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# The value of additive forms at prime arguments

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Résumé. Soit $f(\mathbf{p})$ une forme de degré $k$ en $s$ variables $p_1, p_2, ...,$ $p_s$, où les $p_i$ sont des nombres premiers. On suppose que les coefficients $\lambda_i$ de $f$ sont réels, et qu'au moins l'un des rapports $\lambda_i/\lambda_j$ est irrationnel et algébrique. Si $s$ est assez grand alors $f$ prend des valeurs proches de tous les termes d'une suite arbitraire de nombres bien-espacés. Cela généralise des travaux antérieurs de Brüdern, Cook et Perelli (formes linéaires), et Cook et Fox (formes quadratiques). La preuve de ce résultat dépend du Lemme de Hua et, pour $k \geq 6$ des améliorations dues à Heath-Brown de ce lemme.

Abstract. Let $f(\mathbf{p})$ be an additive form of degree $k$ with $s$ prime variables $p_1, p_2, ..., p_s$. Suppose that $f$ has real coefficients $\lambda_i$ with at least one ratio $\lambda_i/\lambda_j$ algebraic and irrational. If $s$ is large enough then $f$ takes values close to almost all members of any well-spaced sequence. This complements earlier work of Brüdern, Cook and Perelli (linear forms) and Cook and Fox (quadratic forms). The result is based on Hua's Lemma and, for $k \geq 6$, Heath-Brown's improvement on Hua's Lemma.

## 1. Introduction

Montgomery and Vaughan [17] have shown that the exceptional set in Goldbach's problem

$$E(X) = \#\{n \leq X : 2n \neq p_1 + p_2\}$$

satisfies

$$E(X) \ll X^{1-\Delta} \tag{1}$$

for some $\Delta > 0$. Recently Li [15, 16] has shown that we may take $\Delta = 0.079$ and $\Delta = 0.086$. If the Riemann Hypothesis is true for all Dirichlet $L$-functions then (1) holds for any $\Delta < \frac{1}{2}$. This is a classical result due to Hardy and Littlewood [10].

Brüdern, Cook and Perelli [2] obtained an analogous result for binary linear forms with irrational coefficients. Instead of counting exceptional integers, suitably spaced real sequences mimic the role played by even integers in Goldbach's problem. An increasing sequence $v_1 < v_2 < \ldots$ of

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positive real numbers is called *well-spaced* if there exist positive constants $C > c > 0$ such that

$$0 < c < v_{i+1} - v_i < C \text{ for } i = 1, 2, \dots .$$

**Theorem 1** (Brüdern, Cook and Perelli). *Let $\lambda_1, \lambda_2$ be positive real numbers. Suppose that $\lambda_1/\lambda_2$ is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Then the number of $v \in \mathcal{V}$ with $v \leq X$ for which*

$$\left|\lambda_1 p_1 + \lambda_2 p_2 - v\right| < v^{-\delta}$$

*has no solution in primes $p_1, p_2$ does not exceed $O(X^{\frac{2}{3}+2\delta+\epsilon})$, for any $\epsilon > 0$.*

Any well-spaced sequence satisfies

$$X \ll \#\{v \in \mathcal{V} : v \leq X\} \ll X$$

whence the bound in the theorem is non-trivial for $\delta < \frac{1}{6}$. For small values of $\delta$ the exceptional set estimate is much stronger than (1). One reason for this is that the Fourier transform method for inequalities has a single major arc centred at the origin, whereas the major arcs in Goldbach's problem bring in arithmetic considerations involving zeros of $L$-functions, which have to be considered with some uniformity with respect to the modulus.

Hua [12] considered the equation

$$p_1^2 + p_2^2 + p_3^2 = n \tag{2}$$

and showed that the exceptional set $E_1(X)$ (now subject to certain necessary congruence conditions modulo 8, 3 and 5) satisfies

$$E_1(X) \ll X(\log X)^{-B}$$

for some constant $B > 0$ and Schwarz [18] showed that this estimate holds with any constant $B > 0$. Leung and Liu [14] have shown that

$$E_1(X) \ll X^{1-\Delta} \tag{3}$$

for some $\Delta > 0$ and more recently Bauer, Liu and Zhan [1] have shown that we may take

$$\Delta = \frac{3}{80}.$$

Cook and Fox [3] obtained an analogous result for Diophantine inequalities, and again the result for inequalities is stronger than that which can at present be obtained for equations.

**Theorem 2** (Cook and Fox). *Let $\lambda_1, \lambda_2, \lambda_3$ be positive real numbers. Suppose that $\lambda_1/\lambda_2$ is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Then the number $E_2(X)$ of $v \in \mathcal{V}$ with $v \leq X$ for which*

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 - v\right| < v^{-\delta}$$

*has no solution in primes* $p_1, p_2, p_3$ *satisfies*

$$E_2(X) \ll X^{\frac{11}{12}+2\delta+\epsilon} \tag{4}$$

*for any* $\epsilon > 0$.

Here we shall generalize these results to $k$th powers, and in a slightly strengthened form when $k \geq 6$.

