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On normal lattice configurations and simultaneously normal numbers

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Dedicated to Professor Michel Mendès France
in the occasion of his 65th birthday

Résumé. Soient $q, q_1, \dots, q_s \geq 2$ des entiers et $\alpha_1, \alpha_2, \dots$ des nombres réels. Dans cet article, on montre que la borne inférieure de la discrépance de la suite double

$$\left(\{\alpha_m q^n\}, \dots, \{\alpha_{m+s-1} q^n\}\right)_{m,n=1}^{M\ N}$$

coïncide (à un facteur logarithmique près) avec la borne inférieure de la discrépance des suites ordinaires $(x_n)_{n=1}^{MN}$ dans un cube de dimension s (s,M,$N = 1, 2, \dots$). Nous calculons aussi une borne inférieure de la discrépance (à un facteur logarithmique près) de la suite $\left(\{\alpha_1 q_1^n\}, \dots, \{\alpha_s q_s^n\}\right)_{n=1}^{N}$ (problème de Korobov).

Abstract. Let $q, q_1, \dots, q_s \geq 2$ be integers, and let $\alpha_1, \alpha_2, \dots$ be a sequence of real numbers. In this paper we prove that the lower bound of the discrepancy of the double sequence

$$\left(\{\alpha_m q^n\}, \dots, \{\alpha_{m+s-1} q^n\}\right)_{m,n=1}^{M\ N}$$

coincides (up to a logarithmic factor) with the lower bound of the discrepancy of ordinary sequences $(x_n)_{n=1}^{MN}$ in s-dimensional unit cube (s,M,$N = 1, 2, \dots$) . We also find a lower bound of the discrepancy (up to a logarithmic factor) of the sequence $\left(\{\alpha_1 q_1^n\}, \dots, \{\alpha_s q_s^n\}\right)_{n=1}^{N}$ (Korobov's problem).

1. Introduction.

1.1. A number $\alpha \in (0, 1)$ is said to be *normal* to the base q, if in a q-ary expansion of α, $\alpha = d_1 d_2 \cdots \quad (d_i \in \{0, 1, \cdots, q-1\}, i = 1, 2, \cdots)$, each fixed finite block of digits of length k appears with an asymptotic frequency of q^{-k} along the sequence $(d_i)_{i \geq 1}$. Normal numbers were introduced by

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Borel (1909). Champernowne (1935) gave an explicit construction of such a number, namely

$$\theta = .1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ \dots,$$

obtained by successively concatenating all the natural numbers.

1.1.1. We denote by $\mathbb{N}$ the set of non-negative integers. Let $d, q \geq 2$ be two integers, $\mathbb{N}^d = \{(n_1, \dots, n_d) \mid n_i \in \mathbb{N},\ i = 1, \dots, d\}$, $\Delta = \{0, 1, \dots, q-1\}$, $\Omega = \Delta^{\mathbb{N}^d}$.
We shall call $\omega \in \Omega$ a *configuration* (*lattice configuration*). A configuration is a function $\omega : \mathbb{N}^d \to \Delta$. Let $\mathbf{h}, \mathbf{N} \in \mathbb{N}^d$, $\mathbf{h} = (h_1, \dots, h_d)$, $\mathbf{N} = (N_1, \dots, N_d)$. We denote a *rectangular block* by

$$F_{\mathbf{N}} = \{(f_1, \dots, f_d) \in \mathbb{N}^d \mid 0 \leq f_i < N_i,\ i = 1, \dots, d\},$$

$G = G_{\mathbf{h}}$ is a fixed block of digits $G = \{g_{i_1,\dots,i_d} \in \Delta \mid \ i_j \in [0, h_j),\ \ j = 1, \dots, d\}$.
1.1.2. *Definition.* A lattice configuration, $\omega \in \Omega$, is said to be *normal* (*rectangular normal*) if for any $\mathbf{h} \in \mathbb{N}^d$ with $h_1 \cdots h_d \geq 1$ and block of digits $G_{\mathbf{h}}$,

$$(1)\quad \#\big\{\mathbf{n} \in F_{\mathbf{N}} \mid \omega(\mathbf{n} + \mathbf{i}) = g_{i_1,\dots,i_d}\ \ \forall \mathbf{i} \in F_{\mathbf{h}}\big\} - q^{-h_1 \cdots h_d} N_1 \cdots N_d = o(N_1 \cdots N_d),$$

where $\mathbf{i} = (i_1, \dots, i_d)$, and $\max(N_1, \dots, N_d) \to \infty$.
It is evident that almost every $\omega \in \Omega$ is normal. The constructive proof of the existence of the normal lattice configuration is given in [LS1], [LS2].

Below, to simplify the calculations we consider only the case of $d = 2$.
1.1.3. Let $(\mathbf{x}_n)$ be an infinite sequence of points in an $s-$dimensional unit cube $[0,1)^s$; $v = [0, \gamma_1) \times \cdots \times [0, \gamma_s)$ be a box in $[0,1)^s$; and $A_v(N)$ be a number of indexes $n \in [1, N]$ such that $\mathbf{x}_n$ lies in v. The sequence $(\mathbf{x}_n)$ is said to be *uniformly distributed* in $[0,1)^s$ if for every box v, $A_v(N)/N \to \gamma_1 \cdots \gamma_s$. The quantity

$$(2)\quad D(N) = D((\mathbf{x}_n)_{n=1}^N) = D^{(s)}((\mathbf{x}_n)_{n=1}^N) = \sup_{v \in (0,1]^s} \left| \frac{1}{N} A_v(N) - \gamma_1 \cdots \gamma_s \right|$$

is called the *discrepancy* of $(\mathbf{x}_n)_{n=1}^N$.
It is known (Roth, [Ro]) that for any sequence in $[0,1)^s$,

$$\overline{\lim}_{N \to \infty} N D(N) / \log^{s/2} N > 0,$$

and according to the well-known conjecture (see for example [Ni, p. 32,33]),

$$(3)\qquad \overline{\lim}_{N \to \infty} N D(N) / \log^{s} N > 0.$$

1.1.4. The double sequence $(\mathbf{u}_{n,m}) \in [0,1)^s$, $(n,m = 1,2,\dots)$ is said to be uniformly distributed (Cigler, [Ci]) if

$$D^{(s)}(\{\mathbf{u}_{n,m}\}_{n=1,m=1}^{N\ \ M}) = o(1), \quad \text{with} \quad \max(N,M) \to \infty.$$

Kirschenhofer and Tichy [KiTi] investigated double sequences over finite sets (see also references in [DrTi, p.364], [KN, p. 18]).

1.2. It is known (Wall, 1949) that a number α is normal to the base q if and only if the sequence $\{\alpha q^n\}_{n\geq 1}$ is uniformly distributed in $[0,1)$ (see [KN, p. 70]). It is easy to prove similarly (see Appendix of this paper) the following statement:

Proposition 1. *Let $q \geq 2$ be integer, $d_{m,n} \in \{0,1,\dots,q-1\}$, $m,n = 1,2,\dots$. The lattice configuration $(d_{m,n})_{m,n\geq 1}$ is normal if and only if for all $s \geq 1$ the double sequence*

$$\big(\{\alpha_m q^n\},\dots,\{\alpha_{m+s-1}q^n\}\big)_{m,n\geq 1} \tag{4}$$

is uniformly distributed in $[0,1)^s$, where

$$\alpha_m = \sum_{n=1}^{\infty} d_{m,n}/q^n \ . \tag{5}$$

1.2.1. In [Le3] it was proved explicitly that there exists a normal number α with

$$D\big(\{\alpha q^n\}_{n=1}^{N}\big) = O(N^{-1}\log^2 N), \quad N \to \infty \ . \tag{6}$$

The estimate of discrepancy was previously known $O(N^{-2/3}\log^{4/3} N)$ (see [Ko2],[Le2]). According to (3), the estimate (6) cannot be improved essentially.

Our goal is to find a lower bound of discrepancy of the double sequence (4). The main idea of the paper is the using of small discrepancy sequences on the multidimensional unit cube to construct the sequence of reals $(\alpha_m)_{m\geq 1}$ (see (5) and (9)). Here we use a variant of Korobov's $s-$dimensional sequences $(s = 1,2,\dots)$ with optimal coefficients (see [Ko3]). We provide the following construction of a normal lattice configuration:

1.2.2 Construction. Let p_1, p_2 be distinct primes; $(q, p_1p_2) = 1$,

$$k_0 = 0, \quad k_1 = [\log_q(p_1p_2) + 1]p_1, \quad k_i = k_1 p_1^{[\log_{p_1} i]}, \quad i = 2,3,\dots, \tag{7}$$

$$\begin{gathered} t_0 = 1,\ t_i = p_2^{[\log_{p_2}\log_2(i+1)]}\ ,\ i = 1,2,\dots, \\ r(j) = \min_{j\leq t_i p_1^i,\ i=1,2,\dots} i, \quad j = 1,2,\dots, \end{gathered} \tag{8}$$

$$(9)\qquad \alpha_j = \sum_{i=r(j)}^{\infty} \quad \sum_{n=\delta_i^{(j)} p_1^{i-1} k_{i-1}/k_i}^{p_1^i - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_i(2n+\nu)}} \Big\{ \frac{a_{j_2,\nu}^{(i)}(p_2^i n + p_1^i j_1)}{p_1^i p_2^i} \Big\}_{k_i} ,$$

where

$$(10)\qquad \delta_i^{(j)} = 1 \text{ if } \quad i > r(j); \text{ otherwise } \quad \delta_i^{(j)} = 0, \quad \{x\}_k = [q^k\{x\}]/q^k,$$
$$j_2 \in \{0,1,\dots,t_i - 1\}, \quad j_2 \equiv j \bmod t_i \ , \quad j_1 = (j - j_2)/t_i,$$

$$(11)\qquad a_{j_2,\nu}^{(i)} \in \{0,1,\dots,p_1^i p_2^i - 1\}, \quad \nu = 0,1, \qquad j = 1,2,\dots \ .$$

Theorem 1. *There exist integers* $a_{r,\nu}^{(m)}$ $(m, r = 1,2,\dots, \nu = 0,1)$ *satisfying* (11), *such that for all* $s, N, M \geq 1$ *we have*

$$(12)\quad D\Big(\big(\{\alpha_m q^n\}, \dots, \{\alpha_{m+s-1} q^n\}\big)_{1 \leq n \leq N, 0 \leq m < M} \Big) = O\big((MN)^{-1} (\log MN)^{2s+4} \log^2 \log MN\big)$$

with $\max(M,N) \to \infty$, *and the constant implied by* O *only depends on* s.

We note that according to (3), the estimate (12) cannot be improved by more than the power of the logarithmic multiplier.

Corollary. *Let* $s, q \geq 2$. *There exist numbers* $\alpha_1, \dots, \alpha_s$ *(simultaneously normal to the base* q*) such that*

$$D\Big(\big(\{\alpha_1 q^n\}, \dots, \{\alpha_s q^n\}\big)_{n=1}^{N} \Big) = O(N^{-1} \log^{2s+4+\epsilon} N) \ .$$

The discrepancy estimate was previously known as $O(N^{-1/s})$ [Ko1] and $O(N^{-2/3} \log^{s+2} N)$ [Le2].

1.3. Let $s, q_1, \dots, q_s \geq 2$ be integers. Numbers $\alpha_1, \dots, \alpha_s$ are said to be *simultaneously normal to the base* $(q_1, \dots, q_s)$ [Ko1],[Ko3] if the sequence

$$(13)\qquad \big(\{\alpha_1 q_1^n\}, \dots, \{\alpha_s q_s^n\}\big)_{n \geq 1}$$

is uniformly distributed in $[0,1)^s$.

In [Ko1], Korobov obtained the first examples of simultaneously normal numbers using normal periodic systems, completely uniformly distributed sequences, and estimates of trigonometric sums with exponential functions (see also [Ko3]). In [Ko1], Korobov constructed simultaneously normal numbers with

$$D\big((\{\alpha_1 q_1^n\}, \dots, \{\alpha_s q_s^n\})_{n=1}^{N} \big) = O(N^{-1/s})$$

and posed the problem of finding simultaneously normal numbers with a maximum decay of the discrepancy of the sequence (13). In [Le1], simultaneously normal numbers with $D_N = O(N^{-1/2}\log^{s+3/2} N)$ were constructed. Here we find simultaneously normal numbers with the discrepancy estimate $O(N^{-1}\log^{2s+2} N)$. We note that according to (3), this estimate cannot be improved by more than the power of the logarithmic multiplier.

1.3.1 Construction. Let p be prime; $(q_i, p) = 1$, $i = 1, \dots, s$;

$$
\begin{aligned}
& k_1 = p \max_{1\le i\le s}[\log_{q_i} p + 1],\ k_m = k_1 p^{[\log_p m]},\ n_1 = 0, \\
& n_m = n_{m-1} + 2k_{m-1}p^{m-1},\ m = 2, 3, \dots
\end{aligned} \tag{14}
$$

$$
\alpha_i = \sum_{m=1}^{\infty}\sum_{n=0}^{p^m-1}\sum_{\nu=0}^{1} \frac{1}{q_i^{n_m+k_m(2n+\nu)}}\left\{\frac{a_{i,\nu}^{(m)} n}{p^m}\right\}_{k_m,i}, \tag{15}
$$

where $\{x\}_{k,i} = [\{x\}q_i^k]/q_i^k$, $i = 1, \dots, s$, $\quad k = 1, 2, \dots$.

Theorem 2. *Let $s \ge 2$. There exist integers $a_{i,\nu}^{(m)} \in \{0, 1, \dots, p^m - 1\}$ $(i = 1, \dots, s;\quad \nu = 0, 1;\quad m = 1, 2, \dots)$ such that*

$$
D\big((\{\alpha_1 q_1^n\}, \dots, \{\alpha_s q_s^n\})_{n=1}^N\big) = O(N^{-1}\log^{2s+2} N), \quad N \to \infty\ .
$$

We prove this theorem in Section 4. Theorem 1 is proved in Section 3. Section 2 contains auxiliary results.

2. Auxiliary results

First, some further notation is necessary. For integers $d \ge 1$ and $l \ge 2$, let $C_d(l)$ be the set of all nonzero lattice points $(h_1, \dots, h_d) \in \mathbb{Z}^d$ with $-l/2 < h_j \le l/2$ for $1 \le j \le d$; $C(l) = \mathbb{Z} \cap (-l/2, l/2]$. Define

$$
r(h, l) = \begin{cases} l\sin(\pi|h|/l) & \text{for} \quad h \in C_1(l), \\ 1 & \text{for} \quad h = 0, \end{cases}
$$

and

$$
r(\mathbf{h}, l) = \prod_{j=1}^{d} r(h_j, l) \tag{16}
$$

for $\mathbf{h} = (h_1, \dots, h_d) \in C_d(l)$. For real t, the abbreviation $e(t) = e^{2\pi\sqrt{-1}t}$ is used. Subsequently, four known results are stated, which follow from [Ko3, Lemma 2], [Ei, Lemma 3], [Ko3, p.13, Ni, p. 35] and [Ni, Theorem 3.10], respectively.

Lemma 1. *Let $p \geq 2, a$ be integers,*

$$\delta_p(a) = \begin{cases} 1 & \text{if} \quad a \equiv 0 \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\delta_p(a) = \frac{1}{p} \sum_{n=0}^{p-1} e(an/p).$$

Lemma 2. *Let $q \geq 2$ be an integer. Then*

$$\sum_{\substack{\mathbf{h} \in C_d(q) \\ \mathbf{h} \equiv \mathbf{0} (\bmod\ v)}} \frac{1}{r(\mathbf{h}, q)} < \frac{1}{v} \Big(\frac{2}{\pi} \log q + \frac{7}{5} \Big)^d$$

for any divisor v of q with $1 \leq v < q$.

Lemma 3. *Let A, B, T be integers, $1 \leq B \leq T$. Then*

$$\Big| \sum_{n \in [A, A+B)} e(t_n) \Big| \leq \sum_{h_0 \in (-T/2, T/2]} \frac{1}{r(h_0, T)} \Big| \sum_{n=A}^{A+T-1} e\big(t_n + \frac{nh_0}{T}\big) \Big| \ .$$

According to [Ni, p. 35] $|1/T \sum_{n \in [A, A+B)} e(nh_0/T)| \leq 1/r(h_0, T)$. Now the proof of Lemma 3 repeats that of [Ko3, p.13]. □
Applying this lemma twice, we get

Corollary 1. *Let $r \geq 1, M, M_1, N, N_1$ be integers, $M \in [1, p_2^r]$, $N \in [1, p_1^r]$. Then*

$$\Big| \sum_{n=N_1}^{N_1+N-1} \sum_{m=M_1}^{M_1+M-1} e(t_{nm}) \Big| \leq \sum_{h_{-1} \in C(p_1^r)} \sum_{h_{-2} \in C(p_2^r)} \frac{1}{r(h_{-1}, p_1^r) r(h_{-2}, p_2^r)}$$
$$\times \Big| \sum_{n=N_1}^{N_1+p_1^r-1} \sum_{m=M_1}^{M_1+p_2^r-1} e\big(t_{nm} + \frac{nh_{-1}}{p_1^r} + \frac{mh_{-2}}{p_2^r}\big) \Big| \ .$$

Lemma 4. *Let $N \geq 1$ and $P \geq 2$ be integers. Let $\mathbf{t}_n = \mathbf{y}_n/P \in [0,1)^d$ with $\mathbf{y}_n \in \{0, 1, \ldots, P-1\}^d$ for $0 \leq n < N$. Then the discrepancy of the points $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{N-1}$ satisfies*

$$D(\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{N-1}) \leq \frac{d}{P} + \frac{1}{N} \sum_{\mathbf{h} \in C_d(P)} \frac{1}{r(\mathbf{h}, P)} \Big| \sum_{n=0}^{N-1} e\big(\mathbf{h} \cdot \mathbf{t}_n\big) \Big| \ .$$

Corollary 2. *Let $T \geq N \geq 1$ and $P \geq 2$ be integers, $\mathbf{t}_n = \mathbf{y}_n/P \in [0,1)^d$ with $\mathbf{y}_n \in \{0, 1, \ldots, P-1\}^d$ for $0 \leq n < N$. Then*

$$D(\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_{N-1}) \leq \frac{d}{P} + \frac{T}{N} \widetilde{D}_T\big((\mathbf{t}_n)_{n \geq 0}\big) \ , \tag{17}$$

where

$$\widetilde{D}_T\big((\mathbf{t}_n)_{n\geq 0}\big) = \frac{1}{T}\sum_{\mathbf{h}\in C_d(P)}\ \sum_{h_0\in(-T/2,T/2]} \frac{1}{r(\mathbf{h},P)r(h_0,T)}\Big|\sum_{n=0}^{T-1} e(\mathbf{h}\cdot\mathbf{t}_n+\frac{nh_0}{T})\Big| .$$

Corollary 3. *Let $p_1^r \geq N \geq 1$, $p_2^r \geq M \geq 1$, and $p_1, p_2 \geq 2$ be integers, $\mathbf{t}_{n,m} = \mathbf{y}_{n,m}/p_1^r p_2^r \in [0,1)^d$, with $\mathbf{y}_{n,m} \in \{0,1,\dots,p_1^r p_2^r - 1\}^d$ for $n, m = 0,1,\dots$. Then*

$$(18)\quad MND^{(d)}((\mathbf{t}_{nm})_{M_1\leq m<M_1+M,\ N_1\leq n<N_1+N}) \leq \frac{dMN}{p_1^r p_2^r} + \hat{D}_r^{(d+2)}\big(\mathbf{t}_{nm}\big)\ ,$$

where

$$\hat{D}_r^{(d+2)}(\mathbf{t}_{nm}) = \sum_{\mathbf{h}\in C_d(p_1^r p_2^r)}\ \sum_{h_{-1}\in C(p_1^r)}\ \sum_{h_{-2}\in C(p_2^r)} \frac{1}{r(\mathbf{h},p_1^r p_2^r)r(h_{-1},p_1^r)r(h_{-2},p_2^r)} \times\Big|\sum_{n=0}^{p_1^r-1}\sum_{m=0}^{p_2^r-1} e\big(\mathbf{h}\cdot\mathbf{t}_{nm} + \frac{nh_{-1}}{p_1^r} + \frac{mh_{-2}}{p_2^r}\big)\Big| .$$

Lemma 5. *Let $\mathbf{x}_n \in [0,1)^s$, $n = 1,2,\dots,$ $q_1,\dots,q_s \geq 2$ be integers, $q = \min(q_1,\dots,q_s)$, $k^{(1)},\dots,k^{(s)} \geq 1$ be integers, and put $\mathbf{k} = (k^{(1)},\dots,k^{(s)})$, $k = \min(k^{(1)},\dots,k^{(s)})$. Then*

$$(19)\qquad D\big((\mathbf{x}_n)_{n=1}^N\big) \leq D\Big(\big(\{\mathbf{x}_n\}_{\mathbf{k}}\big)_{n=1}^N\Big) + \frac{s}{q^k},$$

$$(20)\quad D\Big(\big(\{\mathbf{x}_n\}_{\mathbf{k}}\big)_{n=1}^N\Big) \leq \frac{s}{q^k} + \max_{c_i\in[1,q_i^{k^{(i)}}],i=1,\dots,s}\Big|\frac{1}{N}\#\Big\{1\leq n\leq N \mid \{\mathbf{x}_n\}_{\mathbf{k}} \in \prod_{i=1}^s[0,\gamma(c_i))\Big\} - \prod_{i=1}^s\gamma(c_i)\Big| \leq \frac{s}{q^k} + D\big((\mathbf{x}_n)_{n=1}^N\big),$$

where $\gamma(c_i) = c_i/q_i^{k^{(i)}}, \{\mathbf{x}_n\}_{\mathbf{k}} = (\{x_{n,1}\}_{k^{(1)},1},\dots,\{x_{n,s}\}_{k^{(s)},s})$, $\{y\}_{m,i} = [q_i^m\{y\}]/q_i^m$, $m = 1,2,\dots$.

