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More on inhomogeneous Diophantine approximation

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Résumé. Pour un nombre irrationnel α et un nombre réel γ, on considère la constante d'approximation non-homogène

$$M(\alpha,\gamma) := \liminf_{|n|\to\infty} |n|||n\alpha-\gamma||$$

en rapport avec le développement en fraction continue négatif semi-régulier de α

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}}$$

et un α-développement adéquat de γ. Nous donnons une majoration de

$$\rho(\alpha) := \sup_{\gamma\notin \mathbf{Z}+\alpha\mathbf{Z}} M(\alpha,\gamma),$$

dans le cas où α est mal approximé, qui s'avère fine lorsque les quotients partiels a_i sont presque tous pairs et supérieurs ou égaux à 4. Lorsque le développement de α est de période 1, on décrit entièrement le spectre des valeurs prises par

$$\mathbf{L}(\alpha) := \{M(\alpha,\gamma) \ : \ \gamma \notin \mathbf{Z}+\alpha\mathbf{Z}\},$$

au-dessus du premier point d'accumulation.

Abstract. For an irrational real number α and real number γ we consider the inhomogeneous approximation constant

$$M(\alpha,\gamma) := \liminf_{|n|\to\infty} |n|||n\alpha-\gamma||$$

via the semi-regular *negative continued fraction expansion* of α

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}}$$

and an appropriate alpha-expansion of γ. We give an upper bound on the case of worst inhomogeneous approximation,

$$\rho(\alpha) := \sup_{\gamma\notin \mathbf{Z}+\alpha\mathbf{Z}} M(\alpha,\gamma),$$

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which is sharp when the partial quotients a_i are almost all even and at least four. When the negative expansion has period one we give a complete description of the spectrum of values

$$\mathbf{L}(\alpha) := \{M(\alpha,\gamma) \ : \ \gamma \notin \mathbf{Z} + \alpha\mathbf{Z}\},$$

above the first limit point.

1. Introduction

For a fixed irrational, real number α and real γ in $[0,1)$ one defines the *two-sided inhomogeneous approximation constant*

$$M(\alpha,\gamma) := \liminf_{|n|\to\infty} |n| ||n\alpha - \gamma||,$$

where $||x||$ denotes the distance from x to the nearest integer. The homogeneous case $\gamma = 0$ is of course classical. Here we shall think of α as fixed and γ varying to obtain an inhomogeneous spectrum of values for α

$$\mathbf{L}(\alpha) := \{M(\alpha,\gamma) \ : \ \gamma \notin \mathbf{Z} + \alpha\mathbf{Z}\}.$$

We shall say that γ and γ' are equivalent (with respect to α), denoted $\gamma{\sim}\gamma'$, if $\gamma = \gamma' + n\alpha + m$ for some integers n, m, where clearly $\gamma{\sim} \pm \gamma'$ implies that $M(\alpha,\gamma) = M(\alpha,\gamma')$. Historically there has been most interest in the case of worst inhomogeneous approximation

$$\rho(\alpha) := \sup_{\gamma \not\sim 0} M(\alpha,\gamma),$$

particularly for quadratic α. It is conjectured that for quadratic α the value of $\rho(\alpha)$ should always be isolated (this would follow from a quadratic forms conjecture of Barnes–Swinnerton-Dyer [1], and may well be equivalent to it). In our previous paper [5] we approached the computation of $M(\alpha,\gamma)$ via the regular continued fraction expansion of α, verifying the isolation of $\rho(\alpha)$ when the regular expansion had period one or two, or the period all even partial quotients. We show here how to alternatively use the *negative continued fraction expansion.* The formulae and bounds obtained this way are similar but simpler to work with (the absence of a sign alternation making the expressions more symmetric). We are thus able to show the isolation of $\rho(\alpha)$ for additional classes of quadratic α having straightforward negative expansions. For example when the partial quotients are all even and at least four we explicitly give the γ achieving $\rho(\alpha)$ (see Theorem 2). In Section 2 we give a complete description of the spectrum above the first limit point when the negative expansion of α has period one (the structure is similar to that of the traditional Lagrange spectrum). As an added advantage the use of the negative expansion leads naturally to a separate consideration of the positive and negative integers, and hence to formulae for the one-sided approximation constants;

$$M_+(\alpha,\gamma) := \liminf_{n\to\infty} n||n\alpha-\gamma||, \quad M_-(\alpha,\gamma) := \liminf_{n\to-\infty} |n|||n\alpha-\gamma||.$$

By the negative expansion we mean that

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}}} =: [0; a_1, a_2, a_3, \dots]^-,$$

where the integers $a_i \geq 2$ are generated by rounding up rather than rounding down in the continued fraction algorithm:

$$\alpha_0 := \{\alpha\}, \quad a_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil, \quad \alpha_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil - \frac{1}{\alpha_n},$$

with corresponding *convergents* $p_n/q_n = [0; a_1, \dots, a_n]^-$ given by

$$\begin{aligned} p_{n+1} &:= a_{n+1}p_n - p_{n-1}, \quad p_0 = 0, \quad p_{-1} = -1, \\ q_{n+1} &:= a_{n+1}q_n - q_{n-1}, \quad q_0 = 1, \quad q_{-1} = 0. \end{aligned}$$

The negative expansion $[0; a_1, a_2, a_3, \dots]^-$ can of course be thought of as a *regular* expansion where the partial quotients are alternately positive and negative integers. Using van der Poorten style identities for dealing with illegal partial quotients,

$$\begin{aligned} [\dots, a, b, c, d, \dots] &= [\dots, a, 0, -1, 1, -1, 0, -b, -c, \dots] \\ &= [\dots, a, 0, 1, -1, 1, 0, -b, -c, \dots], \end{aligned}$$

and $[\dots, a, 0, b, \dots] = [\dots, a+b, \dots]$, to write

$$\begin{aligned} &[\dots, a, b, c, d, e, \dots] = [\dots, a+1, -1, -b+1, -c, -d, -e, \dots] \\ &= [\dots, a+1, \underbrace{-2, 2, \dots, (-2)^{b-1}}_{b-1}, (-1)^b(c+1), (-1)^b d, (-1)^b e, \dots.], \text{etc.}, \end{aligned}$$

it is straightforward to switch between regular and negative expansions:

$$\begin{aligned} &[0; a_1', a_2', a_3', a_4', a_5', a_6', a_7', \dots] \\ &\quad = [0; a_1'+1, \underbrace{2, \dots, 2}_{a_2'-1}, a_3'+2, \underbrace{2, \dots, 2}_{a_4'-1}, a_5'+2, \underbrace{2, \dots, 2}_{a_6'-1}, a_7'+2, \dots]^-. \end{aligned}$$

Writing

$$\overline{\alpha}_i := [0; a_i, a_{i-1}, \dots., a_1]^-, \quad \alpha_i = [0; a_{i+1}, a_{i+2}, \dots]^-,$$

it is readily seen that

$$D_i := q_i\alpha - p_i = \alpha_0 \cdots \alpha_i, \quad q_i = (\overline{\alpha}_1 \cdots \overline{\alpha}_i)^{-1}.$$

For a real number $\gamma < 1$ we generate the coefficients b_i in the *alpha-expansion of* γ by taking

$$\gamma_0 := \{\gamma\}, \quad b_{n+1} := \left\lfloor \frac{\gamma_n}{\alpha_n} \right\rfloor, \quad \gamma_{n+1} := \left\{ \frac{\gamma_n}{\alpha_n} \right\},$$

so that

$$\{\gamma\} = \sum_{i=1}^{n} b_i D_{i-1} + \gamma_n D_{n-1} = \sum_{i=1}^{\infty} b_i D_{i-1} \tag{1.1}$$

gives the unique expansion of γ of the form $\sum_{i=1}^{\infty} b_i D_{i-1}$ such that

(i) $0 \le b_i \le a_i - 1$,

(ii) the sequence of b_i does not contain a block of the form $b_t = a_t - 1$, with $b_j = a_j - 2$ for all $j > t$ or with $b_k = a_k - 1$ for some $k > t$ and $b_j = a_j - 2$ for any $t < j < k$.

