aligned}$$

This, (5) and (6) imply

$$|M_n| \geq \prod_{j=1}^{K} q_{j,n}(|A_P|\frac{c}{a_{P,n+1}q_{P,n}^2} - \sum_{j=P+1}^{K} |A_j|\frac{1}{a_{j,n+1}q_{j,n}^2}).$$

From this last inequality and (1) we obtain

$$|M_n| \geq \prod_{j=1}^{K} q_{j,n}(|A_P|\frac{c}{a_{P,n+1}q_{P,n}^2} - \frac{\sum_{j=P+1}^{K}|A_j|}{a_{P+1,n+1}q_{P+1,n}^2}) \tag{8}$$

$$\geq \frac{\prod_{j=1}^{K} q_{j,n}|A_P|c}{a_{P+1,n+1}q_{P+1,n}^2}(\frac{a_{P+1,n+1}q_{P+1,n}^2}{a_{P,n+1}q_{P,n}^2} - \frac{\sum_{j=P+1}^{K}|A_j|}{|A_P|c})$$

$$= B(\frac{a_{P+1,n+1}q_{P+1,n}^2}{a_{P,n+1}q_{P,n}^2} - C)$$

where B is a positive real number and C is a constant which does not depend on n. We also have

$$\prod_{i=1}^{n} a_{j,i} < q_{j,n} < \prod_{i=1}^{n}(a_{j,i}+1) \tag{9}$$

for every $j = 1, 2, \ldots, K$, $n = 1, 2, \ldots$ which can be proved by mathematical induction using

$$q_{j,n+1} = a_{j,n+1}q_{j,n} + q_{j,n-1}.$$

(This identity can be found, for instance, in [6].) (8) and (9) imply

$$|M_n| \geq B(\frac{a_{P+1,n+1}}{a_{P,n+1}}\prod_{j=1}^{n}(\frac{a_{P+1,j}}{a_{P,j}+1})^2 - C)$$

$$= B(\frac{a_{P+1,n+1}}{a_{P,n+1}}(\prod_{j=1}^{n}(\frac{a_{P+1,j}}{a_{P,j}})\frac{1}{\prod_{j=1}^{n}(1+\frac{1}{a_{P,j}})})^2 - C).$$

This, Lemma 2.1 and (1) imply

$$|M_n| \geq B(E\frac{1+\frac{\epsilon}{(n+1)\log(n+1)}}{(\prod_{j=1}^{\infty}(1+\frac{1}{a_{P,j}}))^2}\prod_{j=1}^{n}(1+\frac{\epsilon}{j\log j})^2 - C) \tag{10}$$

$$> B(D\prod_{j=1}^{n}(1+\frac{\epsilon}{j\log j}) - C)$$

where $D > 0$ is a constant which does not depend on n. From (10) and the fact that $\prod_{j=1}^{\infty}(1+\frac{\epsilon}{n\log n}) = \infty$ we obtain

$$|M_n| > 0 \tag{11}$$

for every sufficiently large positive integer n.

Now we will prove that $|M_n| < 1$ for n sufficiently large. From (7) we obtain

$$|M_n| = \prod_{j=1}^{K} q_{j,n}|\sum_{j=1}^{K} A_j R_{j,n}| \leq \prod_{j=1}^{K} q_{j,n}\sum_{j=1}^{K}|A_j||R_{j,n}|.$$

This and (5) imply

$$|M_n| \le \prod_{j=1}^{K} q_{j,n} \sum_{j=1}^{K} |A_j| \frac{1}{a_{j,n+1} q_{j,n}^2}.$$

From this and (1) we obtain

$$(12) \qquad \begin{aligned} |M_n| &\le \prod_{j=1}^{K} q_{j,n} \sum_{j=1}^{K} |A_j| \frac{1}{a_{1,n+1} q_{1,n}^2} \\ &= \frac{\prod_{j=2}^{K} q_{j,n}}{a_{1,n+1} q_{1,n}} \sum_{j=1}^{K} |A_j| = F \frac{\prod_{j=2}^{K} q_{j,n}}{a_{1,n+1} q_{1,n}} \end{aligned}$$

where $F = \sum_{j=1}^{K} |A_j|$ is a positive real constant which does not depend on n. (9) and (12) imply

$$|M_n| \le F \frac{\prod_{j=2}^{K} q_{j,n}}{a_{1,n+1} q_{1,n}} \le F \frac{\prod_{j=2}^{K} \prod_{i=1}^{n} (a_{j,i}+1)}{\prod_{i=1}^{n+1} a_{1,i}}.$$