Proof. Let $v = [0,\gamma_1)\times\dots\times[0,\gamma_s)$; $v' = \prod_{i=1}^s[0,\{\gamma_i\}_{k^{(i)},i})$. It is easy to see that

$$\#\{1\leq n\leq N \mid \{\mathbf{x}_n\}_{\mathbf{k}}\in v'\} \leq \#\{1\leq n\leq N \mid \{\mathbf{x}_n\}\in v\} \leq \#\{1\leq n\leq N \mid \{\mathbf{x}_n\}_{\mathbf{k}}\in v\}\ .$$

Using (2), we obtain

$$\frac{1}{N}\#\big\{1\leq n\leq N \mid \{\mathbf{x}_n\}_{\mathbf{k}}\in v\big\} \leq \gamma_1\cdots\gamma_s + D\big((\{\mathbf{x}_n\}_{\mathbf{k}})_{n=1}^N\big),$$

and

$$\frac{1}{N}\#\big\{1 \le n \le N \mid \{\mathbf{x}_n\}_{\mathbf{k}} \in v'\big\} \ge \gamma_1 \cdots \gamma_s - D\Big(\big(\{\mathbf{x}_n\}_{\mathbf{k}}\big)_{n=1}^{N}\Big) - \Big|\prod_{i=1}^{s} \gamma_i - \prod_{i=1}^{s} \{\gamma_i\}_{k^{(i)},i}\Big| \,.$$

Hence

$$D\big((\mathbf{x}_n)_{n=1}^{N}\big) \le D\Big(\big(\{\mathbf{x}_n\}_{\mathbf{k}}\big)_{n=1}^{N}\Big) + \sup_{\gamma_1,\dots,\gamma_s \in [0,1)^s} \Big|\prod_{i=1}^{s} \gamma_i - \prod_{i=1}^{s} \{\gamma_i\}_{k^{(i)},i}\Big| \,.$$

Similarly to [Ni, Lemma 3.9], the second sum is not more than

$$1 - \prod_{i=1}^{s} \Big(1 - \frac{1}{q_i^{k^{(i)}}}\Big) \le 1 - \prod_{i=1}^{s} \Big(1 - \frac{1}{q^k}\Big) \le \frac{s}{q^k},$$

and we obtain the first part of the lemma.

Let $v'' = \prod_{i=1}^{s} [0, \{\gamma_i\}_{k^{(i)},i} + 1/q_i^{k^{(i)}})$. It is easy to see that

$$\#\{1 \le n \le N \mid \{\mathbf{x}_n\} \in v'\} \le \#\{1 \le n \le N \mid \{\mathbf{x}_n\}_{\mathbf{k}} \in v\} \le \#\{1 \le n \le N \mid \{\mathbf{x}_n\} \in v''\}$$

and

$$\begin{aligned} &\Big|\frac{1}{N}\#\{1 \le n \le N \mid \{\mathbf{x}_n\} \in v'\} - mes\ v'\Big| - |mes\ v - mes\ v'| \\ &\le \Big|\frac{1}{N}\#\{1 \le n \le N \mid \{\mathbf{x}_n\}_{\mathbf{k}} \in v\} - mes\ v\Big| \\ &\le \Big|\frac{1}{N}\#\{1 \le n \le N \mid \{\mathbf{x}_n\} \in v''\} - mes\ v''\Big| + |mes\ v'' - mes\ v|. \end{aligned}$$

Bearing in mind that $\max((mes\ v - mes\ v'), (mes\ v'' - mes\ v))$

$$\begin{aligned} &\le \max_{c_i \in [0, q_i^{k^{(i)}}), i=1,\dots,s} \prod_{i=1}^{s} \frac{c_i + 1}{q_i^{k^{(i)}}} - \prod_{i=1}^{s} \frac{c_i}{q_i^{k^{(i)}}} \\ &\le 1 - \prod_{i=1}^{s} \Big(1 - \frac{1}{q_i^{k^{(i)}}}\Big) \le \frac{s}{q^k}, \end{aligned}$$

we find that

$$\Big|\frac{1}{N}\#\big\{1 \le n \le N \mid \{\mathbf{x}_n\}_{\mathbf{k}} \in v\big\}\Big| \le \frac{s}{q^k} + \max_{c_i \in [1, q_i^{k^{(i)}}], i=1,\dots,s} \Big|\frac{1}{N}\#\Big\{1 \le n \le N \mid \{\mathbf{x}_n\}_{\mathbf{k}} \in \prod_{i=1}^{s} [0, \frac{c_i}{q_i^{k^{(i)}}})\Big\} - \prod_{i=1}^{s} \frac{c_i}{q_i^{k^{(i)}}}\Big|.$$

Now we obtain from (2) the second part of the lemma. $\square$

Lemma 6. *Let* $q_1, \dots, q_s \ge 2$ *be integers,* $q = \min(q_1, \dots, q_s)$*,* $k^{(1)}, \dots, k^{(s)} \ge 1$ *be integers,* $\ell^{(1)}, \dots, \ell^{(s)}$*; put* $k = \min(k^{(1)}, \dots, k^{(s)})$*,* $\ell = \min(\ell^{(1)}, \dots, \ell^{(s)})$*;* $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$*,* $\mathbf{y}_n = (y_{n,1}, \dots, y_{n,s})$*,* $\mathbf{z}_n = (z_{n,1}, \dots, z_{n,s})$*, and* $y_{n,i} = \{x_{n,i}\}^{(i)}_{k,i} + \frac{1}{q_i^{k^{(i)}}}\{z_{n,i}\}^{(i)}_{\ell,i}$*,* $i = 1, \dots, s$*,* $n = 1, 2, \dots$ *. Then*

$$D^{(s)}\big((\mathbf{y}_n)_{n=1}^N\big) \le 2^s D^{(2s)}\big((\mathbf{x}_n, \mathbf{z}_n)_{n=1}^N\big) + \frac{s}{q^{k+\ell}} \ .$$

Proof. Let $S = \{1, 2, \dots, s\}$, $I \subset S$, $\gamma_i = c_i / q_i^{k^{(i)}+\ell^{(i)}}$, $c_i \in \{1, \dots, q_i^{k^{(i)}+\ell^{(i)}}\}$, $\gamma_i' = q_i^{k^{(i)}}(\gamma_i - \{\gamma_i\}_{k^{(i)},i}) = \{\gamma_i'\}_{\ell^{(i)},i}$, $i = 1, \dots, s$,

$$\gamma_I = \prod_{i \in I} \{\gamma_i\}_{k^{(i)},i} \quad \prod_{i \in S \setminus I} q_i^{-k^{(i)}} \gamma_i' \ ,$$

$$\begin{aligned} B_I = \Big\{1 \le n \le N \mid \Big(\{x_{n,i}\}_{k^{(i)},i} \in [0, \{\gamma_i\}_{k^{(i)},i}), \quad \forall i \in I\Big), \quad \text{and} \\ \Big(\{x_{n,i}\}_{k^{(i)},i} = \{\gamma_i\}_{k^{(i)},i}, \quad \text{and} \quad \{z_{n,i}\}_{\ell^{(i)},i} \in [0, \gamma_i') \ \ \forall i \in S \setminus I\Big)\Big\}, \\ B = \big\{1 \le n \le N \mid \{y_{n,i}\} \in [0, \gamma_i), \ i = 1, \dots, s\big\} \ . \end{aligned}$$

It is easy to see that

$$\begin{aligned} & \prod_{i \in [1,s]} \gamma_i = \sum_{I \subset S} \gamma_I, \qquad B = \bigcup_{I \subset S} B_I, \\ & \Big| \frac{1}{N} \# B - \prod_{i \in [1,s]} \gamma_i \Big| \le \sum_{I \subset S} \Big| \frac{1}{N} \# B_I - \gamma_I \Big|, \end{aligned} \tag{21}$$

and

$$\{x_{n,i}\}_{k^{(i)},i} \in [0, \{\gamma_i\}_{k^{(i)},i}) \Longleftrightarrow \{x_{n,i}\} \in [0, \{\gamma_i\}_{k^{(i)},i});$$

$$\{x_{n,i}\}_{k^{(i)},i} = \{\gamma_i\}_{k^{(i)},i} \Longleftrightarrow \{x_{n,i}\} \in [\{\gamma_i\}_{k^{(i)},i}, \{\gamma_i\}_{k^{(i)},i} + 1/q_i^{k^{(i)}});$$

$$\{z_{n,i}\}_{\ell^{(i)},i} \in [0, \gamma_i') \Longleftrightarrow \{z_{n,i}\} \in [0, \gamma_i') \quad (\gamma_i' = \{\gamma_i'\}_{\ell^{(i)},i}).$$

Applying (2), we obtain:

$$\begin{aligned} \Big|\gamma_I - \frac{1}{N}\#B_I\Big| \le \Big|\gamma_I - \frac{1}{N}\#\Big\{1 \le n \le N \mid \ & \Big(\{x_{n,i}\} \in [0, \{\gamma_i\}_{k^{(i)},i}) \quad \forall i \in I\Big), \\ & \text{and } \Big(\{x_{n,i}\} \in [\{\gamma_i\}_{k^{(i)},i}, \{\gamma_i\}_{k^{(i)},i} + 1/q_i^{k^{(i)}}) \\ & \text{and} \quad \{z_{n,i}\} \in [0, \gamma_i') \ \ \forall i \in S \setminus I\Big)\Big\}\Big| \\ \le D^{(2s-\#I)}\Big(\Big((\{x_{n,i}\})_{i=1}^s, (\{z_{n,i}\})_{i \in S \setminus I}\Big)_{n=1}^N\Big) & \le D^{(2s)}\big((\mathbf{x}_n, \mathbf{z}_n)_{n=1}^N\big) \ . \end{aligned}$$

From (20) and (21) we get :

$$D\Big(\big(\{\mathbf{y}_n\}\big)_{n=1}^{N}\Big) \le \frac{s}{q^{k+\ell}}$$

$$+ \max_{\substack{c_i\in[1,q_i^{k^{(i)}+\ell^{(i)}}]\\ i=1,\dots,s}} \Big|\frac{1}{N}\#\Big\{1\le n\le N \mid \{\mathbf{y}_n\}\in\prod_{i=1}^{s}[0,\gamma(c_i))\Big\} - \prod_{i=1}^{s}\gamma(c_i)\Big|$$

$$\le \frac{s}{q^{k+\ell}} + \max_{\substack{c_i\in[1,q_i^{k^{(i)}+\ell^{(i)}}]\\ i=1,\dots,s}} \sum_{I\subset S}|\frac{1}{N}\#B_I - \gamma_I| \le \frac{s}{q^{k+\ell}} + 2^s D^{(2s)}\big((\mathbf{x}_n,\mathbf{z}_n)_{n=1}^{N}\big) ,$$

where $\gamma(c_i) = c_i/q_i^{k^{(i)}+\ell^{(i)}}$, $i = 1,\dots,s$. □

Let $\varphi_0 = \varphi(p_1p_2)/p_1p_2$, where $\varphi(x)$ is a Euler function,

$$\Delta_m(b) = \{0,1,\dots,b^m-1\} , \qquad (22)$$
$$\Delta_m^*(b) = \{n\in\Delta_m(b) \mid (n,b)=1\} .$$

It is evident that

$$\#\Delta_m^*(p_1p_2) = \varphi(p_1p_2)p_1^{m-1}p_2^{m-1} = \varphi_0 p_1^m p_2^m, \quad m=1,2,\dots . \qquad (23)$$

Now let

$$A(i,m,s,c_0,\dots,c_{2s-1}) = p_1^m p_2^m \qquad (24)$$
$$\times \sum_{h_{-1}\in C(p_1^m)}\sum_{h_{-2}\in C(p_2^m)}\sum_{\mathbf{h}\in C_{2s}(p_1^mp_2^m)} r^{-1}(h_{-1},p_1^m)r^{-1}(h_{-2},p_2^m)r^{-1}(\mathbf{h},p_1^mp_2^m)$$
$$\times \delta_{p_1^mp_2^m}\Big(h_{-1}M_2p_2^m + h_{-2}M_1p_1^m + \sum_{j=0}^{s-1}(q^ih_jc_j + h_{j+s}c_{j+s})\Big) ,$$

where $M_1p_1^m \equiv 1(\bmod\ p_2^m)$ and $M_2p_2^m \equiv 1(\bmod\ p_1^m)$.

Lemma 7. *Let $i\ge 0$; $m,s\ge 1$ be integers. Then*

$$\frac{1}{(\varphi_0p_1^mp_2^m)^{2s}} \sum_{(c_0,\dots,c_{2s-1})\in(\Delta_m^*(p_1p_2))^{2s}} A(i,m,s,c_0,\dots,c_{2s-1}) \qquad (25)$$
$$\le K_1(s)m^{2s+2},$$

with $K_1(s) = 12\varphi_0^{-2s}(\frac{2}{\pi}\log p_1p_2 + \frac{7}{5})^{2s+2}p_1^2p_2^2$.

Proof. We follow [Ko3, p. 191]. We denote the left side of (25) by σ_1. Changing the order of the summation, from (24) we get

$$\sigma_1 \le \sum_{h_{-1}\in C(p_1^m)}\sum_{h_{-2}\in C(p_2^m)}\sum_{\mathbf{h}\in C_{2s}(p_1^mp_2^m)} r^{-1}(h_{-1},p_1^m) \qquad (26)$$
$$\times r^{-1}(h_{-2},p_2^m)r^{-1}(\mathbf{h},p_1^mp_2^m)E(h_{-1},h_{-2},\mathbf{h}),$$

where

$$(27)\quad E(h_{-1},h_{-2},\mathbf{h}) = \frac{1}{(\varphi_0 p_1^m p_2^m)^{2s}} \sum_{(c_0,\dots,c_{2s-1})\in(\Delta_m(p_1p_2))^{2s}} p_1^m p_2^m$$
$$\times\, \delta_{p_1^m p_2^m}\Big(h_{-1}M_2p_2^m + h_{-2}M_1p_1^m + \sum_{j=0}^{s-1}(q^i h_j c_j + h_{j+s}c_{j+s})\Big)\,.$$

Let $(h_0,\dots,h_{2s-1},p_1^m p_2^m) = p_1^{\alpha_1}p_2^{\alpha_2}$, and let

$$(28)\quad h_i = p_1^{\alpha_1+\beta_{1,i}} p_2^{\alpha_2+\beta_{2,i}} h_i', \qquad (h_i', p_1p_2)=1, \quad \beta_{1,i},\beta_{2,i}\ge 0, \quad i=0,\dots,2s-1.$$

This yields that there exist $\mu,\nu\in[0,2s-1]$ such that $\beta_{1,\mu}=0$ and $\beta_{2,\nu}=0$. If $\mu=\nu$, then

$$(29)\quad \sum_{c_\mu\in[0,p_1^m p_2^m)} \delta_{p_1^{m-\alpha_1}p_2^{m-\alpha_2}}(c_\mu h_\mu' v_1 + v_3) = p_1^{\alpha_1}p_2^{\alpha_2} \quad \text{for} \quad (v_1,p_1p_2)=1.$$

Now let $\mu\ne\nu$.
We find for integers v_1, v_2 and v_3 (with $(v_i,p_1p_2)=1,\ \ i=1,2$), that

$$(30)\quad \sum_{c_\mu,c_\nu\in[0,p_1^m p_2^m)} \delta_{p_1^{m-\alpha_1}p_2^{m-\alpha_2}}(c_\mu h_\mu' p_2^{\beta_{2,\mu}} v_1 + c_\nu h_\nu' p_1^{\beta_{1,\nu}} v_2 + v_3)$$
$$= \sum_{c_{\mu,1},c_{\nu,1}\in[0,p_1^m)}\ \sum_{c_{\mu,2},c_{\nu,2}\in[0,p_2^m)} \delta_{p_1^{m-\alpha_1}p_2^{m-\alpha_2}}\Big((c_{\mu,1}M_2p_2^m + c_{\mu,2}M_1p_1^m)h_\mu' p_2^{\beta_{2,\mu}} v_1$$
$$+ (c_{\nu,1}M_2p_2^m + c_{\nu,2}M_1p_1^m)h_\nu' p_1^{\beta_{1,\nu}} v_2 + v_3\Big) = \sigma'\sigma'',$$

where

$$\sigma' = \sum_{c_{\mu,1},c_{\nu,1}\in[0,p_1^m)} \delta_{p_1^{m-\alpha_1}}(c_{\mu,1}M_2p_2^m h_\mu' p_2^{\beta_{2,\mu}} v_1 + c_{\nu,1}M_2p_2^m h_\nu' p_1^{\beta_{1,\nu}} v_2 + v_3),$$

and

$$\sigma'' = \sum_{c_{\mu,2},c_{\nu,2}\in[0,p_2^m)} \delta_{p_2^{m-\alpha_2}}(c_{\mu,2}M_1p_1^m h_\mu' p_2^{\beta_{2,\mu}} v_1 + c_{\nu,2}M_1p_1^m h_\nu' p_1^{\beta_{1,\nu}} v_2 + v_3)\ .$$