We define the integers $Q_k = Q_k(\gamma, \alpha)$ by

$$Q_k := \sum_{i=1}^{k} b_i q_{i-1},$$

and parameters $\xi_k := Q_k/q_k$ so that

$$Q_k = \xi_k q_k, \quad ||Q_k \alpha - \gamma|| = \gamma_k D_{k-1},$$

with

$$0 \le \xi_k, \gamma_k \le 1.$$

We set

$$\lambda(n) = \lambda(n; \alpha, \gamma) := |n| ||n\alpha - \gamma||.$$

In evaluating $M_+(\alpha, \gamma)$ we shall frequently encounter

$$\begin{aligned} \lambda(Q_k) &= \xi_k \gamma_k q_k D_{k-1}, \\ \lambda(Q_k + q_{k-1}) &= (\xi_k + \overline{\alpha}_k)(1 - \gamma_k) q_k D_{k-1}, \end{aligned}$$

and for $M_-(\alpha, \gamma)$

$$\begin{aligned} \lambda(Q_k - (q_k - q_{k-1})) &= |1 - \overline{\alpha}_k - \xi_k||1 - \alpha_k - \gamma_k| q_k D_{k-1}, \\ \lambda(Q_k - q_k) &= (1 - \xi_k)(\alpha_k + \gamma_k) q_k D_{k-1}. \end{aligned}$$

To obtain more symmetrical expressions for these four functions it is often convenient to replace the b_k by the sequence of integers t_k, where

$$b_k = \tfrac{1}{2}(a_k - 2 + t_k),$$

and to define

$$d_k^- := \sum_{1\le j\le k} t_j(q_{j-1}/q_k) = t_k\overline{\alpha}_k + t_{k-1}\overline{\alpha}_k\overline{\alpha}_{k-1} + t_{k-2}\overline{\alpha}_k\overline{\alpha}_{k-1}\overline{\alpha}_{k-2} + \cdots,$$

$$d_k^+ := \sum_{j>k} t_j\left(\frac{D_{j-1}}{D_{k-1}}\right) = t_{k+1}\alpha_k + t_{k+2}\alpha_k\alpha_{k+1} + t_{k+3}\alpha_k\alpha_{k+1}\alpha_{k+2} + \cdots,$$

(use of the negative expansion avoiding a sign alternation in d_k^-). Hence

$$\xi_k = \tfrac{1}{2}\left(1-\overline{\alpha}_k + d_k^- - \frac{1}{q_k}\right), \quad \gamma_k = \tfrac{1}{2}(1-\alpha_k+d_k^+),$$

and as $k\to\infty$ (and $D_{k-1}\to 0$) we can replace $\lambda(Q_k)$, $\lambda(Q_k+q_{k-1})$, $\lambda(Q_k-(q_k-q_{k-1}))$ and $\lambda(Q_k-q_k)$ by

$$\begin{aligned} s_1(k) &:= \tfrac{1}{4}(1-\overline{\alpha}_k+d_k^-)(1-\alpha_k+d_k^+)q_kD_{k-1},\\ s_2(k) &:= \tfrac{1}{4}(1+\overline{\alpha}_k+d_k^-)(1+\alpha_k-d_k^+)q_kD_{k-1},\\ s_3(k) &:= \tfrac{1}{4}|1-\overline{\alpha}_k-d_k^-||1-\alpha_k-d_k^+|q_kD_{k-1},\\ s_4(k) &:= \tfrac{1}{4}(1+\overline{\alpha}_k-d_k^-)(1+\alpha_k+d_k^+)q_kD_{k-1}, \end{aligned}$$

where

$$q_kD_{k-1} = \frac{1}{1-\alpha_k\overline{\alpha}_k}.$$

Of course the t_k are integers with the same parity as a_k and $-(a_k-2)\le t_k\le a_k$. We observe that

$$-(1-\overline{\alpha}_k)\le d_k^- \le (1+\overline{\alpha}_k), \qquad -(1-\alpha_k)\le d_k^+\le(1+\alpha_k),$$

with $d_k^-\ge 1-\overline{\alpha}_k$ (respectively $d_k^+\ge 1-\alpha_k$) iff the sequence $t_k, t_{k-1},\ldots$ (respectively $t_{k+1}, t_{k+2},\ldots$) takes the form $t_i=a_i$ with $t_j=a_j-2$ for any preceding t_j. Notice that if $t_i\ne a_i$ then the expansion of $1-\alpha-\gamma$ is obtained by simply changing the signs of the t_i, where $M_-(\alpha,\gamma)=M_+(\alpha,1-\alpha-\gamma)$, the sign change merely interchanging $s_1(k), s_2(k)$ with $s_3(k), s_4(k)$.

Theorem 1. *For* $\gamma\not\sim 0$

$$\begin{aligned} M_+(\alpha,\gamma) &= \liminf_{k\to\infty}\min\{\lambda(Q_k),\lambda(Q_k+q_{k-1})\}\\ &= \liminf_{k\to\infty}\min\{s_1(k),s_2(k)\}. \end{aligned}$$

If the alpha-expansion of γ *has* $b_i=a_i-1$ *at most finitely many times then,*

$$\begin{aligned} M_-(\alpha,\gamma) &= \liminf_{k\to\infty}\min\{\lambda(Q_k-(q_k-q_{k-1})),\lambda(Q_k-q_k)\}\\ &= \liminf_{k\to\infty}\min\{s_3(k),s_4(k)\}, \end{aligned}$$

and

$$M(\alpha,\gamma) = \liminf_{k\to\infty}\min\{s_1(k),s_2(k),s_3(k),s_4(k)\}. \tag{1.2}$$

We readily deduce the following bound on $\rho(\alpha)$;

Corollary 1. *For $\gamma \not\sim 0$*

$$M(\alpha,\gamma) \le \rho^*(\alpha) := \tfrac{1}{4} \liminf_{k\to\infty} \frac{\max\{\alpha_k, \overline{\alpha}_k, (1-\alpha_k)(1-\overline{\alpha}_k)\}}{(1-\alpha_k\overline{\alpha}_k)}.$$

In particular if $\liminf_{i\to\infty} a_i = R \ge 3$, then

$$M(\alpha,\gamma) \le \tfrac{1}{4}\left(1-\frac{1}{R}\right). \tag{1.3}$$

If the $a_i \ge 3$ for almost all i then, since $\alpha_k, \overline{\alpha}_k \le [0;\overline{3}\,]^- + o(1)$,

$$\rho^*(\alpha) = \tfrac{1}{4} \liminf_{k\to\infty} \frac{(1-\alpha_k)(1-\overline{\alpha}_k)}{(1-\alpha_k\overline{\alpha}_k)}. \tag{1.4}$$

When the a_i are all even and at least four we can achieve this bound by simply taking the $t_i = 0$:

Theorem 2. *Suppose that the negative expansion of α has a_i even for $i \ge N$. Then*

$$\gamma^* = \sum_{i=N}^{\infty} \tfrac{1}{2}(a_i - 2)D_{i-1} = \tfrac{1}{2}\left(D_{N-2} - D_{N-1}\right)$$

has

$$M(\alpha,\gamma^*) = \tfrac{1}{4} \liminf_{k\to\infty} \frac{(1-\alpha_k)(1-\overline{\alpha}_k)}{(1-\alpha_k\overline{\alpha}_k)}.$$

In particular if the $a_i \ge 4$ we have $\rho(\alpha) = M(\alpha,\gamma^*) = \rho^*(\alpha)$. Moreover if α is also quadratic,

$$\alpha = [0; a_1, \dots, a_{N-1}, \overline{a_N, \dots, a_{N+r-1}}\,]^-, \quad 4 \le a_{N+i} \le A, \tag{1.5}$$

then the value of $\rho(\alpha)$ is isolated with

$$M(\alpha,\gamma) \le \left(1 - A^{-2\lceil \frac{r+1}{2} \rceil}\right)^{1/2} \rho(\alpha)$$

for $\gamma \not\sim 0, \gamma^*$.