From this and Lemma 2.1 we obtain

$$(13) \qquad \begin{aligned} |M_n| &\le F \frac{\prod_{j=2}^{K} \prod_{i=1}^{n} (a_{j,i}+1)}{\prod_{i=1}^{n+1} a_{1,i}} \\ &= F \frac{\prod_{j=2}^{K} \prod_{i=1}^{n} a_{j,i}}{\prod_{i=1}^{n+1} a_{1,i}} \prod_{j=2}^{K} \prod_{i=1}^{n} (1 + \frac{1}{a_{j,i}}) \\ &\le F \frac{\prod_{j=2}^{K} \prod_{i=1}^{n} a_{j,i}}{\prod_{i=1}^{n+1} a_{1,i}} \prod_{j=2}^{K} \prod_{i=1}^{\infty} (1 + \frac{1}{a_{j,i}}) \\ &= F \frac{\prod_{j=2}^{K} C_j a_{j,1}}{a_{1,1} a_{1,2}} \frac{\prod_{j=2}^{K} \prod_{i=2}^{n} a_{j,i}}{\prod_{i=3}^{n+1} a_{1,i}} \\ &= H \frac{\prod_{j=2}^{K} \prod_{i=2}^{n} a_{j,i}}{\prod_{i=3}^{n+1} a_{1,i}} \end{aligned}$$

where $H > 0$ is a constant which does not depend on n. (1), (2) and (13) imply

$$\begin{aligned}|M_n| &< H\frac{\prod_{j=2}^{K}\prod_{i=2}^{n}a_{j,i}}{\prod_{i=3}^{n+1}a_{1,i}} \le G\frac{\prod_{j=2}^{K}\prod_{i=2}^{n}a_{K,i}}{\prod_{i=3}^{n+1}a_{1,i}}\\ &= G\frac{\prod_{i=2}^{n}a_{K,i}^{K-1}}{\prod_{i=3}^{n+1}a_{1,i}} = G\prod_{i=2}^{n}\frac{a_{K,i}^{K-1}}{a_{1,i+1}} \le L\prod_{i=2}^{n}\frac{1}{1+\frac{1}{i}}\\ &= \frac{L}{\prod_{i=2}^{n}(1+\frac{1}{i})}\end{aligned}$$

where L is a positive real constant which does not depend on n. It follows that $|M_n| < 1$ for every sufficiently large positive integer n. This and (11) imply that $0 < |M_n| < 1$ for every sufficiently large n, where M_n is an integer. This is impossible therefore the numbers $\alpha_1, \alpha_2, \dots, \alpha_K$ and 1 are linearly independent over the rational numbers. $\square$

3. Conclusion

Example 1. The continued fractions

$$[2^K, 2^{K^2}, 2^{K^3}, \dots], [2.2^K, 2.2^{K^2}, 2.2^{K^3}, \dots], \dots, [K.2^K, K.2^{K^2}, K.2^{K^3}, \dots]$$

and the number 1 are linearly independent over the rational numbers.

Example 2. The continued fractions

$$[3^{K+1}, 3^{K^2+1}, 3^{K^3+1}, \dots], [3^{K+2}, 3^{K^2+2}, 3^{K^3+2}, \dots], \dots,\\ [3^{2K}, 3^{K^2+K}, 3^{K^3+K}, \dots]$$

and the number 1 are linearly independent over the rational numbers.

Example 3. The continued fractions

$$[2^2, 2^{2^2}, 2^{2^3}, 2^{2^4}, \dots], [3^2, 3^{2^2}, 3^{2^3}, 3^{2^4}, \dots]$$

and the number 1 are linearly independent over the rational numbers.

Open Problem. It is not known if the continued fractions

$$[2^2, 2^{2^2}, 2^{2^3}, \dots], [3^2, 3^{2^2}, 3^{2^3}, \dots], [4^2, 4^{2^2}, 4^{2^3}, \dots]$$

and the number 1 are linearly independent or not over the rational numbers.

Example 4. Let $\{G_n\}_{n=1}^{\infty}$ be the linear recurrence sequence of the k-th order such that $G_1, G_2, \dots, G_k, b_0, \dots, b_k$ belong to positive integers, $G_1 < G_2 < \dots < G_k$ and for every positive integer n, $G_{n+k} = G_n b_0 + G_{n+1}b_1 + \dots + G_{n+k-1}b_{k-1}$. If the roots $\alpha_1, \dots, \alpha_s$ of the equation $x^k = b_0 + b_1 x + \dots + b_{k-1}x^{k-1}$ satisfy $|\alpha_1| \ge |\alpha_2| \ge \dots \ge |\alpha_s|$, $|\alpha_1| > 1$ and α_1/α_j is not a root of unity for every $j = 2, 3, \dots, s$, then the continued fractions

$$[G_j G_{k^1}, G_j G_{k^2}, G_j G_{k^3}, \dots]$$

$(j = 1, 2, \dots, k)$ and the number 1 are linearly independent over the rational numbers.

This is an immediate consequence of Theorem 2.1 and the inequality $|\alpha_1|^{n(1-\epsilon)} < G_n < |\alpha_1|^{n(1+\epsilon)}$ which can be found in [7], for instance.

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