Observe that $(M_2p_2^m h_\mu' p_2^{\beta_{2,\mu}} v_1, p_1)=1$, $(M_1p_1^m h_\nu' p_1^{\beta_{1,\nu}} v_2, p_2)=1$,

$$\sum_{c_{\mu,1}\in[0,p_1^m)} \delta_{p_1^{m-\alpha_1}}(c_{\mu,1}M_2p_2^m h_\mu' p_2^{\beta_{2,\mu}} v_1 + v_4) = p_1^{\alpha_1}\ ,$$

and

$$\sum_{c_{\nu,2}\in[0,p_2^m)} \delta_{p_2^{m-\alpha_2}}(c_{\nu,2}M_1p_1^m h_\nu' p_1^{\beta_{1,\nu}} v_2 + v_5) = p_2^{\alpha_2}\ .$$

Hence

$$\sigma' = p_1^{m+\alpha_1} \qquad \text{and} \quad \sigma'' = p_2^{m+\alpha_2}. \tag{31}$$

If $h_{-1} \not\equiv 0 \pmod{p_1^{\alpha_1}}$ or $h_{-2} \not\equiv 0 \pmod{p_2^{\alpha_2}}$, then $E(h_{-1}, h_{-2}, \mathbf{h}) = 0$. Now let

$$h_{-1} = p_1^{\alpha_1} h'_{-1}, \quad h_{-2} = p_2^{\alpha_2} h'_{-2}.$$

From (27) we see, that

$$\begin{aligned} E(p_1^{\alpha_1} h'_{-1}, p_2^{\alpha_2} h'_{-2}, p_1^{\alpha_1} p_2^{\alpha_2}\mathbf{h}') &= \frac{1}{(\varphi_0 p_1^m p_2^m)^{2s}} \\ &\times \sum_{(c_0,\dots,c_{2s-1})\in(\Delta_r(p_1^m p_2^m))^{2s}} p_1^m p_2^m \times \delta_{p_1^{m-\alpha_1} p_2^{m-\alpha_2}} \Big(h'_{-1} M_2 p_2^{m-\alpha_2} \\ &+ h'_{-2} M_1 p_1^{m-\alpha_1} + \sum_{j=0}^{s-1} (q^i h'_j c_j p_1^{\beta_{1,j}} p_2^{\beta_{2,j}} + h'_{j+s} c_{j+s} p_1^{\beta_{1,j+s}} p_2^{\beta_{2,j+s}}) \Big) . \end{aligned} \tag{32}$$

Now, applying (29) for the case of $\mu{=}\nu$ and (30), (31) for the case of $\mu \neq \nu$, we get

$$E(p_1^{\alpha_1} h'_{-1}, p_2^{\alpha_2} h'_{-2}, p_1^{\alpha_1} p_2^{\alpha_2}\mathbf{h}') = \varphi_0^{-2s} p_1^{\alpha_1} p_2^{\alpha_2} .$$

We obtain from (26) that

$$\sigma_1 \leq \varphi_0^{-2s} \sum_{0\leq\alpha_1,\alpha_2\leq m} \sum_{h'_{-1}\in C(p_1^{m-\alpha_1})} \sum_{h'_{-2}\in C(p_2^{m-\alpha_2})} \sum_{\mathbf{h}'\in C_{2s}(p_1^{m-\alpha_1} p_2^{m-\alpha_2})} p_1^{\alpha_1} p_2^{\alpha_2} \times$$

$$\times r^{-1}(h'_{-1} p_1^{\alpha_1}, p_1^m) r^{-1}(h'_{-2} p_2^{\alpha_2}, p_2^m) r^{-1}(\mathbf{h}' p_1^{\alpha_1} p_2^{\alpha_2}, p_1^m p_2^m). \tag{33}$$

Applying Lemma 2, we get

$$\sum_{\mathbf{h}'\in C_{2s}(p_1^{m-\alpha_1} p_2^{m-\alpha_2})} r^{-1}(p_1^{\alpha_1} p_2^{\alpha_2}\mathbf{h}', p_1^m p_2^m) \leq p_1^{-\alpha_1} p_2^{-\alpha_2} \Big(\frac{2m}{\pi} \log p_1 p_2 + \frac{7}{5}\Big)^{2s},$$

and

$$\sum_{h'_{-1}\in C(p_1^{m-\alpha_1})} r^{-1}(p_1^{\alpha_1} h_{-1}, p_1^m) \leq 1 + p_1^{-\alpha_1} \Big(\frac{2m}{\pi} \log p_1 + \frac{7}{5}\Big) . \tag{34}$$

Now, from (33) we obtain :

$$\sigma_1 \le \varphi_0^{-2s} \sum_{0\le\alpha_1,\alpha_2\le m} \Big(\frac{2}{\pi} m \log p_1 p_2 + \frac{7}{5}\Big)^{2s} \Big(1 + p_1^{-\alpha_1}\big(\frac{2}{\pi} m \log p_1 + \frac{7}{5}\big)\Big)$$

$$\times \Big(1 + p_2^{-\alpha_2}\big(\frac{2}{\pi} m \log p_2 + \frac{7}{5}\big)\Big)$$

$$\le \varphi_0^{-2s} \Big(\frac{2}{\pi} m \log p_1 p_2 + \frac{7}{5}\Big)^{2s} \Big((m+1)^2 + (m+1)\big(\frac{2}{\pi} m \log p_1 p_2 + \frac{7}{5}\big)$$

$$\times \big(\frac{p_1}{p_1-1} + \frac{p_2}{p_2-1}\big) + \big(\frac{2}{\pi} m \log p_1 p_2 + \frac{7}{5}\big)^2 \frac{p_1}{p_1-1}\frac{p_2}{p_2-1}\Big)$$

$$\le K_1(s) m^{2s+2} .$$

□

Let

$$\mathbf{b}^{(m)} = (b_{0,0}^{(m)}, \dots, b_{t_m-1,0}^{(m)}, b_{0,1}^{(m)}, \dots, b_{t_m-1,1}^{(m)}),$$

$$(35) \quad B(\nu, i, j, s, \mathbf{b}^{(m)}) = A(i, m, s, b_{j,\nu}^{(m)}, \dots, b_{j+s-1,\nu}^{(m)}, b_{j,\nu+1}^{(m)}, \dots, b_{j+s-1,\nu+1}^{(m)}) ,$$

where $b_{j+t_m,\nu+2}^{(m)} = b_{j,\nu}^{(m)}, \quad \nu \in \{0,1\},\ j \in \{0,1,\dots,t_m-1\}$,

$$(36) \quad \Omega_m = \Big\{\mathbf{b}^{(m)} \in \big(\Delta_m^*(p_1p_2)\big)^{2t_m} \mid \sum_{s\in[1,t_m]} \sum_{j\in[0,t_m)} \times \sum_{i\in[0,k_m-1]} \sum_{\nu\in\{0,1\}} B(\nu, i, j, s, \mathbf{b}^{(m)})(24K_1(s)t_m^2 k_m m^{2s+2})^{-1} > 1\Big\} .$$

Lemma 8. *With the notation defined above we have:*

$$\#\Omega_m \le \frac{1}{12}\big(\#\Delta_m^*(p_1p_2)\big)^{2t_m}, \quad m = 1, 2, \dots .$$

Proof. It follows from Lemma 7 and (35) that

$$\frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}} \sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}} B(\nu, i, j, s, \mathbf{b}^{(m)}) = \frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}}$$

$$\times \sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}} A(i, m, s, b_{j,\nu}^{(m)}, \dots, b_{j+s-1,\nu}^{(m)}, b_{j,\nu+1}^{(m)}, \dots, b_{j+s-1,\nu+1}^{(m)})$$

$$\le K_1(s) m^{2s+2},$$

where $\nu \in \{0,1\}, j \in [0, t_m)$, and $i \in [0, k_m - 1]$.
Hence

$$\sum_{j\in[0,t_m)} \sum_{i\in[0,k_m-1]} \sum_{\nu\in\{0,1\}} \frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}} \sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}} B(\nu, i, j, s, \mathbf{b}^{(m)})$$

$$\leq 2K_1(s)t_m k_m m^{2s+2},$$

and

$$\sum_{s\in[1,t_m]}\sum_{j\in[0,t_m)}\sum_{i\in[0,k_m-1]}\sum_{\nu\in\{0,1\}}\frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}}$$

$$\times\sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}} B(\nu,i,j,s,\mathbf{b}^{(m)})\Big(2K_1(s)t_m^2k_mm^{2s+2}\Big)^{-1}\leq 1.$$

Changing the order of the summation, we find that

$$(37)\quad \frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}}\sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}}\Big(\sum_{s\in[1,t_m]}\Big(24K_1(s)t_m^2k_mm^{2s+2}\Big)^{-1}$$
$$\times\sum_{j\in[0,t_m)}\sum_{i\in[0,k_m-1]}\sum_{\nu\in\{0,1\}}B(\nu,i,j,s,\mathbf{b}^{(m)})\Big)\leq 1/12\ .$$

Now, from (36), we obtain the assertion of the lemma. $\square$

Put

$$(38)\quad A_1(\alpha,i,m,s,c_0,\dots,c_{s-1})=\sum_{h_{-2}\in C(p_2^{m+1})}\ \sum_{\substack{\mathbf{h}\in C_{2s}(p_2^{m+1})\\(h_s,\dots,h_{2s-1},p_2^{m+1})=p_2^\alpha}}$$
$$\times r^{-1}(h_{-2},p_2^{m+1})r^{-1}(\mathbf{h},p_2^{m+1})p_2^\alpha\delta_{p_2^\alpha}\Big(h_{-2}+q^ip_2\sum_{j=0}^{s-1}h_jc_jt_{m+1}/t_m\Big),$$

and for $v\in[1,s-1]$

$$(39)\quad A_2(\alpha,i,m,v,c_0,\dots,c_{2s-1})=\sum_{h_{-1}\in C(p_1^{m+1})}\ \sum_{\substack{\mathbf{h}\in C_{2s}(p_1^{m+1})\\(h_v,\dots,h_{s-1},h_{v+s},\dots,h_{2s-1},p_1^{m+1})=p_1^\alpha}}$$
$$\times r^{-1}(h_{-1},p_1^m)r^{-1}(\mathbf{h},p_1^m)p_1^\alpha\delta_{p_1^\alpha}\Big(h_{-1}+p_1k_{m+1}/k_m\sum_{j=0}^{v-1}(q^ih_jc_j+h_{j+s}c_{j+s})\Big).$$

Lemma 9. *Let $t_m\geq s\geq 1,\alpha\in[0,m+1],v\in[1,s-1],i\in[0,k_{m-1}]$ be integers. Then*

$$(40)\quad \frac{1}{(\#\Delta_m^*(p_2))^s}\sum_{(c_0,\dots,c_{s-1})\in(\Delta_m^*(p_2))^s}A_1(\alpha,i,m,s,c_0,\dots,c_{s-1})$$
$$\leq K_1(s)p_2(m+1)^{2s+1}$$

and

$$(41)\quad \frac{1}{(\#\Delta_m^*(p_1))^{2s}} \sum_{(c_0,\dots,c_{2s-1})\in(\Delta_m^*(p_1))^{2s}} A_2(\alpha,i,m,v,c_0,\dots,c_{2s-1}) \le K_1(s)p_1(m+1)^{2s+1}.$$

Proof. We will prove the statement (41). The proof of (40) repeats that of (41). We denote the left side of (41) by σ_1. Changing the order of the summation, from (39) we get

$$(42)\quad \sigma_1 = \sum_{h_{-1}\in C(p_1^{m+1})} \sum_{\substack{\mathbf{h}\in C_{2s}(p_1^{m+1}) \\ (h_v,\dots,h_{s-1},h_{v+s},\dots,h_{2s-1},p_1^{m+1})=p_1^{\alpha}}} r^{-1}(h_{-1},p_1^{m+1}) \times r^{-1}(\mathbf{h},p_1^{m+1})E(h_{-1},\mathbf{h}),$$

where

$$E(h_{-1},\mathbf{h}) = \frac{1}{(\varphi(p_1)p_1^{m-1})^{2s}} \sum_{(c_0,\dots,c_{2s-1})\in(\Delta_m^*(p_1))^{2s}} p_1^{\alpha} \times \delta_{p_1^{\alpha}}\Big(h_{-1}+p_1k_{m+1}/k_m\sum_{j=0}^{v-1}(q^ih_jc_j+h_{j+s}c_{j+s})\Big).$$

Let $(h_0,\dots,h_{v-1},h_s,\dots,h_{v+s-1},p_1^{m+1}) = p_1^{\alpha_1}$, and let

$$h_{\nu s+i} = p_1^{\alpha_1}h'_{\nu s+i}, \qquad \nu=0,1, \quad i=0,\dots,v-1.$$

Then there is $(\nu_0,i_0)\in\{0,1\}\times\{0,\dots,v-1\}$ with $(h'_{\nu_0 s+i_0},p_1)=1$.
It is easy to see that

$$(43)\quad E(h_{-1},\mathbf{h}) \le \max_{\mathbf{c}\in(\Delta_m^*(p_1))^{2s}} \sigma(\mathbf{c},i_0,\nu_0),$$

where

$$(44)\quad \sigma(\mathbf{c},i_0,\nu_0) = \frac{1}{\varphi(p_1)p_1^{m-1}} \sum_{c_{\nu_0 s+i_0}\in\Delta_{m+1}^*(p_1)} p_1^{\alpha} \times \delta_{p_1^{\alpha}}\Big(h_{-1}+p_1^{\alpha_1+1}k_{m+1}/k_m\sum_{j=0}^{v-1}(q^ih'_jc_j+h'_{j+s}c_{j+s})\Big)\ .$$

We find that if $h_{-1}\not\equiv 0 \pmod{p_1^{\min(\alpha,\alpha_1)}}$, then $E(h_{-1},\mathbf{h})=0$.
Now let $h_{-1}\equiv 0 \pmod{p_1^{\min(\alpha,\alpha_1)}}$, and let $h_{-1}=p_1^{\min(\alpha,\alpha_1)}h'_{-1}$.
From (44) we find that

$$(45)\quad \sigma(\mathbf{c},i_0,\nu_0) = p_1^{\alpha} \qquad \text{for} \quad \alpha_1\ge\alpha\ ,$$

and

$$(46)\quad \sigma(\mathbf{c}, i_0, \nu_0) \le \frac{1}{\varphi(p_1)p_1^{m-1}} \sum_{c_{\nu_0 s+i_0}\in[0,p_1^m)} p_1^{\alpha} \times \delta_{p_1^{\alpha-\alpha_1}}\Big(h'_{-1} + p_1 k_{m+1}/k_m \sum_{j=0}^{v-1}(q^i h'_j c_j + h'_{j+s}c_{j+s})\Big)$$

$$< \frac{1}{\varphi_0 p_1^m} \sum_{c_{\nu_0 s+i_0}\in[0,p_1^m)} p_1^{\alpha}\delta_{p_1^{\alpha-\alpha_1}}\Big(f + p_1 k_{m+1}/k_m q^{(1-\nu_0)i} h'_{\nu_0 s+j_0} c_{\nu_0 s+j_0}\Big) \qquad \text{for} \quad \alpha_1 < \alpha\ ,$$

where

$$f = h'_{-1} + p_1 k_{m+1}/k_m \sum_{\substack{\nu\in\{0,1\},\, j\in[0,v] \\ (\nu,j)\ne(\nu_0,j_0)}} q^{(1-\nu)i} h'_{\nu s+j} c_{\nu s+j}.$$

It is easy to verify that

$$\log_{p_1}(m+1) - \log_{p_1}(m) \le 1, \qquad m = 1,2,\dots,$$

and

$$(47)\qquad [\log_{p_1}(m+1)] - [\log_{p_1}(m)] \le 1, \qquad \text{for} \quad m = 1,2,\dots\ .$$

According to (7), we have

$$(48)\quad k_{m+1}/k_m = p_1^{[\log_{p_1}(m+1)]-[\log_{p_1}(m)]} \in \{1, p_1\} \qquad \text{for} \quad m = 1,2,\dots\ .$$

Bearing in mind that $(q^{(1-\nu_0)i} h'_{\nu_0 s+j_0}, p_1) = 1$, we obtain from (46) that

$$\sigma(\mathbf{c}, i_0, \nu_0) \le \varphi_0^{-1} p_1^{2+\alpha_1} \quad \text{for} \quad \alpha_1 < \alpha\ .$$

Now, (45) and (43) imply that

$$E(h_{-1}, \mathbf{h}) \le \varphi_0^{-1} p_1^{2+\min(\alpha,\alpha_1)} \delta_{p_1^{\min(\alpha,\alpha_1)}}(h_{-1}) \quad \text{for} \quad \alpha_1 \in [0, m+1]\ .$$

From (42) we obtain

$$\sigma_1 \le \sum_{\alpha_1\in[0,m+1]} \sum_{\substack{h_{-1}\in C(p_1^{m+1}) \\ h_{-1}\equiv 0 \pmod{p_1^{\min(\alpha,\alpha_1)}}}} \sum_{\substack{\mathbf{h}\in C_{2s}(p_1^{m+1}) \\ (h_v,\dots,h_{s-1},h_{v+s},\dots,h_{2s-1},p_1^{m+1})=p_1^{\alpha}, \\ (h_0,\dots,h_{v-1},h_s,\dots,h_{s+v-1},p_1^{m+1})=p_1^{\alpha_1}}} \times r^{-1}(h_{-1}, p_1^{m+1}) r^{-1}(\mathbf{h}, p_1^{m+1}) \varphi_0^{-1} p_1^{2+\min(\alpha,\alpha_1)}\ .$$

Applying Lemma 2, we get

$$\begin{aligned}\sigma_1 &\le \varphi_0^{-1} \sum_{\alpha_1 \in [0,m+1]} p_1^{2+\min(\alpha,\alpha_1)} \Big(1 + p_1^{-\min(\alpha,\alpha_1)} \big(\frac{2(m+1)}{\pi} \log p_1 + \frac{7}{5}\big)\Big) \\ &\qquad \times \Big(\frac{2(m+1)}{\pi} \log p_1 + \frac{7}{5}\Big)^{2s} p_1^{-\min(\alpha,\alpha_1)} \\ &\le \varphi_0^{-1} p_1^2 \Big(\frac{2(m+1)}{\pi} \log p_1 + \frac{7}{5}\Big)^{2s} \Big(m + 2 + 2\big(\frac{2(m+1)}{\pi} \log p_1 + \frac{7}{5}\big)\Big) \\ &\le \varphi_0^{-1} p_1^2 (m+1)^{2s+1} \big(\frac{2}{\pi} \log p_1 + \frac{7}{5}\big)^{2s} \Big(\frac{m+2}{m+1} + 2\big(\frac{2}{\pi} \log p_1 + \frac{7}{5}\big)\Big) \\ &\le K_1(s)(m+1)^{2s+1} \ .\end{aligned}$$