We note that the simplified bound (1.4) need not hold when $a_i = 2$ infinitely often (so that the condition $a_i \ge 4$ is needed here). For example if for $i \ge 0$ the $a_{N+2i} = 2$ with $a_{N+2i+1} \ge 4$ even, then

$$\rho(\alpha) = M(\alpha,\gamma^{**}) = \tfrac{1}{4} \liminf_{i\to\infty} \frac{1}{2 - \overline{\alpha}_{N+2i-1} - \alpha_{N+2i}} = \rho^*(\alpha) \tag{1.6}$$

is larger than $M(\alpha,\gamma^*)$, where $\gamma^{**} := D_{N-2} - \frac{1}{2}D_{N-1}$ corresponds to taking $t_{N+2i} = a_{N+2i}$, $t_{N+2i+1} = -2$. Theorem 2 also shows that bound (1.3) can not be improved when R is even (consider period $R, 2A$ with $A \to \infty$).

Finally, the following bound (useful in the explicit computations of Section 2) shows that large $|t_i|$ produce small values for $M(\alpha,\gamma)$.

Lemma 1. *Suppose that $\gamma \not\sim 0$. If $t_k = a_k$ infinitely often then*

$$M(\alpha,\gamma) \le \tfrac{1}{4} \liminf_{\substack{k\to\infty \\ t_k=a_k}} \frac{\overline{\alpha}_k}{(1-\alpha_k\overline{\alpha}_k)}, \tag{1.7}$$

otherwise

$$M(\alpha,\gamma) \le \tfrac{1}{4} \liminf_{k\to\infty} \frac{(a_k-|t_k|)\overline{\alpha}_k}{(1-\alpha_k\overline{\alpha}_k)}. \tag{1.8}$$

2. Period One α

We suppose that α has a period one expansion

$$\alpha = [0; a_1, ..., a_N, \overline{a}\,]^-, \quad a \ge 4, \tag{2.5}$$

and set

$$\theta := [0; \overline{a}\,]^- = \tfrac{1}{2}(a - \sqrt{a^2-4}).$$

From Theorem 1 we can write

$$M(\alpha,\gamma) = \tfrac{1}{4} M^*(\alpha,\gamma)/(1-\theta^2)$$

and evaluate $M^*(\alpha,\gamma)$ using the liminf of the slightly simpler functions

$$\begin{aligned} s_1^*(k) &= (1-\theta+d_k^-)(1-\theta+d_k^+), \\ s_2^*(k) &= (1+\theta+d_k^-)(1+\theta-d_k^+), \\ s_3^*(k) &= |1-\theta-d_k^-||1-\theta-d_k^+|, \\ s_4^*(k) &= (1+\theta-d_k^-)(1+\theta+d_k^+), \end{aligned}$$

with

$$d_k^+ := t_{k+1}\theta + t_{k+2}\theta^2 + \cdots, \quad d_k^- := t_k\theta + t_{k-1}\theta^2 + \cdots.$$

We define sets of $\gamma = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i) D_{i-1}$ whose sequences t_i are eventually periodic:

When a is odd define

$$\begin{aligned} S_0 &:= \{\gamma \,:\, t_i \text{ periodic, period } (-1,1)\}, \\ S_{-3} &:= \{\gamma \,:\, t_i \text{ periodic, period } (-3,3)\}, \\ S_{-2} &:= \{\gamma \,:\, t_i \text{ periodic, period } (-1,-1,1,1)\}, \\ S_{-1} &:= \{\gamma \,:\, t_i \text{ periodic, period } (-1,1,)(-1,1,)(-1,-1,1,1)\}, \\ S_k &:= \left\{\gamma \,:\, t_i \text{ periodic, period } (-1,1,)(-1,-1,1,1,)^k\right\}, \quad k \ge 1, \end{aligned}$$

and when a is even

$$\begin{aligned} S_0 &:= \{\gamma \,:\, t_i \text{ periodic, period } 0\}, \\ S_{-1} &:= \{\gamma \,:\, t_i \text{ periodic, period } (-2,2)\}, \\ S_k &:= \left\{\gamma \,:\, t_i \text{ periodic, period } 0, (-2,2,)^k\right\}, \quad k \ge 1. \end{aligned}$$

When $a=4$ (as for $\alpha=\sqrt{3}$) we interestingly obtain a second sequences of γ with values also tending to the first limit point;

$$S_{-2} := \{\gamma \ : \ t_i \text{ periodic, period } (a,-2)\},$$
$$S_{-k-2} := \left\{\gamma \ : \ t_i \text{ periodic, period } (a,-2,)(2,-2,)^k\right\}$$

We set

$$\delta_k := M(\alpha,\gamma), \quad \gamma \in S_k.$$

Theorem 3. *Suppose that $\alpha=[0;a_1,...,a_N,\overline{a}]^-$ with $a\geq 4$. Then $\rho(\alpha)=\delta_0$.*

When $a\geq 5$ is odd the values of $M(\alpha,\gamma)$, $\gamma\not\sim 0$, greater than

$$\delta_\infty := \tfrac{1}{4}\frac{\left(1-2\theta+\theta^2\right)^2-\frac{\theta^6(1-\theta)^2}{(1+\theta^2)^2}}{(1-\theta^2)}$$
$$= \tfrac{1}{4}\frac{(a-2)}{\sqrt{a^2-4}}\left(1-\frac{1}{a}\right)^2\left(1-\frac{8}{(a-1)(a+\sqrt{a^2-4})^2}\right),$$

are given by

$$\delta_0 = \tfrac{1}{4}\frac{(1-\theta)^2-\frac{\theta^2}{(1+\theta)^2}}{(1-\theta^2)} = \tfrac{1}{4}\frac{a^2-5}{(a+2)\sqrt{a^2-4}},$$
$$\delta_{-3} = \tfrac{1}{4}\frac{\left(1-2\theta+\frac{3\theta^2}{1+\theta}\right)^2}{(1-\theta^2)} = \tfrac{1}{4}\frac{(a-1)^2}{(a+2)\sqrt{a^2-4}}, \quad \textit{if } a\geq 7,$$
$$\delta_{-2} = \tfrac{1}{4}\frac{\left(1-2\theta+\frac{\theta^2(1+\theta)}{(1+\theta^2)}\right)^2}{1-\theta^2} = \tfrac{1}{4}\frac{(a-2)(a-1)^2}{a^2\sqrt{a^2-4}},$$
$$\delta_{-1} = \tfrac{1}{4}\frac{\left(1-2\theta+\theta^2+\frac{\theta^7(1-\theta)(1-\theta^2)}{(1-\theta^8)}\right)^2-\frac{\theta^6(1-\theta)^2(1+\theta^2)^2}{(1-\theta^8)^2}}{(1-\theta^2)},$$

and for $k\geq 1$

$$\delta_k = \tfrac{1}{4}\frac{\left(1-2\theta+\theta^2+\frac{\theta^{4k+1}(1-\theta)(1-\theta^2)}{(1-\theta^{4k+2})}\right)^2-\frac{\theta^6(1-\theta)^2}{(1+\theta^2)^2}\left(\frac{1+\theta^{4k-2}}{1-\theta^{4k+2}}\right)^2}{(1-\theta^2)},$$

with $\delta_k\searrow\delta_\infty$ as $k\to\infty$, and $M(\alpha,\gamma)=\delta_k$ iff $\pm\gamma$ is in S_k.

If $a\geq 6$ is even, then the values of $M(\alpha,\gamma)$, $\gamma\not\sim 0$, greater than

$$\delta_\infty := \tfrac{1}{4}\frac{\left(\frac{1-\theta}{1+\theta}\right)^2-\theta^2}{(1-\theta^2)} = \tfrac{1}{4}\left(\frac{a}{a+2}\right)\left(1-\frac{2}{\sqrt{a^2-4}}\right),$$

are given by

$$\delta_0 = \tfrac{1}{4}\frac{(1-\theta)^2}{(1-\theta^2)} = \tfrac{1}{4}\frac{(a-2)}{\sqrt{a^2-4}},$$

$$\delta_{-1} = \tfrac{1}{4}\frac{(1-\theta)^2 - \frac{4\theta^2}{(1+\theta)^2}}{(1-\theta^2)} = \tfrac{1}{4}\frac{(a^2-8)}{(a+2)\sqrt{a^2-4}},$$

and for $k \geq 1$

$$\delta_k = \tfrac{1}{4}\frac{\left(\frac{1-\theta}{1+\theta}\right)^2\left(\frac{1-\theta^{2k+2}}{1-\theta^{2k+1}}\right)^2 - \theta^2\left(\frac{1-\theta^{2k}}{1-\theta^{2k+1}}\right)^2}{1-\theta^2},$$

with $\delta_k \searrow \delta_\infty$ *as* $k \to \infty$, *and* $M(\alpha,\gamma) = \delta_k$ *iff* $\pm\gamma$ *is in* S_k.

For $a = 4$ *we have the additional values*

$$\delta_{-2} = \tfrac{1}{4}\frac{\theta}{1-\theta^2},$$

$$\delta_{-k-2} = \tfrac{1}{4}\frac{\theta}{1-\theta^2}\left(1 - \tfrac{1}{3}\theta^2\left(\frac{1-\theta^{2k}}{1-\theta^{2k+2}}\right)^2\right),$$

with $\delta_{-k} \searrow \delta_\infty$.

Since $M(\alpha,0) = \theta/(1-\theta^2) = 1/\sqrt{a^2-4}$ the exclusion of the homogeneous case $\gamma \sim 0$ is only relevant when $a \leq 7$, with $M(\alpha,0) \geq \rho(\alpha)$ when $a \leq 6$ (with equality when $a = 6$). We note that δ_∞ is actually a limit point of limit points from below; for example if the expansion t_i for γ consists of blocks $0, (-2,2,)^{k_i}$ or $(-1,1,)(-1,-1,1,1)^{k_i}$, with k_i not eventually constant and $k = \liminf k_i$ then $M(\alpha,\gamma) \nearrow \delta_\infty$ as $k \to \infty$ (with δ_∞ achieved if $k = \infty$, the limit points from taking the k_i to have period $\overline{k,l}$ with $l \to \infty$, tending to δ_∞ as $k \to \infty$). When $a = 5$, the set S_{-3} has $\delta_{-3} < \delta_\infty$ and so is not included in the list. When a is odd the value of δ_{-1} actually lies between δ_1 and δ_2, otherwise the values are given in decreasing order. The value of $\rho(\alpha)$ for odd $a \geq 5$ together with the optimal γ can be deduced from paper I of Barnes–Swinnerton-Dyer [1] (Theorem 1 for $a \geq 7$ and Theorem 3 for $a = 5$). Komatsu [4] has also evaluated $M(\alpha,\gamma)$ for special values of γ (in the regular continued fraction these α of course have period two, $\overline{1,a-2}$). The remaining case $a = 3$ (corresponding to the golden ratio) has been dealt with by Davenport [3], and by Cusick, Rockett and Szüsz [2] who show a similar structure from $\rho(\alpha) = 1/(4\sqrt{5})$ (achieved with t_i of period $(-1,3,-1)$) down to the first limit point $1/(10+2\sqrt{5})$, the intermediate values corresponding to expansions with period $(-1,3,-1,)^k(1,-1)$.

3. Proofs for Section 1

3.1. Proof of Theorem 1.

Observe that any positive integer n, $q_{k-1} \leq n < q_k$, has an expansion

$$n = \sum_{i=1}^{k} z_i q_{i-1}, \quad z_k \geq 1,$$

so that

$$\gamma' := \{n\alpha\} = \sum_{i=1}^{k} z_i D_{i-1}$$

gives the α expansion of $\{n\alpha\}$. This expansion amounts to taking $z_k = \lfloor n/q_{k-1} \rfloor$, repeating this process for $n - z_k q_{k-1}$ and so on. We shall assume that $||n\alpha - \gamma|| = \pm(\{n\alpha\} - \gamma)$ (since otherwise $||n\alpha - \gamma|| = 1 - (\{n\alpha\} - \gamma) > \gamma$ or $||n\alpha - \gamma|| = 1 + (\{n\alpha\} - \gamma) > 1 - \gamma$ and $|n| ||n\alpha - \gamma||$ is unbounded).

We suppose that $n \neq Q_k$ so that $z_s \neq b_s$ for some $1 \leq s \leq k$ with $z_j = b_j$ for any $1 \leq j < s$.

If $z_s < b_s$ then $||n\alpha - \gamma|| = (b_s - z_s + \gamma_s - \gamma'_s)D_{s-1}$. Hence if $s \neq k$

$$n = Q_s + (z_s - b_s)q_{s-1} + \sum_{i=s+1}^{k} z_i q_{i-1} \geq Q_s + q_{k-1} - (a_s - 1)q_{s-1} > Q_s,$$

and

$$||n\alpha - \gamma|| = (b_s - z_s - \gamma'_s)D_{s-1} + ||Q_s\alpha - \gamma|| > ||Q_s\alpha - \gamma||,$$

so that $\lambda(n) > \lambda(Q_s)$ (with the second inequality implying that $Q_s \to \infty$ as $n \to \infty$ if $\lambda(n) \not\to \infty$). Thus it is enough to consider $s = k$, in which case

$$\lambda(n) = (z_k q_{k-1} + Q_{k-1})(b_k - z_k + \gamma_k)D_{k-1}.$$

For $0 \leq z_k \leq b_k$ this is clearly minimised for $z_k = 0$ or $z_k = b_k$ so that $\lambda(n) > \min\{\lambda(Q_k), \lambda(Q_{k-1})\}$.

So suppose that $z_s > b_s$ and

$$||n\alpha - \gamma|| = (z_s - b_s)D_{s-1} + \sum_{i=s+1}^{\infty} (z_i - b_i)D_{i-1}.$$

If $s \neq k$ or $s = k$ and $z_s \geq b_s + 2$ then $n' = n - q_{k-1}$ has $||n'\alpha - \gamma|| = ||n\alpha - \gamma|| - D_{k-1}$ and $\lambda(n') < \lambda(n)$. Hence we can assume that $s = k$, $z_k = b_k + 1$ and $n = Q_k + q_{k-1}$.

If the alpha-expansion of γ has $b_i = a_i - 1$ at most finitely many times then, since $\sum b_i D_{i-1} + \sum (a_i - 2 - b_i)D_{i-1} = 1 - \alpha$, we know that $-\gamma$ is equivalent to a gamma with $b'_i = (a_i - 2 - b_i)$ for almost all i. From this one can readily deduce that $M_-(\alpha, \gamma) = M_+(\alpha, -\gamma) = \liminf_{k\to\infty} \min\{\lambda(Q_k - (q_k - q_{k-1})), \lambda(Q_k - q_k)\}$. $\square$

3.2. Proof of Corollary 1.

Defining

$$w_k(\gamma) := \begin{cases} (1-\alpha_k)(1-\overline{\alpha}_k), & \text{if } d_k^+ \le 1-\alpha_k \text{ and } d_k^- \le 1-\overline{\alpha}_k, \\ \alpha_k, & \text{if } d_k^+ > 1-\alpha_k, \\ \overline{\alpha}_k, & \text{if } d_k^- > 1-\overline{\alpha_k}, \end{cases}$$

Corollary 1 follows from the more precise bound

$$M(\alpha,\gamma) \le \tfrac{1}{4}\liminf_{k\to\infty} w_k(\alpha) q_k D_{k-1}. \tag{3.1}$$

If $d_n^+ \le 1-\alpha_n$ and $d_n^- \le 1-\bar{\alpha}_n$, then

$$\begin{aligned}\min\{s_1(n), s_3(n)\} &\le \sqrt{s_1(n)s_3(n)} \\ &= \tfrac{1}{4} q_n D_{n-1}\left((1-\overline{\alpha}_n)^2 - (d_n^-)^2\right)^{1/2}\left((1-\alpha_n)^2-(d_n^+)^2\right)^{1/2} \\ &\le \tfrac{1}{4} q_n D_{n-1}(1-\overline{\alpha}_n)(1-\alpha_n).\end{aligned}$$