□

Lemma 10. *Let*

$$\Omega_{m,1} = \Big\{ \mathbf{b}^{(m)} \in (\Delta_m^*(p_1 p_2)^{2t_m} \mid \sum_{s=1}^{t_m} \sum_{\alpha=0}^{m+1} \sum_{\mu=0}^{t_m - 1} \sum_{i=0}^{k_m - 1} \times \frac{A_1(\alpha, i, m, s, b_{\mu,1}^{(m)}, \ldots, b_{\mu+s-1,1}^{(m)})}{12 K_1(s) t_m^2 k_m (m+2)(m+1)^{2s+1}} > 1 \Big\}, \tag{49}$$

and

$$\Omega_{m,2} = \Big\{ \mathbf{b}^{(m)} \in (\Delta_m^*(p_1 p_2)^{2t_m} \mid \sum_{s=1}^{t_m} \sum_{\alpha=0}^{m+1} \sum_{\mu \in (t_m - s, t_m)} \sum_{i=0}^{k_m - 1} \sum_{\nu=0}^{1} \times \frac{A_2(\alpha, i, m, t_m - \mu, b_{\mu,\nu}^{(m)}, \ldots, b_{\mu+s-1,\nu}^{(m)}, b_{\mu,\nu+1}^{(m)}, \ldots, b_{\mu+s-1,\nu+1}^{(m)})}{24 K_1(s) t_m^2 k_m (m+2)(m+1)^{2s+1}} > 1 \Big\}. \tag{50}$$

Then

$$\#\Omega_{m,\nu} \le \frac{1}{12} \big(\#\Delta_m^*(p_1 p_2)\big)^{2t_m}, \qquad \nu = 1, 2 \ . \tag{51}$$

Proof. Let $\nu = 2$. It follows from Lemma 9, and (39) that

$$\frac{1}{(\#\Delta_m^*(p_1 p_2))^{2t_m}} \sum_{\mathbf{b}^{(m)} \in (\Delta_m^*(p_1 p_2))^{2t_m}} A_2(\alpha, i, m, t_m - \mu, b_{\mu,\nu}^{(m)}, \ldots, b_{\mu+s-1,\nu}^{(m)}, b_{\mu,\nu+1}^{(m)}, \ldots, b_{\mu+s-1,\nu+1}^{(m)})$$

$$= \frac{1}{(\#\Delta_m^*(p_1))^{2s}} \sum_{(b_{\mu,\nu}^{(m)},\dots,b_{\mu+s-1,\nu+1}^{(m)})\in(\Delta_m^*(p_1))^{2s}}$$

$$A_2(\alpha,i,m,t_m-\mu,b_{\mu,\nu}^{(m)},\dots,b_{\mu+s-1,\nu}^{(m)},b_{\mu,\nu+1}^{(m)},\dots,b_{\mu+s-1,\nu+1}^{(m)})$$

$$\leq K_1(s)p_1(m+1)^{2s+1},$$

for $s\in[1,t_m]$, $\alpha\in[0,m+1]$, $\mu\in(t_m-s,t_m)$, $i\in[0,k_m-1]$, and $\nu\in\{0,1\}$. Hence

$$\sum_{s=1}^{t_m}\sum_{\alpha=0}^{m+1}\sum_{\mu\in(t_m-s,t_m)}\sum_{i=0}^{k_m-1}\sum_{\nu=0}^{1}\frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}}\sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}}$$

$$\times\frac{A_2(\alpha,i,m,t_m-\mu,b_{\mu,\nu}^{(m)},\dots,b_{\mu+s-1,\nu}^{(m)},b_{\mu,\nu+1}^{(m)},\dots,b_{\mu+s-1,\nu+1}^{(m)})}{2K_1(s)p_1t_m^2k_m(m+2)(m+1)^{2s+1}}\leq 1.$$

Changing the order of the summation we obtain that

$$\frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_m}}\sum_{\mathbf{b}^{(m)}\in(\Delta_m^*(p_1p_2))^{2t_m}}\sum_{s=1}^{t_m}\sum_{\alpha=0}^{m+1}\sum_{\mu\in(t_m-s,t_m)}\sum_{i=0}^{k_m-1}\sum_{\nu=0}^{1}$$

$$\times\frac{A_2(\alpha,i,m,t_m-\mu,b_{\mu,\nu}^{(m)},\dots,b_{\mu+s-1,\nu}^{(m)},b_{\mu,\nu+1}^{(m)},\dots,b_{\mu+s-1,\nu+1}^{(m)})}{24K_1(s)p_1t_m^2k_m(m+2)(m+1)^{2s+1}}\leq\frac{1}{12}\,.$$

Now, from (50) we obtain the desired result. Using (49), we similarly obtain (51) for the case of $\nu=1$. $\square$

Now, from Lemmas 8 and 10 we get:

Corollary 4. *Let*

$$\Omega_{m,3}=\Omega_m\cup\Omega_{m,1}\cup\Omega_{m,2}. \tag{52}$$

Then

$$\#\Omega_{m,3}\leq\frac{1}{4}\big(\#\Delta_m^*(p_1p_2)\big)^{2t_m},\quad m=1,2,\dots\ . \tag{53}$$

Put

$$B_1(i,\mu,s,\mathbf{b}^{(m)},\mathbf{c}^{(m+1)})=\sum_{h_{-2}\in C(p_2^{m+1})}\sum_{\mathbf{h}\in C_{2s}(p_2^{m+1})}r^{-1}(h_{-2},p_2^{m+1}) \tag{54}$$

$$\times r^{-1}(\mathbf{h},p_2^{m+1})p_2^{m+1}\delta_{p_2^{m+1}}\Big(h_{-2}+\sum_{j=0}^{s-1}\Big(q^ih_jb_{\mu+j,1}^{(m)}p_2\frac{t_{m+1}}{t_m}+h_{s+j}c_{\mu+j,0}^{(m+1)}\Big)\Big)\,,$$

where $c_{j+t_{m+1},\nu+2}^{(m+1)}=c_{j,\nu}^{(m+1)}$, $\quad\nu\in\{0,1\},\ j\in\{0,1,\dots,t_{m+1}-1\}$.

Lemma 11. *With the notation defined above, we have:*

$$\frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_{m+1}}} \sum_{\mathbf{c}^{(m+1)}\in(\Delta_{m+1}^*(p_1p_2))^{2t_{m+1}}} B_1(i,\mu,s,\mathbf{b}^{(m)},\mathbf{c}^{(m+1)}) \le p_2 \sum_{\alpha=0}^{m+1} A_1(\alpha,i,m,s,b_{\mu,1}^{(m)},\ldots,b_{\mu+s-1,1}^{(m)}) \ . \tag{55}$$

Proof. Let $(h_s,\ldots,h_{2s-1},p_2^{m+1}) = p_2^{\alpha}$, $\quad h_i = h_i^{'}p_2^{\alpha}$, for $i=s,\ldots,2s-1$, and let $(h_{s+i_0}^{'},p_2)=1$ for some $i_0\in[0,s-1]$. We denote the left side of (55) by σ. Changing the order of the summation we get from (54)

$$\sigma = \sum_{\alpha=0}^{m+1} \sum_{h_{-2}\in C(p_2^{m+1})} \sum_{\substack{\mathbf{h}\in C_{2s}(p_2^{m+1})\\ (h_s,\ldots,h_{2s-1},p_2^{m+1})=p_2^{\alpha}}} r^{-1}(h_{-2},p_2^{m+1}) \times r^{-1}(\mathbf{h},p_2^{m+1})W(h_{-2},\mathbf{h}), \tag{56}$$

where

$$W(h_{-2},\mathbf{h}) = \frac{1}{(\#\Delta_{m+1}^*(p_1p_2))^{2t_{m+1}}} \sum_{\mathbf{c}^{(m+1)}\in(\Delta_{m+1}^*(p_1p_2))^{2t_{m+1}}} p_2^{m+1} \times \delta_{p_2^{m+1}}\Big(h_{-2} + \sum_{j=0}^{s-1}\Big(q^ih_jb_{\mu+j,1}^{(m)}p_2t_{m+1}/t_m + h_{s+j}c_{\mu+j,0}^{(m+1)}\Big)\Big).$$

It is easy to see that

$$W(h_{-2},\mathbf{h}) = \frac{1}{(\#\Delta_{m+1}^*(p_1p_2))^{s}} \sum_{(c_0,\ldots,c_{s-1})\in(\Delta_{m+1}^*(p_1p_2))^{s}} p_2^{m+1} \times \delta_{p_2^{m+1}}\Big(f + \sum_{j=0}^{s-1} h_{s+j}c_j\Big),$$

where

$$f = h_{-2} + \sum_{j=0}^{s-1} q^ih_jb_{\mu+j,1}^{(m)}p_2t_{m+1}/t_m.$$

Hence

$$W(h_{-2},\mathbf{h}) = \frac{1}{(\#\Delta_{m+1}^*(p_2))^{s}} \sum_{(c_0,\ldots,c_{s-1})\in(\Delta_{m+1}^*(p_2))^{s}} p_2^{m+1} \times \delta_{p_2^{m+1}}\Big(f + \sum_{j=0}^{s-1} p_2^{\alpha}h_{s+j}^{'}c_j\Big). \tag{57}$$

It is evident that if $f \not\equiv 0 \pmod{p_2^{\alpha}}$, then $W(h_{-2},\mathbf{h}) = 0$.
Let $f \equiv 0 \pmod{p_2^{\alpha}}$, $f = f^{'} p_2^{\alpha}$. Bearing in mind that $(h^{'}_{s+i_0}, p_2) = 1$, we obtain

$$\sum_{c_{i_0}\in[0,p_2^{m+1})} \delta_{p_2^{m+1-\alpha}} \Big(f^{'} + \sum_{i\in[0,s-1],\ i\neq i_0} h^{'}_{s+i} c_i + h^{'}_{s+i_0} c_{i_0} \Big) = p_2^{\alpha}.$$

Equation (57) shows that

$$\begin{aligned} W(h_{-2},\mathbf{h}) &\leq \max_{\mathbf{c}\in(\Delta^*_{m+1}(p_2))^s} \sum_{c_{i_0}\in[0,p_2^{m+1})} (p_2/\varphi(p_2)) \delta_{p_2^{m+1-\alpha}} \Big(f^{'} + \sum_{j=0}^{s-1} h^{'}_{s+j} c_j \Big) \\ &\leq (p_2/\varphi(p_2)) p_2^{\alpha} \delta_{p_2^{\alpha}}(f). \end{aligned}$$

Then equation (56) implies that

$$\begin{aligned} \sigma \leq \sum_{\alpha=0}^{m+1} \sum_{h_{-2}\in C(p_2^{m+1})} \sum_{\substack{\mathbf{h}\in C_{2s}(p_2^{m+1}) \\ (h_s,\dots,h_{2s-1},p_2^{m+1})=p_2^{\alpha}}} & r^{-1}(h_{-2},p_2^{m+1}) r^{-1}(\mathbf{h},p_2^{m+1}) \\ & \times p_2^{\alpha+1} \delta_{p_2^{\alpha}} (h_{-2} + \sum_{j=0}^{s-1} q^i h_j b^{(m)}_{\mu+j,1} p_2 t_{m+1}/t_m) \ . \end{aligned}$$

Now from (38) we obtain the assertion of the lemma. □

Corollary 5. *Let* $\mathbf{b}^{(m)} \in \big(\Delta^*_m(p_1p_2)\big)^{2t_m}$, $\mathbf{b}^{(m)} \notin \Omega_{m,3}$,

$$(58) \quad F_{m+1,1} = \Big\{ \mathbf{b}^{(m+1)} \in (\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}} \mid \sum_{s=1}^{t_m} \sum_{\mu=0}^{t_{m+1}-1} \sum_{i=0}^{k_m-1} \times \frac{B_1(i,\mu,s,\mathbf{b}^{(m)},\mathbf{b}^{(m+1)})}{96K_1(s)p_2t^2_{m+1}k_m(m+1)^{2s+2}} > 1 \Big\} \ .$$

Then

$$(59) \qquad \#F_{m+1,1} \leq \frac{1}{4} \big(\Delta^*_{m+1}(p_1p_2)\big)^{2t_{m+1}} \ .$$

Proof. Changing the order of the summation, from Lemma 11 we have

$$\begin{aligned}(60)\quad & \frac{1}{(\#\Delta_m^*(p_1p_2))^{2t_{m+1}}} \sum_{\mathbf{c}^{(m+1)}\in(\Delta_{m+1}^*(p_1p_2))^{2t_{m+1}}} \sum_{s=1}^{t_m} \sum_{\mu\in[0,t_{m+1})} \sum_{i\in[0,k_m-1]} \\ & \quad \times \frac{B_1(i,\mu,s,\mathbf{b}^{(m)},\mathbf{c}^{(m+1)})}{96K_1(s)p_2t_{m+1}^2k_m(m+1)^{2s+2}} \\ & \le \sum_{s=1}^{t_m} p_2 \sum_{\alpha\in[0,m+1]} \sum_{\mu\in[0,t_{m+1})} \sum_{i\in[0,k_m-1]} \frac{A_1(\alpha,i,m,s,b_{\mu,1}^{(m)},\dots,b_{\mu+s-1,1}^{(m)})}{96K_1(s)p_2t_{m+1}^2k_m(m+1)^{2s+2}}.\end{aligned}$$

We denote the right side of (60) by σ. Bearing in mind (8) and that $b_{\mu+t_m,1}^{(m)} = b_{\mu,1}^{(m)}, \quad \mu = 0,1,\dots,$ we find that

$$\begin{aligned}\sigma \le \frac{m+2}{8(m+1)} \sum_{s=1}^{t_m} \sum_{\alpha\in[0,m+1]} \sum_{\mu\in[0,t_m)} \sum_{i\in[0,k_m-1]} \\ \times \frac{A_1(\alpha,i,m,s,b_{\mu,1}^{(m)},\dots,b_{\mu+s-1,1}^{(m)})}{12K_1(s)p_2t_m^2k_m(m+2)(m+1)^{2s+1}}.\end{aligned}$$

Taking into account (49), (52), and that $\mathbf{b}^{(m)} \notin \Omega_{m,3}$, we deduce that $\sigma \le (m+2)/8(m+1) \le 1/4$.
Now, from (58) and (60), we obtain the desired result. □

Let x_2,x_3,x_5,x_6,y_2,y_4 be integers, $y_2 \in (t_m - s, t_m)$,

$$\begin{aligned}(61)\quad & B_2(x_2,x_3,x_5,x_6,y_2,y_4,s,\mathbf{b}^{(m)},\mathbf{c}^{(m+1)}) = \sum_{h_{-1}\in C(p_1^{m+1})} \sum_{\mathbf{h}\in C_{2s}(p_1^{m+1})} \\ & \quad \times r^{-1}(h_{-1},p_1^{m+1}) r^{-1}(\mathbf{h},p_1^{m+1}) p_1^{m+1} \\ & \times \delta_{p_1^{m+1}}\Big(h_{-1} + p_1k_{m+1}/k_m \sum_{0\le j<t_m-y_2} (q^{x_3}h_jb_{y_2+j,x_2}^{(m)} + h_{s+j}b_{y_2+j,x_2+1}^{(m)}) \\ & \quad + \sum_{t_m-y_2\le j<s} (q^{x_6}h_jc_{y_4+j,x_5}^{(m+1)} + h_{s+j}c_{y_4+j,x_5+1}^{(m+1)})\Big).\end{aligned}$$

Lemma 12. *Let* $s \leq t_m$, $y_2 \in (t_m - s, t_m)$, $\mathbf{b}^{(m)} \notin \Omega_{m,3}$. *Then*

$$\begin{aligned}(62)\quad & \frac{1}{(\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}} \sum_{\mathbf{c}^{(m+1)} \in (\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}} \\ & \qquad B_2(x_2, x_3, x_5, x_6, y_2, y_4, s, \mathbf{b}^{(m)}, \mathbf{c}^{(m+1)}) \\ & \leq p_1 \sum_{\alpha=0}^{m+1} A_2\big(\alpha, x_3, m, t_m - y_2, b^{(m)}_{y_2,x_2}, \dots, b^{(m)}_{y_2+s-1,x_2}, b^{(m)}_{y_2,x_2+1}, \\ & \qquad \dots, b^{(m)}_{y_2+s-1,x_2+1}\big) \ .\end{aligned}$$

Proof. Let $(h_{t_m-y_2}, \dots, h_{s-1}, h_{t_m-y_2+s}, \dots, h_{2s-1}, p_1^{m+1}) = p_1^\alpha$, and let $h_i = h_i^{'} p_1^\alpha$, $i \in [t_m - y_2, s) \cup [t_m - y_2 + s, 2s)$.
Then $(h_{j_0}, p_1) = 1$ for some $j_0 = j_1 + sj_2$, with $j_1 \in [t_m - y_2, s)$, and $j_2 \in \{0, 1\}$.
We denote the left side of (62) by σ. Changing the order of the summation, we obtain from (61) that

$$\begin{aligned}(63)\quad \sigma = \sum_{\alpha=0}^{m+1} \sum_{h_{-1} \in C(p_1^{m+1})} & \sum_{\substack{\mathbf{h} \in C_{2s}(p_1^{m+1}), \\ (h_{t_m-y_2}, \dots, h_{s-1}, h_{t_m-y_2+s}, \dots, h_{2s-1}, p_1^{m+1}) = p_1^\alpha}} \\ & \times r^{-1}(h_{-1}, p_1^{m+1}) r^{-1}(\mathbf{h}, p_1^{m+1}) W(h_{-1}, \mathbf{h}),\end{aligned}$$

where

$$\begin{aligned} W(h_{-1}, \mathbf{h}) = & \frac{1}{(\#\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}} \sum_{\mathbf{c}^{(m+1)} \in (\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}} p_1^{m+1} \\ \times \delta_{p_1^{m+1}} \Big(& h_{-1} + p_1 k_{m+1}/k_m \sum_{0 \leq j < t_m - y_2} (q^{x_3} h_j b^{(m)}_{y_2+j,x_2} + h_{s+j} b^{(m)}_{y_2+j,x_2+1}) \\ & + \sum_{t_m - y_2 \leq j < s} (q^{x_6} h_j c^{(m+1)}_{y_4+j,x_5} + h_{s+j} c^{(m+1)}_{y_4+j,x_5+1}) \Big).\end{aligned}$$

We now obtain

$$\begin{aligned} W(h_{-1}, \mathbf{h}) \leq & \max_{\mathbf{c}^{(m+1)} \in (\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}} \frac{1}{\#\Delta^*_{m+1}(p_1)} \sum_{c_{y_4+j_1, x_5+j_2} \in \Delta^*_{m+1}(p_1)} \\ & \times p_1^{m+1} \delta_{p_1^{m+1}} \Big(f + p_1^\alpha \sum_{t_m - y_2 \leq j < s} (q^{x_6} h_j^{'} c_{y_4+j,x_5} + h_{s+j}^{'} c_{y_4+j,x_5+1}) \Big) \ ,\end{aligned}$$

where

$$f = h_{-1} + p_1 k_{m+1}/k_m \sum_{0 \leq j < t_m - y_2} (q^{x_3} h_j b^{(m)}_{y_2+j,x_2} + h_{s+j} b^{(m)}_{y_2+j,x_2+1}) \ .$$