If $d_n^- \le 1-\overline{\alpha}_n$ and $d_n^+ > 1-\alpha_n$ then

$$\sqrt{s_3(n)s_2(n)} = \tfrac{1}{4} q_n D_{n-1}(1-(\overline{\alpha}_n + d_n^-)^2)^{\frac{1}{2}}(\alpha_n^2 - (1-d_n^+)^2)^{\frac{1}{2}},$$

and in the same way if $d_n^- > 1-\overline{\alpha}_n$ and $d_n^+ \le 1-\alpha_n$ then

$$\sqrt{s_3(n)s_4(n)} = \tfrac{1}{4} q_n D_{n-1}(\overline{\alpha}_n^2 - (1-d_n^-)^2)^{\frac{1}{2}}(1-(\alpha_n + d_n^+)^2)^{\frac{1}{2}}.$$

Bound (1.3) is immediate from (1.4) and the observation that $(1-\alpha_i) < (1-\alpha_i\overline{\alpha}_i)$ with $\overline{\alpha}_i = 1/(R-\overline{\alpha}_{i-1}) \ge 1/R$ infinitely often. □

3.3. Proof of Theorem 2 and (1.6).

Assume that a_i is even for $i \ge N$. For $\gamma = \gamma^*$ or γ^{**} we have $\gamma \sim -\gamma$ and $M(\alpha,\gamma) = \liminf_{k\to\infty}\min\{s_1(k), s_2(k)\}$. For γ^* we have $t_{N+i} = 0$ giving $d_k^+, d_k^- \to 0$ and

$$\begin{aligned} M(\alpha,\gamma^*) &= \tfrac{1}{4}\liminf_{k\to\infty}\min\left\{\frac{(1-\overline{\alpha}_k)(1-\alpha_k)}{(1-\alpha_k\overline{\alpha}_k)}, \frac{(1+\overline{\alpha}_k)(1+\alpha_k)}{(1-\alpha_k\overline{\alpha}_k)}\right\} \\ &= \tfrac{1}{4}\liminf_{k\to\infty}\frac{(1-\overline{\alpha}_k)(1-\alpha_k)}{(1-\alpha_k\overline{\alpha}_k)}.\end{aligned}$$

Suppose now that α is also of the form (1.5). Notice that $a_i \ge 4$ for almost all i gives $\alpha_i, \overline{\alpha}_i \le [0;\overline{4}]^- + o(1) = (2-\sqrt{3}+o(1))$. Hence if γ has $t_i = a_i$ infinitely often then (1.7) gives $M(\alpha,\gamma) \le \frac{1}{4}\limsup_{i\to\infty}\frac{\overline{\alpha}_i}{1-\alpha_i\overline{\alpha}_i} \le \frac{1}{8\sqrt{3}}$, while $M(\alpha,\gamma^*) = \frac{1}{4}\liminf_{i\to\infty}\frac{(1-\alpha_i)(1-\overline{\alpha}_i)}{1-\alpha_i\overline{\alpha}_i} > \frac{1}{4\sqrt{3}}$, and $M(\alpha,\gamma) \le \frac{1}{2}M(\alpha,\gamma^*)$. Hence we can assume that $t_i = a_i$ at most finitely many times. Set $l := \lfloor \frac{r}{2} \rfloor$. Suppose that $\gamma \not\sim \pm\gamma^*, 0$. Then, for each $i = 0, \ldots, r-1$, there will be infinitely $n \equiv i \pmod r$ with $t_m \ne 0$ for some m with $n-l \le m \le n$ or $n+1 \le m \le n+1+l$ (and $t_j = 0$ for any j closer to n or $n+1$ than m). Now if $m \le n$ then $|d_n^-| q_n \ge 2q_{m-1} - |d_{m-1}^-| q_{m-1} \ge q_{m-1}$ and

$|d_n^-| \geq \overline{\alpha}_n \cdots \overline{\alpha}_m \geq A^{-(l+1)}$ (likewise if $m > n$ the $|d_n^+| \geq A^{-(l+1)}$). Hence, as in the proof of Corollary 1,

$$\begin{aligned}\min\{s_1(n), s_3(n)\} &\leq \tfrac{1}{4} q_n D_{n-1}(1-\overline{\alpha}_n)(1-\alpha_n)(1-(d_k^+)^2)^{\frac{1}{2}}(1-(d_k^-)^2)^{\frac{1}{2}} \\ &\leq \tfrac{1}{4} q_n D_{n-1}(1-\overline{\alpha}_n)(1-\alpha_n)(1-A^{-2(l+1)})^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}M(\alpha,\gamma) &\leq \tfrac{1}{4}\left(1-A^{-2\lceil\frac{r+1}{2}\rceil}\right)^{\frac{1}{2}} \min_{i=0,\ldots,r-1} \lim_{\substack{q\to\infty \\ n=qr+i}} (1-\overline{\alpha}_n)(1-\alpha_n) q_n D_{n-1} \\ &= \left(1-A^{-2\lceil\frac{r+1}{2}\rceil}\right)^{\frac{1}{2}} \rho(\alpha).\end{aligned}$$

Suppose now that $a_{N+2i} = 2$ and $a_{N+2i+1} \geq 4$ for all $i \geq 0$. For γ^{**} we have $d_{N+2i}^-, d_{N+2i-1}^+ \to 1$, $d_{N+2i-1}^- \to -\overline{\alpha}_{N+2i-1}$, $d_{N+2i}^+ \to -\alpha_{N+2i}$. with $\alpha_{N+2i}, \overline{\alpha}_{N+2i-1} \leq [0; \overline{4,2}\,]^- + o(1) = \frac{1}{2}(2-\sqrt{2}) + o(1)$, and writing $\mu_i := \frac{1}{4} q_{N+2i-1} D_{N+2i-1}$,

$$\begin{aligned}s_2(N+2i-1) &\to \mu_i, \\ s_2(N+2i) \geq s_1(N+2i) &\to \mu_i(3-2\overline{\alpha}_{N+2i-1})(1-2\alpha_{N+2i}) \geq \mu_i + o(1), \\ s_1(N+2i-1) &\to \mu_i(3-2\alpha_{N+2i})(1-2\overline{\alpha}_{N+2i-1}) \geq \mu_i + o(1),\end{aligned}$$

and $M(\alpha, \gamma^{**}) = \liminf_{i\to\infty} \mu_i = \rho^*(\alpha)$. □

3.4. Proof of Lemma 1.

Bound (1.7) follows at once from bound (3.1). Bound (1.8) follows on observing (for $\pm\gamma$) that the minimum of

$$\begin{aligned}\lambda(Q_k) &= q_{k-1} D_{k-1}(b_k + \xi_{k-1})\gamma_k, \\ \lambda(Q_{k-1}) &= q_{k-1} D_{k-1} \xi_{k-1}(b_k + \gamma_k), \\ \lambda(Q_{k-1} - q_{k-1}) &= q_{k-1} D_{k-1}(b_k + 1 + \gamma_k)(1 - \xi_{k-1}), \\ \lambda(Q_k + q_{k-1}) &= q_{k-1} D_{k-1}(1 + b_k + \xi_{k-1})(1 - \gamma_k),\end{aligned}$$

is certainly no more than

$$\tfrac{1}{4}(\lambda(Q_k) + \lambda(Q_{k-1}) + \lambda(Q_{k-1} - q_{k-1}) + \lambda(Q_k + q_{k-1})) = \tfrac{1}{4}(a_k + t_k) q_{k-1} D_{k-1}.$$

□

4. Proof of Theorem 3

4.1. Evaluating the δ_k.

We evaluate $\delta_k^* = 4(1-\theta^2)\delta_k$. Apart from the δ_{-k}, $k \geq 2$ when $a = 4$ (which have some $t_i = a$) we can assume that

$$\delta_k^* = \liminf_{n\to\infty} \min\{s_1^*(n), s_2^*(n), s_3^*(n), s_4^*(n)\}$$

with

$$s_1^*(n), s_3^*(n) = (1-\theta \pm d_n^-)(1-\theta \pm d_n^+), \quad s_2^*(n), s_4^*(n) = (1+\theta \pm d_n^-)(1+\theta \mp d_n^+).$$

Except for δ_{-3}, a odd, we have $|d_n^-|, |d_n^+| < 2\theta$ so that $s_2^*(n), s_4^*(n) > (1-\theta)^2$ and we need only evaluate the $s_1^*(n), s_3^*(n)$. For γ in S_0 with a even the $d_n^+, d_n^- \sim 0$ and plainly $\delta_0^* = (1-\theta)^2$ (the largest possible value). For γ having t_i of period $(t,-t)$, $t = 1,2,3$ we have $\{d_n^+, d_n^-\} \sim \pm t\theta/(1+\theta)$ and $s_1^*(n), s_3^*(n) \sim (1-\theta)^2 - (t\theta/(1+\theta))^2$, giving the value of δ_0^*, a odd and δ_{-1}^*, a even (and $\delta_{-3}^* < \delta_\infty^*$ if $a = 5$). When $t = 3$, $a \geq 7$ odd, $\min\{s_2^*(n), s_4^*(n)\} \sim (1+\theta - t\theta/(1+\theta))^2$ is smaller and gives δ_{-3}^*.

Now if the t_i have period $0, (-2, 2,)^k$ the smallest pair $\{d_n^-, d_n^+\}$ (and smallest $\{-d_n^-, -d_n^+\}$) are asymptotically

$$\left\{-\frac{2\theta(1-\theta^{2k})}{(1+\theta)(1-\theta^{2k+1})}, \frac{2\theta^2(1-\theta^{2k})}{(1+\theta)(1-\theta^{2k+1})}\right\}$$

occurring when $t_n = 0$ (or $t_{n+1} = 0$) giving the smallest $s_1^*(n)$ (or $s_3^*(n)$) and the value claimed for δ_k^*, $k \geq 1$, when a is even.

For $a \geq 5$ odd and δ_{-1}^*, δ_{-2}^*, δ_k^*, $k \geq 1$ we note that the values claimed are certainly less than $(1-2\theta+\theta^2+\theta^3)^2 \leq 1-4\theta+6\theta^2$. Now if $t_n, t_{n+1} = 1, -1$ (or vice versa) then $\theta+\theta^2 \geq d_n^-, -d_n^+ \geq \theta-\theta^2-\theta^3$ producing $s_1^*(n), s_3^*(n) \geq (1-\theta^2-\theta^3)(1-2\theta-\theta^2) > 1-2\theta-2\theta^2$. Hence it is enough to consider $s_1^*(n)$ when $t_n, t_{n+1} = -1, -1$ (for both γ and its negative). For δ_{-2}^* these n have $d_n^+, d_n^- \sim (-\theta+\theta^2)/(1+\theta^2)$ and $s_1^*(n)$ gives the value claimed. For $\pm\gamma$ in S_{-1} these n have $\{d_n^-, d_n^+\}$ asymptotic to

$$\left\{\frac{-\theta+\theta^2+\theta^3-\theta^4+\theta^5-\theta^6+\theta^7-\theta^8}{1-\theta^8}, \right.$$
$$\left. \frac{-\theta+\theta^2-\theta^3+\theta^4-\theta^5+\theta^6+\theta^7-\theta^8}{1-\theta^8}\right\}$$

with $s_1^*(n)$ giving the value claimed for δ_{-1}^*. For the δ_k, $k \geq 1$ when the $-1, -1$ occurs in a block $-1, 1, -1, -1, 1, 1$ or $1, 1, -1, -1, 1, -1$ we have $\{d_n^-, d_n^+\}$ tending to

$$\left\{\frac{-\theta(1-\theta) - \frac{\theta^3(1-\theta)(1-\theta^{4k})}{1+\theta^2}}{1-\theta^{4k+2}}, \quad \frac{-\frac{\theta(1-\theta)(1-\theta^{4k})}{1+\theta^2} + \theta^{4k+1}(1-\theta)}{1-\theta^{4k+2}}\right\}$$

with $s_1^*(n)$ asymptotically giving the value claimed for δ_k^*, with this certainly less that $(1-2\theta+\theta^2+\theta^5)^2$. When the $-1, -1$ occur inside blocks $1, 1, -1, -1, 1, 1$ we have $d_n^-, d_n^+ \geq -\theta+\theta^2+\theta^3-\theta^4-\theta^5$ giving a larger and hence irrelevant $s_1^*(n) \geq (1-2\theta+\theta^2+\theta^3-\theta^4-\theta^5)^2$.

Finally we deal with the δ_{-k}, $k \geq 2$ when $a = 4$. In this case we need check $s_1^*(n), s_2^*(n)$ for both γ and its negative. If $t_n = a$ then $d_n^- \geq a\theta - 2\theta^2$,

$d_n^+ \geq -2\theta$ and $s_1^*(n) \geq (2-\theta-\theta^2)(1-3\theta) > \theta$ (likewise if $t_{n+1} = a$, this merely reversing the roles of d_n^+ and d_n^-). Moreover when $t_n = a$ we have $d_n^- > 0$, $d_n^+ < 0$ and $s_2^*(n) > 1$ and when $t_{n+1} = a$

$$s_2^*(n) = \theta(1 + (\theta + d_n^-))(1 - (\theta + d_{n+1}^+)).$$

Hence we can ignore the $t_n = a$ and when $t_{n+1} = a$ merely check $s_2^*(n)$. For γ in S_{-2} and $t_{n+1} = a$ we have $d_n^-, d_{n+1}^+ \sim (-2\theta + a\theta^2)/(1-\theta^2) = -\theta$ giving $\delta^*_{-2} = \theta$. For γ with period $2, -2, a, -2, (2, -2,)^{k-1}$ and $t_{n+1} = a$ we have

$$d_n^-, d_{n+1}^+ \sim -\frac{2\theta}{(1+\theta)} + \frac{(a-2)\theta^{2k+2}}{(1-\theta^{2k+2})}$$

and $s_2^*(n) \to \theta\left(1 - \left(\theta - \frac{2\theta^2}{1+\theta} - \frac{2\theta^{2k+2}}{1-\theta^{2k+2}}\right)^2\right)$, the value claimed for δ^*_{-k-2}. For the negative of this $-2, 0, a, 0, (-2, 2,)^{k-1}$ we have $d_n^-, d_{n+1}^+ \sim -2\theta^2/(1+\theta) - 2\theta^{2k+2}/(1-\theta^{2k+2})$ asymptotically giving the same value. For the remaining n with $t_n, t_{n+1} \neq a$ we have $|d_n^-|, |d_n^+| < 2\theta$ and again we need only consider $s_1^*(n)$. Now if $t_n = -2$, $t_{n+1} \neq a$ (or vice versa) the bounds $d_n^- \geq -2\theta + a\theta^3 - 2\theta^5 \geq -2\theta + \theta^2$, $d_n^+ \geq a\theta^2 - 2\theta^4 > \theta$ give $s_1^*(n) > 1 - 3\theta + \theta^2 = \theta$ so these do not affect the value of δ^*_{-k-2}. □

4.2. Proof of the Theorem when a is odd.

Writing $\delta_\infty^* = 4(1-\theta^2)\delta_\infty$ we suppose that γ has $M^*(\alpha,\gamma) > \delta_\infty^*$. We note the rough bounds

$$\delta_\infty^* = (1-2\theta+\theta^2)^2 - \frac{\theta^6(1-\theta)^2}{(1+\theta^2)^2} > (1-2\theta+\theta^2)^2 - \theta^6 > (1-4\theta+5\theta^2)$$

and $\delta_\infty^* > (1-2\theta+\theta^2-\frac{1}{2}\theta^5)^2$. Now if γ has $t_i = a$ infinitely often then, from (1.7),

$$M^*(\alpha,\gamma) \leq \theta < \delta_\infty^* - (1-5\theta+5\theta^2) \leq \delta_\infty^* - 4\theta^2,$$

and if $|t_i| \geq 5$ infinitely often, then from (1.8)

$$M^*(\alpha,\gamma) \leq (a-5)\theta = 1 - 5\theta + \theta^2 < \delta_\infty^* - \theta.$$

Hence it is enough to consider γ with $t_i = \pm 1, \pm 3$ for all i (if we were only interested in $\rho(\alpha)$ we could similarly rule out $t_i = \pm 3$ infinitely often).