It is easy to see that if $f \not\equiv 0 \mod (p_1^\alpha)$, then $W(h_{-1}, \mathbf{h}) = 0$. Let $f \equiv 0 \mod (p_1^\alpha)$, $f = f' p_1^\alpha$. Bearing in mind that $(q^{x_6(1-j_2)} h'_{j_2 s+j_1}, p_1) = 1$, we have

$$\sum_{c_{y_4+j_1,x_5+j_2} \in [0,p_1^{m+1})} \delta_{p_1^{m+1-\alpha}} \Big(f' + q^{x_6(1-j_2)} h'_{j_2 s+j_1} c_{y_4+j_1,x_5+j_2} \\ + \sum_{\substack{(\nu,j)\in\{0,1\}\times[t_m-y_2,s) \\ (\nu,j)\neq(j_2,j_1)}} q^{x_6(1-\nu)} h'_{\nu s+j} c_{y_4+j,x_5+\nu} \Big) = p_1^\alpha .$$

Hence

$$W(h_{-1}, \mathbf{h}) \le (p_1/\varphi(p_1)) p_1^\alpha \delta_{p_1^\alpha}(f)$$

$$\le p_1^{\alpha+1} \delta_{p_1^\alpha} \Big(h_{-1} + p_1 k_{m+1}/k_m \sum_{0\le j<t_m-y_2} (q^{x_3} h_j b^{(m)}_{y_2+j,x_2} + h_{s+j} b^{(m)}_{y_2+j,x_2+1}) \Big) .$$

Now, from (63) and (39), we obtain the assertion of the lemma. $\square$

Corollary 6. *Let* $\mathbf{b}^{(m)} \in (\Delta_m^*(p_1p_2))^{2t_m}$, $\mathbf{b}^{(m)} \notin \Omega_{m,3}$, $x_2 = 2\{([x_6/k_m] + x_5 k_{m+1}/k_m)/2\}$, $x_3 = k_m\{x_6/k_m\}$, $y_4 = t_{m+1}\{(t_m(p_1^m - 1) + y_2)/t_{m+1}\}$, $m = 1, 2, \dots,$

$$F_{m+1,2} = \Big\{ \mathbf{b}^{(m+1)} \in \big(\Delta_{m+1}^*(p_1p_2)\big)^{2t_{m+1}} \,\Big|\, \sum_{s=1}^{t_m} \sum_{x_5=0}^{1} \sum_{x_6=0}^{k_{m+1}-1} \\ \times \sum_{y_2\in(t_m-s,t_m)} \frac{B_2(x_2,x_3,x_5,x_6,y_2,y_4,s,\mathbf{b}^{(m)},\mathbf{b}^{(m+1)})}{192K_1(s)p_1t_m^2k_{m+1}(m+1)^{2s+2}} > 1 \Big\}. \tag{64}$$

Then

$$\#F_{m+1,2} \le \frac{1}{4}\big(\#\Delta_{m+1}^*(p_1p_2)\big)^{2t_{m+1}} . \tag{65}$$

Proof. Changing the order of the summation, from Lemma 12 we have

$$\frac{1}{(\Delta_{m+1}^*(p_1p_2))^{2t_{m+1}}} \sum_{\mathbf{c}^{(m+1)}\in(\Delta_{m+1}^*(p_1p_2))^{2t_{m+1}}} \sum_{x_5=0}^{1} \sum_{x_6=0}^{k_{m+1}-1} \\ \times \sum_{y_2\in(t_m-s,t_m)} B_2(x_2,x_3,x_5,x_6,y_2,y_4,s,\mathbf{b}^{(m)},\mathbf{c}^{(m+1)}) \\ \le p_1 \sum_{\alpha=0}^{m+1} \sum_{x_5=0}^{1} \sum_{x_6=0}^{k_{m+1}-1} \sum_{y_2\in(t_m-s,t_m)} \\ \times A_2(\alpha, x_3, m, t_m - y_2, b^{(m)}_{y_2,x_2}, \dots, b^{(m)}_{y_2+s-1,x_2}, b^{(m)}_{y_2,x_2+1}, \dots, b^{(m)}_{y_2+s-1,x_2+1}) . \tag{66}$$

According to (7), $k_{m+1}/k_m = p_1^{[\log_{p_1}(m+1)]-[\log_{p_1} m]}$.

Let $p_1 = 2$ and $k_{m+1}/k_m > 1$. Then $x_2 = 2\{[x_6/k_m]/2\}$, $x_3 = k_m\{x_6/k_m\}$. If x_6 passes the set $\{0,1,\dots,k_{m+1}-1\}$, then (x_2,x_3) passes $k_{m+1}/(2k_m)$ times the set $\{0,1\}\times\{0,1,\dots,k_m-1\}$.
Now let $p_1 \geq 3$ or $p_1 = 2$ and $k_{m+1}/k_m = 1$. We find that $x_2 = 2\{([x_6/k_m]+x_5)/2\}$, $x_3 = k_m\{x_6/k_m\}$. If (x_5,x_6) passes the set $\{0,1\}\times\{0,1,\dots,k_{m+1}-1\}$, then (x_2,x_3) passes k_{m+1}/k_m times the set $\{0,1\}\times\{0,1,\dots,k_m-1\}$. Therefore, for both of the cases, if (x_5,x_6) passes the set $\{0,1\}\times\{0,1,\dots,k_{m+1}-1\}$, then (x_2,x_3) passes k_{m+1}/k_m times the set $\{0,1\}\times\{0,1,\dots,k_m-1\}$. Hence

$$\frac{1}{(\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}}\sum_{\mathbf{c}^{(m+1)}\in(\Delta^*_{m+1}(p_1p_2))^{2t_{m+1}}}\sum_{s=1}^{t_m}\sum_{x_5=0}^{1}\sum_{x_6=0}^{k_{m+1}-1}\tag{67}$$
$$\times\sum_{y_2\in(t_m-s,t_m)}\frac{B_2(x_2,x_3,x_5,x_6,y_2,y_4,s,\mathbf{b}^{(m)},\mathbf{c}^{(m+1)})}{192K_1(s)p_1t_m^2k_{m+1}(m+1)^{2s+2}}$$
$$\leq\frac{(m+2)}{8(m+1)}\sum_{s=1}^{t_m}\sum_{\alpha=0}^{m+1}\sum_{x_2=0}^{1}\sum_{x_3=0}^{k_m-1}\sum_{y_2\in(t_m-s,t_m)}$$
$$\times\frac{p_1A_2(\alpha,x_3,m,t_m-y_2,b^{(m)}_{y_2,x_2},\dots,b^{(m)}_{y_2+s-1,x_2},b^{(m)}_{y_2,x_2+1},\dots,b^{(m)}_{y_2+s-1,x_2+1})}{24K_1(s)p_1t_m^2k_m(m+2)(m+1)^{2s+1}}\,.$$

Bearing in mind (50), (52), and that $\mathbf{b}^{(m)}\notin\Omega_{m,3}$, we deduce that the left side of (67) is less than $(m+2)/(8(m+1))\leq 1/4$. Then from (64) we obtain the desired result. $\square$

Now we choose vectors $\mathbf{a}^{(m)}$ $(m=1,2,\dots)$ for the construction in (9) in the following way:
According to (53),

$$\#\Big(\big(\Delta^*_m(p_1p_2)\big)^{2t_m}\backslash\Omega_{m,3}\Big)>0\ .$$

We take $\mathbf{a}^{(1)}$ arbitrarily from the set $\big(\Delta^*_1(p_1p_2)\big)^{2t_1}\backslash\Omega_{1,3}$.
Taking into account (53),(59), and (65), we obtain that the set

$$F_{m+1,3}=\big(\Delta^*_{m+1}(p_1p_2)\big)^{2t_{m+1}}\setminus\Big(\Omega_{m+1,3}\cup F_{m+1,1}\cup F_{m+1,2}\Big)\ .\tag{68}$$

is not empty. Let $\mathbf{a}^{(1)},\dots,\mathbf{a}^{(m)}$ be chosen. Then we choose $\mathbf{a}^{(m+1)}$ arbitrarily so that

$$\mathbf{a}^{(m+1)}\in F_{m+1,3}\ .\tag{69}$$

The sequence of vectors $\mathbf{a}^{(m)}$ $(m=1,2,\dots)$ is constructed inductively.
Next we fix $s\geq 1$, and we consider integers m such that $t_m\geq s$.

Main Lemma. *With the notation defined above, we have:*

$$\sum_{0\le x_2\le 1}\sum_{0\le x_3<k_m}\sum_{0\le y_2<t_m} B(x_2,x_3,y_2,s,\mathbf{a}^{(m)}) = O(m^{2s+3}\log^2 m)\ , \tag{70}$$

$$\sum_{0\le x_3<k_m}\sum_{0\le y_4<t_{m+1}} B_1(x_3,y_4,s,\mathbf{a}^{(m)},\mathbf{a}^{(m+1)}) = O(m^{2s+3}\log^2 m)\ , \tag{71}$$

$$\sum_{0\le x_5\le 1}\sum_{0\le x_6<k_{m+1}}\sum_{y_2\in(t_m-s,t_m)} B_2(x_2,x_3,x_5,x_6,y_2,y_4,s,\mathbf{a}^{(m)},\mathbf{a}^{(m+1)}) = O(m^{2s+3}\log^2 m)\ , \tag{72}$$

where $m=1,2,\dots$, $x_2 = 2\{([x_6/k_m]+x_5k_{m+1}/k_m)/2\}$, $x_3 = k_m\{x_6/k_m\}$, $y_4 = t_{m+1}\{(t_m(p_1^m-1)+y_2)/t_{m+1}\}$, and $s\le t_m$.

Proof. The proof follows from (7), (8), (36), (52), (58), (64), (68) and (69). □

3. Proof of Theorem 1.

In this section, the integer s is fixed. For $m=1,2,\dots$, let

$$V_{m,1} = [0,2k_mp_1^m)\times[t_{m-1}p_2^{m-1},t_mp_2^m),$$
$$V_{m,2} = [2k_{m-1}p_1^{m-1},2k_mp_1^m)\times[1,t_mp_2^m),$$

$$V_m = V_{m,1}\cup V_{m,2},\qquad G_{M,N} = [0,M)\times[1,N], \tag{73}$$

$$G_1^{(m)} = G_{M,N}\cap V_{m,1},\ G_2^{(m)} = G_{M,N}\cap(V_{m,2}\setminus V_{m,1}), \tag{74}$$

and let

$$D_E = \#ED^{(s)}\Big(\big(\{\alpha_yq^x\},\dots,\{\alpha_{y+s-1}q^x\}\big)_{(x,y)\in E}\Big)\ . \tag{75}$$

It is easy to see that $G_i^{(m)}$ is rectangular domain ($i=1,2$) and

$$D_{G_{M,N}} \le \sum_{m\ge1}\sum_{i=1,2} D_{G_i^{(m)}}\ .$$

Thus, to compute $D_{G_{M,N}}$, it is sufficient to find the estimate of D_F, where F is an arbitrary rectangular domain in V_m, $m=1,2,\dots$. According to (9), the analytic expression of $\{\alpha_yq^x\}$ depends on the position of (x,y) in V_m. We will consider three possibilities for the position of F in V_m: the middle, the right bourne and the upper bourne. Next we will consider sub-domains $F_{i,j} = \{(x,y)\in F \mid x\equiv i \bmod (2k_m),\quad y\equiv j \bmod (t_m)\}$ of rectangular domain F, and we will compute the discrepancy on the sub-domain $F_{i,j}$ for all $(i,j)\in[0,2k_m)\times[0,t_m)$, with $m=1,2,\dots$.

First we obtain a simple expression for $\{\alpha_y q^x\}$, where (x,y) belongs to a middle domain of V_m:

Lemma 13. *Let $(x,y) \in V_m$; $x = 2k_m x_1 + k_m x_2 + x_3, y = t_m y_1 + y_2$, with $x_2 \in \{0,1\}$, $x_3 \in \{0,\dots,k_m-1\}$, $y_1 \in \{0,\dots,p_2^m-1\}$, $y_2 \in \{0,\dots,t_m-1\}$, $x_1 + x_2 \le p_1^m - 1$, and $t_m > s$. Then*

$$(76)\quad \{\alpha_y q^x\}_{2k_m - x_3} = \Big\{ \frac{q^{x_3} a^{(m)}_{y_2,x_2}(x_1 p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m - x_3} + \frac{1}{q^{k_m - x_3}} \Big\{ \frac{a^{(m)}_{y_2,x_2+1}((x_1+x_2)p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m}.$$

Proof. From (8) we obtain

$$(77)\qquad j \in [t_{r(j)-1} p_1^{r(j)-1}, t_{r(j)} p_1^{r(j)}), \qquad j = 1,2,\dots\ .$$

Let $(x,y) \in [0, 2k_m p_1^m) \times [t_{m-1} p_2^{m-1}, t_m p_2^m) = V_{m,1}$.
Then equation (8) implies that $r(y) = m$.
According to (9), we find that

$$(78)\quad \alpha_y q^x = q^x \sum_{n=0}^{p_1^m - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_m(2n+\nu)}} \Big\{ \frac{a^{(m)}_{y_2,\nu}(p_2^m n + p_1^m y_1)}{p_1^m p_2^m} \Big\}_{k_m} + q^x \sum_{i=m+1}^{\infty} \sum_{n=p_1^{i-1} k_{i-1}/k_i}^{p_1^i - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_i(2n+\nu)}} \Big\{ \frac{a^{(i)}_{y_2(i),\nu}(p_2^i n + p_1^i y_1(i))}{p_1^i p_2^i} \Big\}_{k_i},$$

where $y = t_m y_1 + y_2$; $y_2(i) \in \{0,1,\dots,t_i - 1\}$, $y_2(i) \equiv y (\mathrm{mod}\ t_i)$, and $y_1(i) = (y - y_2(i))/t_i$, $i = m+1, \dots$.
Bearing in mind that $k_i(2n+\nu) - x \ge 2k_{i-1} p_1^i - x \ge 2k_m p_1^m - x$ for $i \ge m+1$ and $n \ge p_1^{i-1} k_{i-1}/k_i$, we obtain that

$$\{\alpha_y q^x\}_{2k_m - x_3} = \Big\{ q^{2k_m x_1 + k_m x_2 + x_3} \sum_{n=0}^{p_1^m - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_m(2n+\nu)}} \Big\{ \frac{a^{(m)}_{y_2,\nu}(p_2^m n + p_1^m y_1)}{p_1^m p_2^m} \Big\}_{k_m} \Big\}_{2k_m - x_3}.$$

Hence

$$(79)\quad \{\alpha_y q^x\}_{2k_m - x_3} = \Big\{ q^{x_3} \Big\{ \frac{a^{(m)}_{y_2,x_2}(p_2^m x_1 + p_1^m y_1)}{p_1^m p_2^m} \Big\}_{k_m} + q^{x_3} \sum_{\substack{n=x_1 \text{ for } x_2=0 \\ n=x_1+1 \text{ for } x_2=1}} \times \sum_{\substack{\nu=1, \text{ for } x_2=0 \\ \nu=0, \text{ for } x_2=1}} \frac{1}{q^{k_m(2n+\nu-2x_1-x_2)}} \Big\{ \frac{a^{(m)}_{y_2,\nu}(p_2^m n + p_1^m y_1)}{p_1^m p_2^m} \Big\}_{k_m} \Big\}_{2k_m - x_3}$$

$$= \Big\{ \Big\{ \frac{q^{x_3} a^{(m)}_{y_2,x_2}(x_1 p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m - x_3} + \frac{1}{q^{k_m - x_3}} \Big\{ \frac{a^{(m)}_{y_2,x_2+1}((x_1+x_2) p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m} \Big\}_{2k_m - x_3}$$

$$= \Big\{ \frac{q^{x_3} a^{(m)}_{y_2,x_2}(x_1 p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m - x_3} + \frac{1}{q^{k_m - x_3}} \Big\{ \frac{a^{(m)}_{y_2,x_2+1}((x_1+x_2) p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m}.$$

Now let $(x,y) \in [2k_{m-1} p_1^{m-1}, 2k_m p_1^m) \times [1, t_{m-1} p_2^{m-1}) = V_{m,2} \setminus V_{m,1}$. Then equation (77) and the conditions of the lemma show that $r(y) < m$. From (9) we get

$$(80) \quad \alpha_y q^x = f_{x,y} + q^x \sum_{n=p_1^{m-1} k_{m-1}/k_m}^{p_1^m - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_m(2n+\nu)}} \Big\{ \frac{a^{(m)}_{y_2,\nu}(p_2^m n + p_1^m y_1)}{p_1^m p_2^m} \Big\}_{k_m} + q^x \sum_{i=m+1}^{\infty} \sum_{n=p_1^{i-1} k_{i-1}/k_i}^{p_1^i - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_i(2n+\nu)}} \Big\{ \frac{a^{(i)}_{y_2(i),\nu}(p_2^i n + p_1^i y_1(i))}{p_1^i p_2^i} \Big\}_{k_i},$$

where $f_{x,y} \geq 0$ is an integer. It is easy to see that, here,

$$x \geq 2k_{m-1} p_1^{m-1} = \min_{n \geq p_1^{m-1} k_{m-1}/k_m, \quad \nu = 0,1} (k_m(2n+\nu)).$$

Hence we can repeat the calculations (77) - (79). Thus we obtain the desired result. □

Define $G_1 =$

$$(81) \quad G_1(m, x_2, x_3, y_2, K_1, K, L_1, L) = \Big\{ (x,y) \mid x = 2k_m x_1 + k_m x_2 + x_3, \; y = t_m y_1 + y_2, x_1 \in [K_1, K_1 + K), \quad y_1 \in [L_1, L_1 + L] \Big\},$$

with $x_2 \in \{0,1\}$, $x_3 \in \{0, \dots, k_m - 1\}$ and $y_2 \in \{0, \dots, t_m - 1\}$.

Lemma 14. *Let $G_1 \subset V_m$; $K, K_1, L, L_1 \geq 0$ be integers; $K_1 + K + x_2 \leq p_1^m$; $t_m(L_1 + L) + y_2 \leq t_m p_2^m - s$; $t_m > s$. Then*

$$(82) \quad \# G_1 D^{(s)} \Big(\big(\{\alpha_y q^x\}, \dots, \{\alpha_{y+s-1} q^x\} \big)_{(x,y) \in G_1} \Big) \leq 2^s B(x_2, x_3, y_2, s, \mathbf{a}^{(m)}) + s 2^{s+2}.$$

Proof. Let $(x,y) \in G_1$,

$$\theta_m(y_2,i) = \begin{cases} 1 & \text{if} \quad y_2+i \geq t_m \\ 0 & \text{otherwise,} \end{cases} \tag{83}$$

and let

$$a^{(m)}_{i+t_m,v+2} = a^{(m)}_{i,v}, \quad i,v = 0,1,2,\dots \ .$$

(We continue periodically the coordinates of vector $\mathbf{a}^{(m)}$.) From Lemma 13 we have that

$$\{\alpha_{y+i}q^x\}_{2k_m-x_3} = \Big\{ \frac{q^{x_3}a^{(m)}_{y_2+i,x_2}(x_1p_2^m + (y_1+\theta_m(y_2,i))p_1^m)}{p_1^m p_2^m} \Big\}_{k_m-x_3} \tag{84}$$
$$+ \frac{1}{q^{k_m-x_3}} \Big\{ \frac{a^{(m)}_{y_2+i,x_2+1}((x_1+x_2)p_2^m + (y_1+\theta_m(y_2,i))p_1^m)}{p_1^m p_2^m} \Big\}_{k_m},$$
$$i = 0,1,\dots,s-1.$$

We denote the left side of (82) by σ. Applying Lemma 5 with $k = k^{(i)} = 2k_m - x_3$ and $q_i = q$ $(i = 1,\dots,s)$, we get $\sigma \leq$

$$\#G_1\Big(\frac{s}{q^{2k_m-x_3}} + D^{(s)}\Big(\big(\{\alpha_y q^x\}_{2k_m-x_3},\dots,\{\alpha_{y+s-1}q^x\}_{2k_m-x_3}\big)_{(x,y)\in G_1}\Big)\Big).$$

Using Lemma 6 with $k = k^{(i)} = k_m - x_3$, $\ell = \ell^{(i)} = k_m$ and $q = q_i$ $(i = 1,\dots,s)$, we obtain from (84)

$$\sigma \leq \#G_1\Big(\frac{2s}{q^{2k_m-x_3}} \tag{85}$$
$$+ 2^s D^{(2s)}\Big(\Big(\Big(\Big\{\frac{q^{x_3}a^{(m)}_{y_2+i,x_2}(x_1p_2^m + \big(y_1+\theta_m(y_2,i)\big)p_1^m)}{p_1^m p_2^m}\Big\},$$
$$\Big\{\frac{a^{(m)}_{y_2+i,x_2+1}\big((x_1+x_2)p_2^m + (y_1+\theta_m(y_2,i))p_1^m\big)}{p_1^m p_2^m}\Big\}\Big)_{0\leq i\leq s-1}\Big)_{(x,y)\in G_1}\Big)\Big) .$$

Then Corollary 3 (with $M = K, M_1 = K_1, N = L+1, N_1 = L_1, d = 2s$, and $r = m$) implies that

$$\sigma \leq \#G_1\Big(\frac{2s}{q^{2k_m-x_3}} + \frac{2s2^s}{p_1^m p_2^m}\Big) + 2^s \sum_{h_{-1}\in C(p_1^m)} \sum_{h_{-2}\in C(p_2^m)} \tag{86}$$
$$\times \sum_{\mathbf{h}\in C_{2s}(p_1^m p_2^m)} r^{-1}(h_{-1},p_1^m) r^{-1}(h_{-2},p_2^m) r^{-1}(\mathbf{h},p_1^m p_2^m)|\sigma_1(\mathbf{h})|,$$

where $\#G_1 = K(L+1) \le p_1^m p_2^m$ and

$$\sigma_1(\mathbf{h}) = \sum_{x_1=0}^{p_1^m-1} \sum_{y_1=0}^{p_2^m-1} e\Big(\frac{h_{-1}x_1}{p_1^m} + \frac{h_{-2}y_1}{p_2^m} + \sum_{i=0}^{s-1} \Big(q^{x_3} h_i a^{(m)}_{y_2+i,x_2}\big(x_1 p_2^m + \big(y_1+\theta_m(y_2,i)\big)p_1^m\big) + h_{s+i} a^{(m)}_{y_2+i,x_2+1}\Big((x_1+x_2)p_2^m + \big(y_1+\theta_m(y_2,i)\big)p_1^m\Big)\Big)/p_1^m p_2^m\Big) .$$

It is easy to see that

$$h_{-1}x_1p_2^m + h_{-2}y_1p_1^m \equiv (x_1p_2^m + y_1p_1^m)(h_{-1}p_2^m M_2 + h_{-2}p_1^m M_1)(\mathrm{mod} p_1^m p_2^m) ,$$

where $M_1 p_1^m \equiv 1(\mathrm{mod} p_2^m)$ and $M_2 p_2^m \equiv 1(\mathrm{mod} p_1^m)$. Taking into account that $x_1p_2^m + y_1p_1^m$ passes the complete residue system $(\mathrm{mod} p_1^m p_2^m)$ and using Lemma 1, we have that

$$|\sigma_1(h)| = p_1^m p_2^m \delta_{p_1^m p_2^m}\Big(h_{-1}p_2^m M_2 + h_{-2}p_1^m M_1 + \sum_{i=0}^{s-1}(q^{x_3}h_i a^{(m)}_{y_2+i,x_2} + h_{s+i}a^{(m)}_{y_2+i,x_2+1})\Big).$$

Now from (24),(35) and (86) we find that

$$\sigma \le \big(2sq^{x_3-2k_m} + s2^{s+1}p_1^{-m}p_2^{-m}\big)\#G_1 + 2^s B(x_2,x_3,y_2,s,\mathbf{a}^{(m)}) .$$

Applying (7), (81) and the condition of the lemma, we obtain the desired result. □

We can now use Lemma 14 to compute the discrepancy of the considered double sequence in the rectangular domain $E \subset V_m$.

Lemma 15. *Let $K_1, K_6, L_1, L_6 \ge 0$ be integers, $E = [2k_mK_1, K_6) \times [t_mL_1, L_6] \subset V_m$, $2k_mK_1 < K_6 \le k_m(2p_1^m - 1)$, $t_mL_1 \le L_6 \le t_mp_2^m - s$; $t_m > s$. Then*

$$D_E = O(m^{2s+3}\log^2 m) , \tag{87}$$

where the constant implied by O only depends on s.

Proof. We consider following rectangular domains:

$$\begin{aligned} E_1 &= [2k_mK_1, K_5) \times [t_mL_1, L_5), \quad E_2 = [2k_mK_1, K_5) \times [L_5, L_6], \\ E_3 &= [K_5, K_6) \times [L_5, L_6], \quad E_4 = [K_5, K_6) \times [t_mL_1, L_5) , \end{aligned} \tag{88}$$

where $K, K_2, K_3, K_5,\ L, L_2, L_5 \geq 0$ are integers, and

$$
\begin{gathered}
K_2 \in \{0,1\}, K_3 \in [0,k_m);\ K_1, K, L, L_1 \geq 0,\ L_2 \in [0,t_m), \\
K = [K_6/2k_m] - K_1,\ K_5 = 2k_m(K_1+K), \\
K_6 = K_5 + k_m K_2 + K_3\ , \\
L = [L_6/t_m] - L_1,\ L_5 = t_m L_1 + t_m L,\ L_6 = L_5 + L_2.
\end{gathered} \tag{89}
$$

It is evident that

$$E = E_1 \cup E_2 \cup E_3 \cup E_4, \qquad \text{and} \quad E_i \cap E_j = \emptyset \quad \text{for} \quad i \neq j\ .$$

Using (2) and (75), we obtain

$$D_E \leq \sum_{1 \leq i \leq 4} D_{E_i}\ . \tag{90}$$

From (81) we obtain :

$$
\begin{aligned}
E_1 &= \bigcup_{\substack{0 \leq x_2 \leq 1 \\ 0 \leq x_3 < k_m \\ 0 \leq y_2 < t_m}} G_1(m, x_2, x_3, y_2, K_1, K, L_1, L)\ , \\
E_2 &= \bigcup_{\substack{0 \leq x_2 \leq 1 \\ 0 \leq x_3 < k_m \\ 0 \leq y_2 < L_2}} G_1(m, x_2, x_3, y_2, K_1, K, L_1 + L, 0), \\
E_4 &= \bigcup_{\substack{0 \leq x_2 k_m + x_3 < K_2 k_m + K_3 \\ 0 \leq y_2 < t_m}} G_1(m, x_2, x_3, y_2, K_1 + K, 1, L_1, L).
\end{aligned} \tag{91}
$$

We obtain from (89),(88),(7), and (8) that

$$\# E_3 \leq 2 k_m t_m = O(m \log m)\ .$$

Applying (75), (70), and Lemma 14, we get from (7) and (91) that

$$
\begin{aligned}
D_{E_1} \leq & \sum_{0 \leq x_2 \leq 1} \sum_{0 \leq x_3 < k_m} \sum_{0 \leq y_2 < t_m} \# G_1(m, x_2, x_3, y_2, K_1, K, L_1, L) \\
& \times D^{(s)}\Big((\{\alpha_y q^x\}, \ldots, \{\alpha_{y+s-1} q^x\})_{(x,y) \in G_1(m, x_2, x_3, y_2, K_1, K, L_1, L)} \Big) \\
& \leq 2^s \sum_{0 \leq x_2 \leq 1} \sum_{0 \leq x_3 < k_m} \sum_{0 \leq y_2 < t_m} \Big(B(x_2, x_3, y_2, s, \mathbf{a}^{(m)}) + 4s \Big) \\
& = O(m^{2s+3} \log^2 m)\ .
\end{aligned}
$$

Similarly, estimates are valid for the cases of sets E_2 and E_4. Thus

$$\max_{1 \leq i \leq 4} D_{E_i} = O(m^{2s+3} \log^2 m)\ .$$

Now from (90) we obtain the assertion of the lemma. □

Consider the right bourne of V_m. Define

$$G_2 = G_2(m, x_3, y_4, L_2, L_3) = \big\{(x,y) \mid x = k_m(2p_1^m - 1) + x_3, \\ y = t_{m+1}y_3 + y_4, y_3 \in [L_2, L_2 + L_3] \subset [0, p_2^m)\big\} \ . \tag{92}$$

Lemma 16. *Let $G_2 \subset V_m$; $L_2, L_3 \geq 0$ be integers; $t_{m+1}(L_2 + L_3) + y_4 \leq t_m p_2^m - s$; $y_4 \in [0, t_{m+1})$, $x_3 \in [0, k_m)$, $t_m > s$. Then*

$$\#G_2 D^{(s)}\Big(\big(\{\alpha_y q^x\}, \dots, \{\alpha_{y+s-1} q^x\}\big)_{(x,y)\in G_2}\Big) \\ \leq 2^s B_1(x_3, y_4, s, \mathbf{a}^{(m)}, \mathbf{a}^{(m+1)}) + s 2^{s+3}. \tag{93}$$

Proof. Let $(x,y) \in G_2$. Thus $y < t_m p_2^m$, and from (77) we find that $r(y) \leq m$. Similarly to (78) and (80) we have that

$$\{\alpha_y q^x\} = \Big\{ q^x \sum_{n=\delta_m^{(y)} p_1^{m-1} k_{m-1}/k_m}^{p_1^m - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_m(2n+\nu)}} \Big\{ \frac{a_{y_2,\nu}^{(m)}(p_2^m n + p_1^m y_1)}{p_1^m p_2^m} \Big\}_{k_m} \\ + q^x \sum_{i=m+1}^{\infty} \sum_{n=p_1^{i-1} k_{i-1}/k_i}^{p_1^i - 1} \sum_{\nu=0}^{1} \frac{1}{q^{k_i(2n+\nu)}} \Big\{ \frac{a_{y_2(i),\nu}^{(i)}(p_2^i n + p_1^i y_1(i))}{p_1^i p_2^i} \Big\}_{k_i} \Big\}.$$

Hence

$$\{\alpha_y q^{2k_m(p_1^m - 1) + k_m + x_3}\} = \Big\{ q^{x_3} \Big\{ \frac{a_{y_2,1}^{(m)}((p_1^m - 1)p_2^m + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m} \\ + \frac{1}{q^{k_m - x_3}} \Big\{ \frac{a_{y_4,0}^{(m+1)}(n_m p_2^{m+1} + y_3 p_1^{m+1})}{p_1^{m+1} p_2^{m+1}} \Big\}_{k_{m+1}} + \frac{\varepsilon_{x,y}}{q^{k_{m+1} + k_m - x_3}} \Big\},$$

with $y_2 = t_m\{y_4/t_m\}$, $y_1 = y_3 t_{m+1}/t_m + [y_4/t_m]$, $n_m = p_1^m k_m / k_{m+1}$, and $\varepsilon_{x,y} \in [0,1)$.
Therefore

$$\{\alpha_y q^x\}_{k_{m+1} + k_m - x_3} = \Big\{ \frac{q^{x_3} a_{y_2,1}^{(m)}(x_0 + y_1 p_1^m)}{p_1^m p_2^m} \Big\}_{k_m - x_3} \\ + \frac{1}{q^{k_m - x_3}} \Big\{ \frac{a_{y_4,0}^{(m+1)}(n_m p_2^{m+1} + y_3 p_1^{m+1})}{p_1^{m+1} p_2^{m+1}} \Big\}_{k_{m+1}} , \tag{94}$$

where $x = 2k_m(p_1^m - 1) + k_m + x_3$, and $x_0 = (p_1^m - 1)p_2^m$. We have for $i \in [0, s-1]$ that $(x, y+i) \in V_m$, and that $r(y+i) \leq m$. Thus we can apply (94) for the pair $(x, y+i)$. Bearing in mind (83), we obtain

$$(95)\quad \{\alpha_{y+i}q^x\}_{k_{m+1}+k_m-x_3} = \Big\{\frac{q^{x_3}a^{(m)}_{y_4+i,1}(x_0+(y_1+\theta_m(y_2,i))p_1^m)}{p_1^m p_2^m}\Big\}_{k_m-x_3}$$
$$+\frac{1}{q^{k_m-x_3}}\Big\{\frac{a^{(m+1)}_{y_4+i,0}(n_m p_2^{m+1}+(y_3+\theta_{m+1}(y_4,i))p_1^{m+1})}{p_1^{m+1}p_2^{m+1}}\Big\}_{k_{m+1}},$$

with $y_2 = t_m\{y_4/t_m\}$, $y_1 = y_3 t_{m+1}/t_m + [y_4/t_m]$, $x = 2k_m(p_1^m-1)+k_m+x_3$, and $x_0 = (p_1^m-1)p_2^m$. We denote the left side of (93) by σ. Applying Lemma 5 with $k = k^{(i)} = k_{m+1}+k_m-x_3$ and $q_i = q$ $(i=1,\dots,s)$, we obtain from (19)

$$\sigma \le \#G_2\Big(\frac{s}{q^{k_{m+1}+k_m-x_3}}$$
$$+D^{(s)}\Big(\big(\{\alpha_y q^x\}_{k_{m+1}+k_m-x_3},\dots,\{\alpha_{y+s-1}q^x\}_{k_{m+1}+k_m-x_3}\big)_{(x,y)\in G_2}\Big)\Big).$$

Using Lemma 6 with $k = k^{(i)} = k_m - x_3, \ell^{(i)} = k_m$ and $q_i = q$ $(i=1,\dots,s)$, we obtain from (95)

$$\sigma \le \#G_2\Big(\frac{2s}{q^{k_{m+1}+k_m-x_3}}$$
$$+2^s D^{(2s)}\Big(\Big(\Big(\Big\{\frac{q^{x_3}a^{(m)}_{y_4+i,1}(x_0+(y_3t_{m+1}/t_m+[y_4/t_m]}{p_1^m p_2^m}\frac{+\theta_m(y_2,i))p_1^m)}{p_1^m p_2^m}\Big\},$$
$$\Big\{\frac{a^{(m+1)}_{y_4+i,0}(n_m p_2^{m+1}+(y_3+\theta_{m+1}(y_4,i))p_1^{m+1})}{p_1^{m+1}p_2^{m+1}}\Big\}\Big)_{0\le i\le s-1}\Big)_{L_2\le y_3\le L_2+L_3}\Big)\Big).$$

Here we now replace fractions with denominators $p_1^m p_2^m$ and $p_1^{m+1}p_2^{m+1}$ with fractions with denominator p_2^{m+1}, and we again apply Lemma 5 with the parameters $2s$ instead of s, p_2 instead of q_i, and $m+1$ instead of $k^{(i)}$ $(i=1,\dots,2s)$:

$$(96)\quad \sigma \le \#G_2\Big(\frac{2s}{q^{k_{m+1}+k_m-x_3}}+\frac{2s\cdot 2^s}{p_2^{m+1}}$$
$$+2^s D^{(2s)}\Big(\Big(\Big(\Big\{\frac{p_2t_{m+1}/t_m y_3 q^{x_3}a^{(m)}_{y_4+i,1}+d_i}{p_2^{m+1}}\Big\},$$
$$\Big\{\frac{y_3 a^{(m+1)}_{y_4+i,0}+f_i}{p_2^{m+1}}\Big\}\Big)_{0\le i<s}\Big)_{L_2\le y_3\le L_2+L_3}\Big)\Big),$$

where $d_i = [p_2^{m+1}\{q^{x_3}a^{(m)}_{y_4+i,1}(-p_1^{-m}+([y_4/t_m]\}]+\theta_m(y_2,i))p_2$, and $f_i = [p_2^{m+1}\{a^{(m+1)}_{y_4+i,0}n_m p_1^{-m-1}\}]+\theta_{m+1}(y_4,i),\quad i=0,\dots,s-1.$

Then Corollary 2 (with $d = 2s, T = P = p_2^{m+1}, N = \#G_2 = L_3 + 1$) implies that

$$\sigma \leq \#G_2\Big(\frac{2s}{q^{k_{m+1}+k_m-x_3}} + \frac{4s2^s}{p_2^{m+1}}\Big)$$

$$+ 2^s \sum_{h_{-2}\in C(p_2^{m+1})} \sum_{\mathbf{h}\in C_{2s}(p_2^{m+1})} r^{-1}(h_{-2}, p_2^{m+1}) r^{-1}(\mathbf{h}, p_2^{m+1}) \Big| \sum_{y_3=0}^{p_2^{m+1}-1} e\Big(\frac{1}{p_2^{m+1}}$$

$$\times \Big(h_{-2}y_3 + \sum_{i=0}^{s-1}(h_i(p_2 t_{m+1}/t_m y_3 q^{x_3} a_{y_4+i,1}^{(m)} + d_i) + h_{s+i}(y_3 a_{y_4+i,0}^{(m+1)} + f_i))\Big)\Big)\Big| \ .$$

Taking into account (7), (54), (92), Lemma 1, and the conditions of the lemma we obtain :

$$\sigma \leq s2^{s+3} + 2^s \sum_{h_{-2}\in C(p_2^{m+1})} \sum_{\mathbf{h}\in C_{2s}(p_2^{m+1})} r^{-1}(h_{-2}, p_2^{m+1}) r^{-1}(\mathbf{h}, p_2^{m+1})$$

$$\times p_2^{m+1} \delta_{p_2^{m+1}} \Big(h_{-2} + \sum_{i=0}^{s-1}(p_2 t_{m+1}/t_m q^{x_3} h_i a_{y_4+i,1}^{(m)} + h_{s+i} a_{y_4+i,0}^{(m+1)})\Big)$$

$$\leq s2^{s+3} + 2^s B_1(x_3, y_4, s, \mathbf{a}^{(m)}, \mathbf{a}^{(m+1)}) \ .$$