Now if $t_n = -3$ and $t_{n+1} \leq -1$ (or vice versa) infinitely often, then

$$s_1^*(n) \leq \left(1 - \theta - \theta + \frac{3\theta^2}{1-\theta}\right)\left(1 - \theta - 3\theta + \frac{3\theta^2}{1-\theta}\right)$$

and

$$M^*(\alpha,\gamma) \leq \left(1 - 3\theta + \frac{3\theta^2}{1-\theta}\right)^2 - \theta^2 < \delta_\infty^* - \theta^2.$$

Similarly we can dismiss $t_i = 3$, $t_{i\pm 1} \geq 1$ (by considering the negative $(1-\alpha-\gamma)$). So if $\{t_n, t_{n+1}\} = \{-3, 1\}$ then

$$s_1^*(n) \leq \left(1 - \theta + \frac{\theta}{1-\theta}\right)\left(1 - \theta - 3\theta + 3\theta^2\right) < \delta_\infty^* - \theta^2.$$

Likewise if $\{t_n, t_{n+1}\} = \{3, -1\}$. Hence apart from the period $-3, 3$ elements of S_{-3} we can assume that $t_i = \pm 1$ for all i. We assume that $\gamma \notin S_0$ so that γ (or its negative) has infinitely many blocks $t_n, t_{n+1} = -1, -1$ with $s_1^*(n) = (1 - 2\theta + \theta d_{n-1}^-)(1 - 2\theta + \theta d_{n+1}^+)$. Now we can rule out blocks $-1, -1, -1$ and $-1, 1, -1, -1, 1, -1$ since $t_{n+2} = -1$ would give $d_{n+1}^+ < 0$, $d_{n-1}^- \leq \theta/(1-\theta)$ and $s_1^*(n) \leq (1-2\theta)^2 + \theta^2 < \delta_\infty^*$, and $t_{n-1}, t_{n-2} = t_{n+2}, t_{n+3} = 1, -1$ would give $d_n^+, d_n^- \leq \theta - \theta^2 + \theta^3/(1-\theta)$ and $s_1^*(n) \leq (1 - 2\theta + \theta^2 - \theta^3 + \theta^4/(1-\theta))^2 < \delta_\infty^*$. Hence we can assume the blocks $-1, -1$ and $1, 1$ occur inside blocks $-1, 1, 1, -1, -1, 1$ or $1, -1, -1, 1, 1, -1$. Suppose that $t_{n-1}, \ldots, t_{n+4} = 1, -1, -1, 1, 1, -1$ then

$$s_1^*(n) = (1 - 2\theta + \theta^2 + \theta^2 d_{n-2}^-)(1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^4(-d_{n+4}^+))$$
$$s_3^*(n+2) = (1 - 2\theta + \theta^2 + \theta^2(-d_{n+4}^+))(1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^4 d_{n-2}^-).$$

Now we can rule out blocks $t_{n+1}, t_n, t_{n-1}, \ldots = -1, -1, 1, -1, -1, \ldots$ or

$$-1, -1, 1, -1, 1, -1, -1, \ldots \text{ or } -1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

(or $t_{n+2}, t_{n+3}, t_{n+4}, \ldots$ having the negative of these) since these would give $d = \min\{d_{n-2}^-, -d_{n+4}^+\} \leq -\theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \theta^6/(1-\theta)$, and

$$\min\{s_1^*(n), s_3^*(n+2)\} \leq (1 - 2\theta + \theta^2 + \theta^2 d)(1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^4 d)$$
$$\leq (1 - 2\theta + \theta^2 + \theta^9)^2 - (\theta^3 - \theta^4 + \theta^5 - \theta^6)^2 < \delta_\infty^*.$$

Hence we can assume that the sequence $t_{n+1}, t_n, t_{n-1}, \ldots$ takes the form

$$-1, -1, 1, 1, \ldots \text{ or } -1, -1, 1, -1, 1, 1, -1, -1, \ldots$$

or

$$-1, -1, 1, -1, 1, -1, 1, 1, -1, -1, \ldots$$

(and the $t_{n+2}, t_{n+3}, \ldots$ the form $1, 1, -1, -1, \ldots$ or $1, 1, -1, 1, -1, -1, 1, 1, \ldots$ or $1, 1, -1, 1, -1, 1, -1, -1, 1, 1, \ldots$). Moreover if

$$t_{n+1}, t_n, \ldots = -1, -1, 1, -1, 1, -1, \ldots$$

and $t_{n+2}, t_{n+3}, \ldots \neq 1, 1, -1, 1, -1, 1, \ldots$ then

$$d_{n-2}^- \leq -\theta + \theta^2 - \theta^3 + \theta^4 + \theta^5 - \theta^6 - \theta^7 + \theta^8/(1-\theta), \quad d_{n+4}^+ \leq \theta - \theta^2 - \theta^3 + \theta^4 + \theta^5,$$

and $s_1^*(n) \leq (1 - 2\theta + \theta^2 + \theta^{10}/(1-\theta))^2 - (\theta^3 - \theta^4 + \theta^5 - \theta^6 - \theta^7)^2 < \delta_\infty^*$. Hence excluding the γ in S_{-1} with period $-1, -1, 1, -1, 1, -1, 1, 1$ or its negative, and the γ in S_{-2} with period $-1, -1, 1, 1$, we can assume that we have infinitely many blocks $-1, 1, -1$ or $1, -1, 1$ with these contained in blocks $1,1,-1,-1,1,-1,1,1,-1,-1$ or $-1,-1,1,1,-1,1,-1,-1,1,1$. Suppose that $t_n,...,t_{n+5}=-1,-1,1,-1,1,1$ then

$$s_1^*(n)=(1-2\theta+\theta d_{n-1}^-)(1-2\theta+\theta^2-\theta^3+\theta^4+\theta^5-\theta^5(-d_{n+5}^+))$$
$$s_3^*(n+4)=(1-2\theta+\theta(-d_{n+5}^+))(1-2\theta+\theta^2-\theta^3+\theta^4+\theta^5-\theta^5 d_{n-1}^-).$$

Now we can rule out $d=\min\{d_{n-1}^-,-d_{n+5}^+\}\le(\theta+\theta^2)/(1+\theta^2)$ else

$\min\{s_1^*(n),s_3^*(n+4)\}\le(1-2\theta+\theta d)(1-2\theta+\theta^2-\theta^3+\theta^4+\theta^5-\theta^5 d)\le\delta_\infty^*$.

Hence we can assume that the sequence consists solely of blocks

$$1,-1,(1,1,-1,-1,)^{k_i}$$

with $k_i\ge 1$ (or solely of their negatives $-1,1,(-1,-1,1,1,)^{k_i}$). Now if we have a block $...t_{n-1}|t_n...$ of the form

$$..,1,-1,1,1,(-1,-1,1,1)^l|-1,-1,1,-1,1,1,(-1,-1,1,1)^k-1,-1,1,-1,..$$

with $l>k\ge 0$ then

$$d_{n-1}^-\le\frac{\theta+\theta^2}{1+\theta^2}+2\theta^{4+4l}(1-\theta-\theta^2+\theta^3+\theta^4),$$
$$d_{n+5}^+\le-\frac{\theta+\theta^2}{1+\theta^2}-2\theta^{4+4k}(1-\theta-\theta^2),$$

and

$$\begin{aligned}s_1^*(n)\le&\left(1-2\theta+\theta^2+\frac{\theta^3(1-\theta)}{(1+\theta^2)}+2\theta^{9+4k}(1-\theta-\theta^2)+2\theta^{12+4k}(1+\theta)\right)\\&\left(1-2\theta+\theta^2-\frac{\theta^3(1-\theta)}{(1+\theta^2)}-2\theta^{9+4k}(1-\theta-\theta^2)\right)\\\le&\delta_\infty^*-2\theta^{12+4k}\left(2(1-\theta-\theta^2)\frac{(1-\theta)}{(1+\theta^2)}-(1+\theta)(1-2\theta+\theta^2)\right).\end{aligned}$$

This leaves only the periodic expansions of elements in S_k. □

4.3. Proof of the Theorem when a is even.

Suppose that $M^*(\alpha,\gamma)>\delta_\infty^*$ where

$$\begin{aligned}\delta_\infty^*&=\left(1-\theta-\frac{2\theta}{1+\theta}\right)\left(1-\theta+\frac{2\theta^2}{1+\theta}\right)\\&=1-4\theta+5\theta^2+\frac{2\theta^3(a-4)}{1+\theta}-\frac{2\theta^3(1+\theta^2)}{(1+\theta)^2}.\end{aligned}$$

We suppose first that all the $t_i=0,\pm 2$. We can certainly assume this when $a\ge 6$ since if $t_i=a$ infinitely often then $M^*(\alpha,\gamma)<\theta$ and if $|t_i|\ge 4$ infinitely often $M^*(\alpha,\gamma)\le(a-4)\theta=1-4\theta+\theta^2<\delta_\infty^*$ (when $a=4$ we consider separately the γ with $t_i=a$ infinitely often). We can rule out infinitely blocks $(t_n, t_{n+1}) = (-2,-2)$ (or their negative $(2,2)$) since these give

$$(4.1) \qquad s_1^*(n) \le \left(1-\theta-2\theta+\frac{2\theta^2}{1-\theta}\right)^2 < (1-2\theta)^2 < \delta_\infty^*.$$

Also when $t_n = 0$ we must have $|d_{n-1}^-|, |d_n^+| < 2\theta/(1+\theta)$ ruling out blocks $0,(-2,2,)^k-2,0$ or $0,(2,-2,)^k 2,0$, since if $d = \min\{\pm d_{n-1}^-, \pm d_n^+\} < -2\theta/(1+\theta)$ then the minimum of

$$\begin{aligned} s_1^*(n) &= (1-\theta-\theta(-d_{n-1}^-))(1-\theta+d_n^+),\\ s_3^*(n-1) &= (1-\theta-\theta d_n^+)(1-\theta+(-d_{n-1}^-)),\\ s_3^*(n) &= (1-\theta-\theta d_{n-1}^-)(1-\theta+(-d_n^+)),\\ s_1^*(n-1) &= (1-\theta-\theta(-d_n^+))(1-\theta+d_{n-1}^-), \end{aligned}$$

is certainly at most

$$(4.2) \qquad (1-\theta-\theta d)(1-\theta+d) < \delta_\infty^*.$$

Moreover if γ does not have period 0 and $t_n = 0$ then d_n^+ and d_{n-1}^- must be of opposite signs, since if for example $t_n, t_{n+1} = 0, -2$ with $d_{n-1}^- < 0$ then (using that $d_{n+3}^+ \le 2\theta/(1+\theta)$ if $t_{n+3} = 0$)

$$\begin{aligned} s_1^*(n) &\le (1-\theta)\left(1-\theta-2\theta+2\theta^2+\frac{2\theta^4}{1+\theta}\right)\\ &= 1-4\theta+5\theta^2-\frac{2\theta^3(1+\theta^2)}{1+\theta} < \delta_\infty^*. \end{aligned}$$

Hence we can assume that the sequence of t_i has period 0 or $(-2,2)$ or consists only of blocks $0,(-2,2,)^{l_i}$, $l_i \ge 0$ (or only of its negative $0,(2,-2,)^{l_i}$). Now if we have a block $\dots, t_{n-1}, t_n, |t_{n+1}, \dots = \dots, 0, (-2,2,)^l 0, |(-2,2,)^k 0, \dots$ with $0 \le l < k$ then

$$d_n^+ \le -\frac{2\theta}{1+\theta}+\frac{2\theta^{2k+1}}{1+\theta}, \qquad d_{n-1}^- < \frac{2\theta}{1+\theta}-2\theta^{2l+1}+\frac{4\theta^{2l+2}}{1+\theta},$$

and

$$\begin{aligned} s_1^*(n) &\le \left(1-\theta+\frac{2\theta^2}{1+\theta}-2(1-\theta)\frac{\theta^{2k}}{1+\theta}\right)\left(1-\theta-\frac{2\theta}{1+\theta}+\frac{2\theta^{2k+1}}{1+\theta}\right)\\ &= \delta_\infty^*-\frac{2\theta^{2k}}{(1+\theta)^2}\left(1-4\theta+\theta^2+2(1-\theta)\theta^{2k+1}\right) < \delta_\infty^*. \end{aligned}$$

This leaves only the periodic elements of S_k.

It remains to check the case $a=4$ when γ has $t_i=a$ infinitely often. Observe that if $t_n=a$ then

$$s_4^*(n)=\theta(1+(\theta+d_n^+))(1-(\theta+d_{n-1}^-)),$$
$$s_2^*(n-1)=\theta(1+(\theta+d_{n-1}^-))(1-(\theta+d_n^+)),$$

and we can assume that

$$d=\max\{|\theta+d_n^+|,|\theta+d_{n-1}^-|\}\le\theta-\frac{2\theta^2}{1+\theta}, \tag{4.3}$$

since otherwise the minimum of these is certainly no more than

$$\theta(1-d^2)<\theta\left(1-\left(\theta-\frac{2\theta^2}{1+\theta}\right)^2\right)=\delta_\infty^*.$$

Hence $d_{n\pm1},d_{n\pm2},\ldots$ must take the form $0,-2,\ldots$ or $-2,2,\ldots$ or $-2,a,\ldots$ (since $t_{n+1}=2$, or $t_{n+1}=0,t_{n+2}\ge 0$ would give $d_n^+\ge -2\theta^3/(1-\theta)$ and $t_{n+1}=-2,t_{n+2}\le 0$, would give $d_n^+\le -2\theta+a\theta^3$). We can rule out blocks $(-2,-2)$ just as in (4.1). Hence condition (4.3) forces the $(a,0,-2)$ to lie inside blocks $t_n,\ldots,t_k=a,0,-2,(2,-2,)^l0$. But if $t_k=0$ and $d_k^+,d_{k-1}^-<1-\theta$ then $|d_k^+|,|d_{k-1}^-|\le 2\theta/(1+\theta)$ just as in (4.2). So the $a,0,-2$ lie inside blocks $a,0,-2,(2,-2,)^l0,a$. Now if we have $d_n=a$ and $d_{n+1},d_{n+2},\ldots=0,-2,\ldots$ then $d_{n-1},d_{n-2},\ldots=0,-2,\ldots$ (and vice versa) since $d_{n-1}=-2$ would give $d_n^+\ge-2\theta^2$, $d_{n-1}^-\le -2\theta+a\theta^2$ and $s_2^*(n-1)<\delta_\infty^*$. Hence, since going from γ to $1-\gamma$ interchanges the blocks $a,0,-2$ and $a,-2,2$ (and fixes the $a,-2,a$) we can assume that either γ has period $a,-2$ or consists entirely of blocks $a,0,-2,(2,-2,)^{l_i}0$, $l_i\ge 0$ (or its negative composed entirely of blocks $a,-2,2,(-2,2,)^{l_i}-2$). Now if we had a block $\ldots.t_{n-1},|t_n,\ldots$ of the form

$$\ldots,a,0,(-2,2,)^k-2,0,|a,0,-2,(2,-2,)^l0,a,\ldots$$

with $l<k$ then

$$d_{n-1}^-\ge-\frac{2\theta^2}{1+\theta}-\theta^{3+2k}\frac{(2-(a+2)\theta+2\theta^2)}{1-\theta^4},$$
$$d_n^+\le-\frac{2\theta^2}{1+\theta}-\theta^{3+2l}(2-(a+2)\theta+2\theta^2),$$

and

$$\begin{aligned}s_4^*(n)&\le\theta\left(1+\theta-\frac{2\theta^2}{1+\theta}-2\theta^{4+2l}\right)\left(1-\theta+\frac{2\theta^2}{1+\theta}+\frac{2\theta^{6+2l}}{1-\theta^4}\right)\\&\le\delta_\infty^*-2\theta^{5+2l}\left(1-\theta-\theta^2(1+\theta)\right)<\delta_\infty^*.\end{aligned}$$

This leaves only the periodic elements of $S_{-(k+2)}$. $\square$

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