Thus we obtain the assertion of the lemma. □

Corollary 7. *Let $E = [k_m(2p_1^m - 1), K_6) \times [t_m L_1, L_6] \subset V_m$; $0 \leq t_m L_1 \leq L_6 \leq t_m p_2^m - s$, $k_m(2p_1^m - 1) < K_6 \leq 2k_m p_1^m$, $t_m \geq s$. Then*

$$D_E = O(m^{2s+3} \log^2 m) \ . \tag{97}$$

Proof. Let $K_4 = k_m(2p_1^m - 1)$, $L_2 = [t_m L_1/t_{m+1}] + 1$, $L_3 = [L_6/t_{m+1}] - L_2 - 1$. It is easy to see that

$$t_{m+1}(L_2 + L_3) + 1 < L_6 \leq t_m p_2^m - s. \tag{98}$$

If $L_3 \leq 0$, then $D_E \leq 4k_m t_{m+1} = O(m \log m)$. Let $L_3 > 0$, and let $E_1 = [K_4, K_6) \times [t_m L_1, t_{m+1} L_2)$, $E_2 = [K_4, K_6) \times [t_{m+1} L_2, t_{m+1}(L_2 + L_3)]$, $E_3 = [K_4, K_6) \times (t_{m+1}(L_2 + L_3), L_6]$.
It is evident that

$$E = E_1 \cup E_2 \cup E_3, \qquad \text{and} \quad E_i \cap E_j = \emptyset \quad \text{for} \quad i \neq j \ .$$

Using (2) and (75), we find that

$$D_E \leq D_{E_1} + D_{E_2} + D_{E_3} \ . \tag{99}$$

From (7) and (8) we obtain :

$$D_{E_i} \leq \#E_i \leq 2k_m t_{m+1} = O(m^2), \qquad i = 1, 3 \ . \tag{100}$$

It follows from (75) that

$$D_{E_2} \le \sum_{0\le x_3<z}\ \sum_{0\le y_4<t_{m+1}} \#G_2 D^{(s)}\Big(\big(\{\alpha_y q^x\},\dots,\{\alpha_{y+s-1}q^x\}\big)_{(x,y)\in G_2}\Big),$$

where $G_2 = G_2(m,x_3,y_4,L_2,L_3)$, and $z = K_6 - K_4 \le k_m$. Bearing in mind that (98) is true, we can apply Lemma 16:

$$D_{E_2} \le \sum_{0\le x_3<k_m}\ \sum_{0\le y_4<t_{m+1}} (2^s B_1(x_3,y_4,s,\mathbf{a}^{(m)},\mathbf{a}^{(m+1)}) + s2^{s+3})$$

Now from (71), (99), and (100) we obtain the assertion of the corollary. $\square$

We now consider the upper bourne of V_m:

Lemma 17. *Let* $K_2,K_3 \ge 0$, $K_2+K_3 \le p_1^{m-1}-2$, $x_5\in\{0,1\}$, $x_6 \in \{0,1,\dots,k_{m+1}-1\}$, $y_2\in(t_m-s,t_m)$, $t_m>s$,

$$\begin{aligned}(101)\quad G_3 = G_3(m,x_5,x_6,y_2,K_2,K_3) = \big\{(x,y)\mid x = 2k_{m+1}x_4 + \\ + k_{m+1}x_5 + x_6,\ y = t_m(p_2^m-1)+y_2,\ K_2\le x_4<K_2+K_3\big\}\subset V_m,\end{aligned}$$

for $m=1,2,\dots$.
Then

$$\begin{aligned}(102)\quad \#G_3 D^{(s)}\Big(\big(\{\alpha_y q^x\},\dots,\{\alpha_{y+s-1}q^x\}\big)_{(x,y)\in G_3}\Big) \le s2^{s+3} \\ + 2^s B_2(x_2,x_3,x_5,x_6,y_2,y_4,s,\mathbf{a}^{(m)},\mathbf{a}^{(m+1)}),\end{aligned}$$

where $x_2 = 2\{([x_6/k_m]+x_5k_{m+1}/k_m)/2\}$, $x_3 = k_m\{x_6/k_m\}$, *and* $y_4 = t_{m+1}\{(t_m(p_2^m-1)+y_2)/t_{m+1}\}$.

Proof. Let $(x,y)\in G_3$ and let $i\in[0,t_m-y_2)$. The equality (83) and the conditions of the lemma show, that $\theta_m(y_2,i)=0$. The pair $(x,y+i)$ satisfies the conditions of Lemma 13. The equation (84) implies that

$$\begin{aligned}(103)\quad \{\alpha_{y+i}q^x\}_{2k_m-x_3} = \Big\{\frac{q^{x_3}a^{(m)}_{y_2+i,x_2}(x_1p_2^m+(p_2^m-1)p_1^m)}{p_1^mp_2^m}\Big\}_{k_m-x_3} \\ + \frac{1}{q^{k_m-x_3}}\Big\{\frac{a^{(m)}_{y_2+i,x_2+1}((x_1+x_2)p_2^m+(p_2^m-1)p_1^m}{p_1^mp_2^m}\Big\}_{k_m},\end{aligned}$$

where

$$(104)\quad \begin{gathered} x = 2k_{m+1}x_4+k_{m+1}x_5+x_6 = 2k_mx_1+k_mx_2+x_3,\\ x_3 = k_m\{x_6/k_m\}\in\{0,1,\dots,k_m-1\},\\ x_1 = x_4k_{m+1}/k_m + [(k_{m+1}x_5+x_6)/k_m],\\ x_2\in\{0,1\},\quad x_2\equiv[x_6/k_m]+x_5k_{m+1}/k_m \pmod 2.\end{gathered}$$

Now let $i \in [t_m - y_2, s-1]$. Then $y+i > t_m p_2^m$, and $r(y+i) = m+1$ (see (8) and (77)). The pair $(x, y+i)$ satisfies the conditions of Lemma 13 (with $m+1$ instead of m). Hence, we have from (84) and (104) that

$$(105) \quad \{\alpha_{y+i} q^x\}_{2k_{m+1}-x_6} = \Big\{ \frac{q^{x_6} a^{(m+1)}_{y_4+i,x_5} (x_4 p_2^{m+1} + (y_3 + \theta_{m+1}(y_4, i)) p_1^{m+1})}{p_1^{m+1} p_2^{m+1}} \Big\}_{k_{m+1}-x_6} + \frac{1}{q^{k_{m+1}-x_6}} \Big\{ \frac{a^{(m+1)}_{y_4+i,x_5+1}((x_4+x_5)p_2^{m+1} + (y_3 + \theta_{m+1}(y_4,i))p_1^{m+1})}{p_1^{m+1}p_2^{m+1}} \Big\}_{k_{m+1}},$$

where

$$y = t_m(p_1^m - 1) + y_2 = t_{m+1} y_3 + y_4, \quad y_4 \in \{0, 1, \dots, t_{m+1} - 1\},$$

$$y_4 \equiv t_m(p_1^m - 1) + y_2 (\bmod t_{m+1}), \quad y_3 = (t_m(p_1^m - 1) + y_2 - y_4)/t_{m+1} .$$

It is easy to see that if $k_{m+1} = k_m$, then $x_3 = x_6$. Otherwise $k_{m+1} \geq p_1 k_m \geq 2k_m$, and $2k_{m+1} - x_6 \geq k_{m+1} \geq 2k_m \geq 2k_m - x_3$.

We denote the left side of (102) by σ. Applying Lemma 5 with $k = 2k_m - x_3$, $k^{(i)} = 2k_m - x_3$ for $i \in [0, t_m - y_2)$; $k^{(i)} = 2k_{m+1} - x_6$ for $i \in [t_m - y_2, s-1]$), and $q_i = q$, for $i = 1, \dots, s$, we get

$$\sigma \leq \# G_3 \Big(\frac{s}{q^{2k_m - x_3}} + D^{(s)} \Big(((\{\alpha_{y+i} q^x\}_{k^{(i)}})_{i \in [0,s-1]})_{(x,y) \in G_3} \Big) \Big).$$

Using Lemma 6 with $k = 2k_m - x_3$, $k^{(i)} = 2k_m - x_3$, $\ell^{(i)} = k_m$ for $i \in [0, t_m - y_2)$, and $k^{(i)} = 2k_{m+1} - x_6$, $\ell^{(i)} = k_{m+1}$ for $i \in [t_m - y_2, s-1]$, we obtain from (103), and (105)

$$\sigma \leq \# G_3 \Big(\frac{2s}{q^{2k_m - x_3}} + 2^s D^{(2s)} \Big(\Big(\Big(\Big\{ \frac{q^{x_3} a^{(m)}_{y_2+i,x_2} (x_1 p_2^m + (p_2^m - 1) p_1^m)}{p_1^m p_2^m} \Big\}, \Big\{ \frac{a^{(m)}_{y_2+i,x_2+1}((x_1+x_2)p_2^m + (p_2^m - 1)p_1^m}{p_1^m p_2^m} \Big\} \Big)_{i \in [0, t_m - y_2)}, \Big(\Big\{ \frac{q^{x_6} a^{(m+1)}_{y_4+i,x_5} (x_4 p_2^{m+1} + (y_3 + \theta_{m+1}(y_4,i)) p_1^{m+1})}{p_1^{m+1} p_2^{m+1}} \Big\}, \Big\{ \frac{a^{(m+1)}_{y_4+i,x_5+1}((x_4+x_5)p_2^{m+1} + (y_3 + \theta_{m+1}(y_4,i))p_1^{m+1})}{p_1^{m+1}p_2^{m+1}} \Big\} \Big)_{i \in [t_m - y_2, s-1]} \Big)_{K_2 \leq x_4 < K_2 + K_3} \Big) \Big) ,$$

where $x_1 = x_4 k_{m+1}/k_m + [(k_{m+1} x_5 + x_6)/k_m]$.
Here we now replace fractions with denominators $p_1^m p_2^m$ and $p_1^{m+1} p_2^{m+1}$ with fractions with denominator p_1^{m+1}, and we again apply Lemma 5 with

the parameters $2s$ instead of s, p_1 instead of q_i, and $m+1$ instead of $k^{(i)}$ $(i=1,\dots,2s)$:

$$\sigma \le \#G_3\Big(\frac{2s}{q^{2k_m-x_3}} + \frac{2s2^s}{p_1^{m+1}} + 2^s D^{(2s)}\Big(\Big(\Big(\Big\{\frac{p_1 q^{x_3} a^{(m)}_{y_2+i,x_2} x_4 k_{m+1}/k_m + d_i}{p_1^{m+1}}\Big\},$$
$$\Big\{\frac{p_1 a^{(m)}_{y_2+i,x_2+1} x_4 k_{m+1}/k_m + d_i^{'}}{p_1^{m+1}}\Big\}\Big)_{i\in[0,t_m-y_2)}, \Big(\Big\{\frac{q^{x_6} a^{(m+1)}_{y_4+i,x_5} x_4 + f_i}{p_1^{m+1}}\Big\},$$
$$\Big\{\frac{a^{(m+1)}_{y_4+i,x_5+1} x_4 + f_i^{'}}{p_1^{m+1}}\Big\}\Big)_{i\in[t_m-y_2,s-1]}\Big)_{K_2\le x_4<K_2+K_3}\Big)\Big)\ ,$$

where for $i=0,\dots,t-y_2-1$

$$d_i = [q^{x_3} a^{(m)}_{y_2+i,x_2}(p_1[(k_{m+1}x_5+x_6)/k_m] - p_1^{m+1}p_2^{-m})],$$
$$d_i^{'} = [a^{(m)}_{y_2+i,x_2+1}(p_1(x_2+[(k_{m+1}x_5+x_6)/k_m]) - p_1^{m+1}p_2^{-m})],$$

and for $i=t-y_2,\dots,s-1$

$$f_i = [(p_1/p_2)^{m+1} q^{x_6} a^{(m+1)}_{y_4+i,x_5}(y_3+\theta_{m+1}(y_4,i))],$$
$$f_i^{'} = [a^{(m+1)}_{y_4+i,x_5+1}((p_1/p_2)^{m+1}(y_3+\theta_{m+1}(y_4,i))+x_5)].$$

Then Corollary 2 (with $N=\#G_3=K_3$, $T=P=p_1^{m+1}$, and $d=2s$) implies that $\sigma \le$

$$\#G_3\Big(\frac{2s}{q^{k_m}} + \frac{4s2^s}{p_1^{m+1}}\Big) + 2^s \sum_{h_{-1}\in C(p_1^{m+1})} \sum_{\mathbf{h}\in C_{2s}(p_1^{m+1})} r^{-1}(h_{-1},p_1^{m+1}) r^{-1}(\mathbf{h},p_1^{m+1})$$
$$\times \Big|\sum_{x_4=0}^{p_1^{m+1}-1} e\Big(\Big(h_{-1}x_4 + \sum_{0\le i<t_m-y_2}(x_4 p_1 k_{m+1}/k_m q^{x_3} a^{(m)}_{y_2+i,x_2} + d_i)h_i$$
$$+ \sum_{0\le i<t_m-y_2}(x_4 p_1 k_{m+1}/k_m a^{(m)}_{y_2+i,x_2+1} + d_i^{'})h_{s+i}$$
$$+ \sum_{t_m-y_2\le i<s}(q^{x_6} a^{(m+1)}_{y_4+i,x_5} + f_i)h_i + \sum_{t_m-y_2\le i<s}(x_4 a^{(m+1)}_{y_4+i,x_5+1} + f_i^{'})h_{s+i}\Big)/p_1^{m+1}\Big)\Big|.$$

Using Lemma 1, (7), (101), and the conditions of the lemma, we find that

$$\sigma \le 8s2^s + 2^s \sum_{h_{-1}\in C(p_1^{m+1})} \sum_{\mathbf{h}\in C_{2s}(p_1^{m+1})} r^{-1}(h_{-1},p_1^{m+1}) r^{-1}(\mathbf{h},p_1^{m+1}) p_1^{m+1}$$
$$\times \delta_{p_1^{m+1}}\Big(h_{-1} + \sum_{0\le i<t_m-y_2}(q^{x_3}h_i a^{(m)}_{y_2+i,x_2} + h_{s+i}a^{(m)}_{y_2+i,x_2+1})p_1 k_{m+1}/k_m$$
$$+ \sum_{t_m-y_2\le i<s}(q^{x_6}h_i a^{(m+1)}_{y_4+i,x_5} + h_{s+i}a^{(m+1)}_{y_4+i,x_5+1})\Big).$$

We obtain from (61) that

$$\sigma \leq s2^{s+3} + 2^s B_2(x_2, x_3, x_5, x_6, y_2, y_4, s, \mathbf{a}^{(m)}, \mathbf{a}^{(m+1)}) \ .$$

Thus we derive the desired result. $\square$

Corollary 8. *Let* $E = [2k_m K_1, K_6) \times [t_m p_2^m - s + 1, L_6) \subset V_m$; $2k_m K_1 < K_6 \leq k_m(2p_1^m - 1)$, $t_m p_2^m - s + 1 < L_6 \leq t_m p_2^m$, $t_m > s$. *Then*

$$D_E = O(m^{2s+3} \log^2 m) \ . \tag{106}$$

Proof. Let $L_4 = t_m p_2^m - s + 1$, $K_2 = [k_m K_1 / k_{m+1}] + 1$, $K_3 = [K_6 / k_{m+1}] - K_2 - 1$. It is easy to see that

$$k_{m+1}(K_2 + K_3 + 1) - 1 < K_6 \leq k_m(2p_1^m - 1). \tag{107}$$

If $K_3 \leq 0$, then

$$D_E \leq 2sk_{m+1} = O(m).$$

Now let $K_3 > 0$. Put

$$E_1 = [2k_m K_1, 2k_{m+1} K_2) \times [L_4, L_6),$$
$$E_2 = [2k_{m+1} K_2, 2k_{m+1}(K_2 + K_3)) \times [L_4, L_6),$$
$$E_3 = [2k_{m+1}(K_2 + K_3), K_6) \times [L_4, L_6).$$

It is evident that

$$E = E_1 \cup E_2 \cup E_3, \qquad \text{and} \quad E_i \cap E_j = \emptyset \quad \text{for} \quad i \neq j \ .$$

Using (2) and (75), we find that

$$D_E \leq D_{E_1} + D_{E_2} + D_{E_3} \ . \tag{108}$$

From (7) and (8) we obtain :

$$D_{E_i} \leq \# E_i \leq 4sk_{m+1} = O(m), \qquad i = 1, 3 \ . \tag{109}$$

It follows from (75) and (101), that

$$D_{E_2} \leq \sum_{0 \leq x_5 \leq 1} \sum_{0 \leq x_6 < k_{m+1}} \sum_{y_2 \in (t_m - s, t_m)} \# G_3$$
$$\times D^{(s)} \Big((\{\alpha_y q^x\}, \ldots, \{\alpha_{y+s-1} q^x\})_{(x,y) \in G_3} \Big),$$

where $G_3 = G_3(m, x_5, x_6, y_2, K_2, K_3)$.
Because (107), we can apply Lemma 17. Thus from (72), (108), and (109), we obtain (106). $\square$

Now, combining (87),(97), and (106) we obtain :

Lemma 18. *Let* $K_1, K_6, L_1, L_6 \geq 0$ *be integers and* $E = [2k_m K_1, K_6) \times [t_m L_1, L_6) \subset V_m$. *Then*

$$D_E = O(m^{2s+3} \log^2 m),$$

where the constant implied by O *only depends on* s.

Proof. Let

$$E_1 = [2k_m K_1, k_m(2p_1^m - 1)) \times [t_m L_1, t_m p_2^m - s],$$
$$E_2 = [k_m(2p_1^m - 1), 2k_m p_1^m) \times [t_m L_1, t_m p_2^m - s],$$
$$E_3 = [2k_m K_1, k_m(2p_1^m - 1)) \times [t_m p_2^m - s + 1, t_m p_2^m),$$
$$E_4 = [k_m(2p_1^m - 1), 2k_m p_1^m) \times [t_m p_2^m - s + 1, t_m p_2^m),$$

and let

$$E_i^{'} = E \cap E_i, \qquad i = 1, \dots, 4.$$

It is easy to see that

$$E = \cup_{i=1}^4 E_i^{'}, \qquad \text{and} \quad E_i^{'} \cap E_j^{'} = \emptyset \quad \text{for} \quad i \neq j. \tag{110}$$

Now equations (7) and (75) imply that

$$D_{E_4^{'}} \leq sk_m = O(m).$$

From Lemma 15, Corollary 7, and Corollary 8, we get

$$D_{E_i^{'}} = O(m^{2s+3} \log^2 m), \qquad i = 1, 2, 3.$$

From (110), we obtain the assertion of the lemma. □

End of the proof of Theorem 1. We use notations (73), (74), and (75). Let

$$G^{(m)} = G_{M,N} \cap V_m \neq \emptyset \quad \text{for} \quad m = 1, \dots, r \quad \text{and} \quad G^{(r+1)} = \emptyset\ . \tag{111}$$

It is evident that

$$G^{(m)} = G_1^{(m)} \cup G_2^{(m)}, \quad m = 1, 2, \dots \qquad G^{(m)} \cap G^{(n)} = \emptyset \quad \text{for} \quad m \neq n,$$

and $G_i^{(m)}$ is the rectangular domain $(i = 1, 2)$. Let $(K_\nu^{(m,i)}, L_\mu^{(m,i)})_{\nu,\mu=1,2}$ be coordinates of the vertex of $G_i^{(m)}$:

$$G_i^{(m)} = [K_1^{(m,i)}, K_2^{(m,i)}) \times [L_1^{(m,i)}, L_2^{(m,i)}) \qquad i = 1, 2, \ \ m = 1, 2, \dots\ .$$

From (7), (8), and (48), we see that

$$k_{m-1} p_1^{m-1} \equiv 0 \bmod k_m \text{ and } \ t_{m-1} p_2^{m-1} \equiv 0 \bmod t_m \quad m = 2, 3, \dots\ .$$

Hence

$$K_1^{(m,i)} \equiv 0 \bmod 2k_m \text{ and } \ L_1^{(m,i)} \equiv 0 \bmod t_m, \quad m = 2, 3, \dots\ .$$

Applying Lemma 18, we obtain

$$D_{G_i^{(m)}} = O(m^{2s+3} \log^2 m), \qquad i = 1, 2, \quad m = 2, 3, \dots\ .$$

Using (2), (75), and (111), we find that

$$MND\Big(\big(\{\alpha_m q^n\},\ldots,\{\alpha_{m+s-1}q^n\}\big)_{1\le n\le N, 0\le m<M}\Big) = D_{G_{M,N}}$$
$$\le \sum_{m=1}^{r}\sum_{i=1}^{2} D_{G_i^{(m)}} = O\Big(\sum_{m=1}^{r} m^{2s+3}\log^2 m\Big) = O(r^{2s+4}\log^2 r)$$
$$= O\big((\log MN)^{2s+4}\log^2\log MN\big)\ .$$

Thus we obtain the assertion of Theorem 1. □

Remark. Using (76), (95), and (103) we can find specifically digits $d_{i,j}$ $(i,j=1,2,\ldots)$ of a normal lattice configuration (see (5)). With the notations defined above, if $(x,y) = (2k_m x_1 + k_m x_2 + x_3, t_m y_1 + y_2) \in V_m$, then

$$d_{x,y} = \Big[q\Big\{\frac{q^{x_3} a^{(m)}_{y_2,x_2}(x_1 p_2^m + y_1 p_1^m)}{p_1^m p_2^m}\Big\}\Big]\ .$$

4. Proof of Theorem 2

Let

$$(112)\quad A_3(j,m,\nu,c_{1,0},\ldots,c_{s,0},c_{1,1},\ldots,c_{s,1}) = p^m \sum_{h_0\in C(p^m)}\ \sum_{\mathbf{h}\in C_{2s}(p^m)}$$
$$\times\, r^{-1}(h_0,p^m) r^{-1}(\mathbf{h},p^m)\delta_{p_m}\Big(h_0 + \sum_{i=1}^{s}(q_i^j h_i c_{i,\nu} + h_{i+s}c_{i,\nu+1}\Big)\ ,$$

$$(113)\quad A_4(j,m,\nu,c_{1,0},\ldots,c_{s,0},c_{1,1},\ldots,c_{s,1})$$
$$= p^m \sum_{\mathbf{h}\in C_{2s}(p^m)} r^{-1}(\mathbf{h},p^m)\delta_{p_m}\Big(\sum_{i=1}^{s}(q_i^j h_i c_{i,\nu} + h_{i+s}c_{i,\nu+1}\Big)\ ,$$

with $c_{i,\nu+2} = c_{i,\nu}$, $\nu = 0,1$, $i = 1,\ldots,s$.

Lemma 19. *Let* $\nu\in\{0,1\}$, $j\in[0,k_m)$, $\varphi_0 = \varphi(p)/p$, *and*

$$K(s) = 4p/\varphi(p)\Big(\frac{2}{\pi}\log p + \frac{7}{5}\Big)^{2s+1}.$$

Then

$$(114)\quad \frac{1}{\varphi_0^{2s}p^{2ms}} \sum_{(c_{1,0},\ldots,c_{s,1})\in(\Delta_m^*(p))^{2s}} A_3(j,m,\nu,c_{1,0},\ldots,c_{s,1}) < K(s)m^{2s+1},$$

and

$$(115)\quad \frac{1}{\varphi_0^{2s}p^{2ms}} \sum_{(c_{1,0},\ldots,c_{s,1})\in(\Delta_m^*(p))^{2s}} A_4(j,m,\nu,c_{1,0},\ldots,c_{s,1}) < K(s)m^{2s}\ .$$

Proof. We will prove the inequality (114). The proof of (115) repeats that of (114). We denote the left side of (114) by σ. Changing the order of the summation, from (112) we obtain

$$\sigma \leq \sum_{h_0 \in C(p^m)} \sum_{\mathbf{h} \in C_{2s}(p^m)} r^{-1}(h_0, p^m)(\mathbf{h}, p^m) E(h_0, \mathbf{h}),$$

where

$$E(h_0, \mathbf{h}) = \frac{1}{\varphi_0^{2s} p^{2ms}} \sum_{(c_1, \ldots, c_{2s}) \in (\Delta_m(p))^{2s}} \times p^m \delta_{p^m} \Big(h_0 + \sum_{j=1}^{s} (q_i^j h_j c_j + h_{j+s} c_{j+s}) \Big) .$$

Let $(h_1, \ldots, h_{2s}, p^m) = p^\alpha$, let $h_i = p^\alpha h_i^{'}$, $i = 1, \ldots, 2s$, and $(h^{'}_{\nu_0 s + i_0}, p) = 1$, $\nu_0 = 0, 1$, $i_0 = 1, \ldots, s$. If $h_0 \not\equiv 0 \pmod{p^\alpha}$, then $E(h_0, \mathbf{h}) = 0$.
Now let $h_0 = p^\alpha h_0'$. Hence

$$E(h_0, \mathbf{h}) \leq \max_{(c_1, \ldots, c_{2s}) \in (\Delta_m(p^m))^{2s}} p^m \frac{p}{\varphi(p)} \sum_{c_{s\nu_0 + i_0} = 0}^{p^m - 1} \times \delta_{p^{m-\alpha}} \Big(h_0' + \sum_{i=1}^{s} (q_i^j h_i' c_i + h_{i+s}' c_{i+s}) \Big) .$$

Bearing in mind that $(q_{i_0}^j h^{'}_{s\nu_0 + i_0}, p) = 1$, and

$$\sum_{c_{s\nu_0 + i_0} = 0}^{p^m - 1} \delta_{p^{m-\alpha}} \big(f + q_{i_0}^j h'_{s\nu_0 + i_0} c_{s\nu_0 + i_0} \big) = p^\alpha ,$$

we find that

$$E(h_0, \mathbf{h}) \leq p^{\alpha+1} / \varphi(p).$$

Applying Lemma 2, we obtain

$$\begin{aligned}
\sigma &\leq \sum_{\alpha=0}^{m} \sum_{\substack{h_0 \in C(p^m) \\ (h_0, p^m) = p^\alpha}} \sum_{\substack{\mathbf{h} \in C_{2s}(p^m) \\ (h_1, \ldots, h_{2s}, p^m) = p^\alpha}} r^{-1}(h_0, p^m)(\mathbf{h}, p^m) p^{\alpha+1} / \varphi(p) \\
&\leq p/\varphi(p) \sum_{\alpha=0}^{m} \Big(\frac{2}{\pi} m \log p + \frac{7}{5} \Big)^{2s} \Big(1 + p^{-\alpha} \big(\frac{2}{\pi} m \log p + \frac{7}{5} \big) \Big) \\
&< m^{2s+1} p/\varphi(p) \Big(\frac{2}{\pi} \log p + \frac{7}{5} \Big)^{2s} \Big(\frac{m+1}{m} + \frac{p}{p-1} \big(\frac{2}{\pi} \log p + \frac{7}{5} \big) \Big) \\
&\leq K(s) m^{2s+1}.
\end{aligned}$$

□

Corollary 9. *There exist integers* $a_{1,0}^{(m)},\dots,a_{s,0}^{(m)},a_{1,1}^{(m)},\dots,a_{s,1}^{(m)}$ *such that*

$$\sum_{j=0}^{k_m-1}\sum_{\nu=0}^{1} A_3(j,m,\nu,a_{1,0}^{(m)},\dots,a_{s,0}^{(m)},a_{1,1}^{(m)},\dots,a_{s,1}^{(m)}) \le 4k_mK(s)m^{2s+1}\ , \tag{116}$$

and

$$\sum_{j=0}^{k_m-1}\sum_{\nu=0}^{1} A_4(j,m,\nu,a_{1,0}^{(m)},\dots,a_{s,0}^{(m)},a_{1,1}^{(m)},\dots,a_{s,1}^{(m)}) \le 4k_mK(s)m^{2s},\ m=1,2,\dots\ . \tag{117}$$

We use such integers $a_{1,0}^{(m)},\dots,a_{s,1}^{(m)}$ $(m=1,2,\dots)$ to construct the real numbers $\alpha_1,\dots,\alpha_s$ (15).

Lemma 20. *Let* $1\le M\le p^m$. *Then for* $m=1,2,\dots$

$$2k_mMD\Big(\big(\{\alpha_1q_1^x\},\dots,\{\alpha_sq_s^x\}\big)_{x\in[n_m,n_m+2k_mM)}\Big)=O(m^{2s+2}), \tag{118}$$

$$2k_mp^mD\Big(\big(\{\alpha_1q_1^x\},\dots,\{\alpha_sq_s^x\}\big)_{x\in[n_m,n_{m+1})}\Big)=O(m^{2s+1}). \tag{119}$$

Proof. We denote the left side of (118) by σ. Equation (2) implies that

$$\sigma\le\sum_{x_3=0}^{k_m-1}\sum_{x_2=0}^{1}\sigma(x_2,x_3), \tag{120}$$

with

$$\sigma(x_2,x_3)=MD^{(s)}\Big(\big(\{\alpha_1q_1^{n_m+2k_mx_1+k_mx_2+x_3}\},\dots,\{\alpha_sq_s^{n_m+2k_mx_1+k_mx_2+x_3}\}\big)_{x_1\in[0,M)}\Big)\ .$$

We apply $k^{(i)}=2k_m-x_3,\ i=1,\dots,s$ to Lemma 5:

$$\sigma(x_2,x_3)\le\frac{sM}{q^{2k_m-x_3}}+MD^{(s)}\Big(\Big(\big(\{\alpha_iq_i^{n_m+2k_mx_1+k_mx_2+x_3}\}_{2k_m-x_3}\big)_{i=1,\dots,s}\Big)_{x_1\in[0,M)}\Big)\ ,$$

where $q=\min(q_1,\dots,q_s)$. Using (15), we obtain, similarly to (79), that

$$\{\alpha_iq_i^{n_m+2k_mx_1+k_mx_2+x_3}\}_{2k_m-x_3}=\Big\{\frac{q_i^{x_3}a_{i,x_2}^{(m)}x_1}{p^m}\Big\}_{k_m-x_{3,i}}+\frac{1}{q_i^{k_m-x_3}}\Big\{\frac{a_{i,x_2+1}^{(m)}(x_1+x_2)}{p^m}\Big\}_{k_m,i}\ .$$

Then Lemma 6 (with $k_i = k_m - x_3$ and $l_i = k_m,\ i = 1, \dots, s$) shows that

$$\sigma(x_2, x_3) \le \frac{2sM}{q^{2k_m - x_3}} + 2^s M D^{(2s)}\Big(\Big(\Big(\Big\{\frac{q_i^{x_3} a_{i,x_2}^{(m)} x_1}{p^m}\Big\}\Big)_{i=1,\dots,s}, \Big(\Big\{\frac{a_{i,x_2+1}^{(m)}(x_1+x_2)}{p^m}\Big\}\Big)_{i=1,\dots,s}\Big)_{x_1\in[0,M)}\Big).$$

Now Corollary 2 (with $T = P = p^m$) implies that

$$\sigma(x_2, x_3) \le \frac{2sM}{q^{k_m}} + \frac{2s2^sM}{p^m} + 2^s \sum_{h_0\in C(p^m)} \sum_{\mathbf{h}\in C_{2s}(p^m)} r^{-1}(h_0, p^m) r^{-1}(\mathbf{h}, p^m)$$

$$\times \Big| \sum_{x_1=0}^{p^m-1} e\Big(\Big(h_0 x_1 + \sum_{i=1}^{s} \Big(q_i^{x_3} h_i a_{i,x_2}^{(m)} x_1 + h_{s+i} a_{i,x_2+1}^{(m)}(x_1+x_2)\Big)\Big)/p^m\Big)\Big| .$$

Applying Lemma 1, (14), and (112), we obtain

$$\sigma(x_2, x_3) \le 4s2^{2s} + 2^s A_3(x_3, m, x_2, a_{1,0}^{(m)}, \dots, a_{s,1}^{(m)}).$$

From (14), (116), (120), and Corollary 9, we get (118). Using (117), we similarly obtain (119). $\square$

End of the proof of Theorem 2. Let N be in $[n_r, n_{r+1})$. Define $D(N_1, N_2) = 0$ for $N_2 \le 0$, and

$$D(N_1, N_2) = N_2 D\Big(\big(\{\alpha_1 q_1^x\}, \dots, \{\alpha_s q_s^x\}\big)_{x\in[N_1, N_1+N_2)}\Big) \quad \text{for} \quad N_2 > 0 . \tag{121}$$

Using (2), (14), and Lemma 20 we have for $M \in [1, 2k_m p^m]$ that

$$D(n_m, M) \le D\big(n_m, 2k_m[M/2k_m]\big) + 2k_m = O(m^{2s+2}) . \tag{122}$$

Applying (2),(14), (118), (119), (121), and (122), we get the assertion of Theorem 2:

$$\begin{aligned} D(1,N) &\le D(1, 2k_1 p) + \sum_{m=2}^{r} D(n_m, 2k_m p^m) + D(n_r, N+1-n_r) \\ &= O\Big(\sum_{m=1}^{r} m^{2s+1} + r^{2s+2}\Big) = O(r^{2s+2}) = O(\log^{2s+2} N) . \end{aligned}$$ $\square$

5. Appendix

The proof of Proposition 1. We follow [KN p.70]. Let $s_1, s_2 \ge 1$ be integers. Consider a box $v \subset [0,1)^s$, a block of digits $G_{s_1,s_2} = \{g_{i,j} \in$

$\{0,\dots,q-1\} \mid \ i=0,\dots,s_1-1, \ \ j=1,\dots,s_2\}$, a configuration $\omega = (d_{i,j})_{i,j\geq 1}$ with $d_{i,j}\in\{0,\dots,q-1\}$ and real numbers

$$\alpha_m = \sum_{n=1}^{\infty} d_{m,n}/q^n \ .$$

Now let

$$A_v(M,N) = \#\big\{(m,n)\in[1,M]\times[1,N] \mid (\{\alpha_m q^n\},\dots,\{\alpha_{m+s-1}q^n\})\in v\big\},$$

$$S(M,N,G_{s_1,s_2}) = \#\big\{(m,n)\in[1,M]\times[1,N] \mid d_{m+i,n+j}=g_{i,j}, \\ i\in[0,s_1), \ \ j\in[1,s_2]\big\}.$$

The block $(d_{m+i,n+j})_{i=0,\ j=1}^{s_1-1\ \ s_2}$ is identical with G_{s_1,s_2} if and only if

$$\alpha_{m+i} = [\alpha_{m+i}] + \sum_{k=1}^{n}\frac{d_{m+i,k}}{q^k} + \frac{g_{i,1}}{q^{n+1}} + \cdots + \frac{g_{i,s_2}}{q^{n+s_2}} + \sum_{k=n+s_2+1}^{\infty}\frac{d_{m+i,k}}{q^k},$$

or

$$\{\alpha_{m+i}q^n\} = \frac{g_{i,1}}{q} + \cdots + \frac{g_{i,s_2}}{q^{s_2}} + \sum_{k=s_2+1}^{\infty}\frac{d_{m+i,k}}{q^k},$$

or

$$\{\alpha_{m+i}q^n\}\in\Big[\frac{g_{i,1}q^{s_2-1}+\cdots+g_{i,s_2}}{q^{s_2}}, \frac{g_{i,1}q^{s_2-1}+\cdots+g_{i,s_2}+1}{q^{s_2}}\Big) \\ = \Delta_i(G_{s_1,s_2}), \quad i\in[0,s_1).$$

It follows that

$$S(M,N,G_{s_1,s_2}) = A_v(M,N), \qquad \text{with} \quad v = \prod_{i=0}^{s_1-1}\Delta_i(G_{s_1,s_2}).$$

Now suppose that the double sequence $\big(\{\alpha_m q^n\},\dots,\{\alpha_{m+s_1-1}q^n\}\big)_{m,n\geq 1}$ is uniformly distributed in $[0,1)^{s_1}$. Then

$$A_v(M,N) = q^{-s_1 s_2}MN + o(MN), \qquad \text{with} \quad \max(M,N)\to\infty,$$

and so ω is a normal lattice configuration.

Conversely, if ω is a normal configuration, then

$$A_v(M,N) = S(M,N,G_{s_1,s_2}) = MN mes\ v + o(MN),$$

where $v=\prod_{i=0}^{s_1-1}\Delta_i(G_{s_1,s_2})$, $mes\ v = q^{-s_1 s_2}$ and $\max(M,N)\to\infty$. This holds for all G_{s_1,s_2}. Therefore

$$A_v(M,N) = MN mes\ v + o(MN), \tag{123}$$

for all boxes $v=\prod_{i=1}^{s_1}[0,\gamma_i)$ with $\gamma_i = h_i/q^{s_2}$, $\quad h_i\in\{0,\dots,q^{s_2}-1\}, s_2\geq 1, i=1,\dots,s_1$, and $\max(M,N)\to\infty$.

Now let v be a box with arbitrary $\gamma_i \in (0,1]$ $(i = 1,\dots,s_1)$, and let $\epsilon \in (0,1)$ be given. Put $s_2 = 1+[-\log_q \epsilon]$ and put $h_i = [\gamma_i q^{s_2}]$, $i = 1,\dots,s_1$. Then

$$A_{v_1}(M,N) \le A_v(M,N) \le A_{v_2}(M,N), \tag{124}$$

where $v_1 = \prod_{i=1}^{s_1}[0, h_i/q^{s_2})$ and $v_2 = \prod_{i=1}^{s_1}[0,(h_i+1)/q^{s_2})$.
It is easy to see that $\gamma_i \in [h_i/q^{s_2},(h_i+1)/q^{s_2})$, with $i = 1,\dots,s_1$, $1/q^{s_2} < \epsilon$. According to [Ni, Lemma 3.9]

$$\max(|mes\ v_1 - mes\ v|, |mes\ v_2 - mes\ v|) \le 1-(1-1/q^{s_2})^{s_1} \le s_1 q^{-s_2} \le s_1\epsilon\,.$$

From (123) and (124) we deduce that

$$A_v(M,N) = MN(mes\ v + \epsilon_1) + o(MN),$$

where $|\epsilon_1| \le s_1\epsilon$, and $\max(M,N) \to \infty$. Hence for all $s_1 \ge 1$ and all boxes $v \subset [0,1)^{s_1}$, $A_v(M,N) = MNmes\ v + o(MN)$, with $\max(M,N) \to \infty$, and so $\left(\{\alpha_m q^n\},\dots,\{\alpha_{m+s_1-1}q^n\}\right)_{m,n\ge 1}$ is a uniformly distributed double sequence in $[0,1)^{s_1}$. $\square$

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