**Theorem 3.** *Let* $k \geq 3$ *be an integer and either*

$$s = 2^{k-1} + 1 \tag{a}$$

*or*

$$k \geq 6 \quad \text{and} \quad s = \frac{7}{8}2^{k-1} + 1. \tag{b}$$

*Let*

$$\gamma = 4^{1-k}.$$

*Let* $\lambda_1, \ldots, \lambda_s$ *be positive real numbers. Suppose that* $\lambda_1/\lambda_2$ *is irrational and algebraic. Let* $\mathcal{V}$ *be a well-spaced sequence. Let* $\delta > 0$. *Then the number* $E_0(X)$ *of* $v \in \mathcal{V}$ *with* $v \leq X$ *for which*

$$\left|\lambda_1 p_1^k + \ldots + \lambda_s p_s^k - v\right| < v^{-\delta}$$

*has no solution in primes* $p_1, \ldots, p_s$ *satisfies*

$$E_0(X) \ll X^{1-\frac{2\gamma}{3k}+2\delta+\epsilon} \tag{5}$$

*for any* $\epsilon > 0$.

We may suppose that $\delta < \frac{\gamma}{3k}$ since otherwise the result is trivial; in particular we have $\delta < \frac{1}{144}$. Most of the conditions in the theorem can be relaxed. We write $\lambda = \lambda_1/\lambda_2$. We only require that $\lambda$ should not be too well-approximable. For algebraic $\lambda$ this holds by Roth's theorem, but is also true for almost all $\lambda$ in the sense of Lebesgue measure; see Cook and Fox [3] for a more general setting of the results. The referee has kindly pointed out the following two immediate consequences of Theorem 3, which enable the result to be stated without reference to the sequence $\mathcal{V}$. To prove Corollary 1 consider the cases $m$ odd and $m$ even separately and choose a suitable $v$ from each interval, an exceptional value if possible, to produce a well-spaced sequence.

**Corollary 1.** *Let* $k \geq 3$ *be an integer and either*

$$s = 2^{k-1} + 1 \tag{a}$$

*or*

$$k \geq 6 \quad \text{and} \quad s = \frac{7}{8}2^{k-1} + 1. \tag{b}$$

*Let*

$$\gamma = 4^{1-k}.$$

*Let* $\lambda_1, \dots, \lambda_s$ *be positive real numbers. Suppose that* $\lambda_1/\lambda_2$ *is irrational and algebraic. Let* $\delta > 0$. *Then the number of intervals* $(m, m+1]$ *with an integer* $m$ *satisfying* $0 \le m \le X$ *and containing an exceptional value* $v$ *for which the inequality is insoluble satisfies*

$$O(X^{1-\frac{2\gamma}{3k}+2\delta+\epsilon}).$$

Alternatively, the result can be stated in terms of the measure of the set of exceptional values $v$; if the measure of the set was larger than the bound stated the set could not fit into the intervals provided by Corollary 1.

**Corollary 2.** *Let* $k \ge 3$ *be an integer and either*

(*a*) $$s = 2^{k-1} + 1$$

*or*

(*b*) $$k \ge 6 \quad \textit{and} \quad s = \frac{7}{8}2^{k-1} + 1.$$

*Let*

$$\gamma = 4^{1-k}.$$

*Let* $\lambda_1, \dots, \lambda_s$ *be positive real numbers. Suppose that* $\lambda_1/\lambda_2$ *is irrational and algebraic. Let* $\delta > 0$. *Then the total measure of the set of exceptional values* $v$ *in the interval* $0 \le v \le X$ *is*

$$O(X^{1-\frac{2\gamma}{3k}+2\delta+\epsilon}).$$

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## 2. Preliminaries

In this section we begin our approach to the theorem via a Fourier transform method originating in the work of Davenport and Heilbronn [6]. We shall actually prove more than the theorem.

We regard $k, \lambda_1, \dots, \lambda_s$ and $\delta$ as constant. We take $P = X^{1/k}$ and $\eta$ as a fixed small positive number. Let $0 < \tau < 1$, we shall eventually take $\tau = X^{-\delta}$. For any $v \in \mathbb{R}$ it is our intention to estimate the sum

$$(6) \qquad \mathcal{N}_v = \sum_{\eta P < p_1, \dots, p_s \le P} (\log p_1) \dots (\log p_s) W(\lambda_1 p_1^k + \dots + \lambda_s p_s^k - v)$$

with a non-negative weight $W(\xi) = W_\tau(\xi)$ such that $W(\xi) = 0$ for $|\xi| \ge \tau$.

Hence $\mathcal{N}_v$ counts the solutions of

$$\left|\lambda_1 p_1^k + \dots + \lambda_s p_s^k - v\right| \le \tau$$

with a non-negative weight. The key idea of Davenport and Heilbronn was to use the Dirichlet kernel

$$K_0(\alpha) = \Big(\frac{\sin \pi\alpha}{\pi\alpha}\Big)^2,$$

which satisfies

$$\int_{-\infty}^{\infty} K_0(\alpha) e(\alpha\xi)\, d\alpha = \max(0, 1- \mid \xi \mid)$$

and

$$K_0(\alpha) \ll \min(1, \mid \alpha \mid^{-2}).$$

We shall follow a variant of this method, introduced by Davenport [4]. He constructed a kernel which vanished rapidly using an iterative process, starting with the square-wave kernel

$$2\Big(\frac{\sin 2\pi\alpha}{2\pi\alpha}\Big).$$

We shall need a kernel which is non-negative so start the iterative process with $K_0(\alpha)$ and, following Lemma 1 of Davenport [4], obtain the kernel

$$K_1(\alpha) = \frac{2}{3}\Big(\frac{\sin 2\pi\alpha/3}{2\pi\alpha/3}\Big)^2 \Big(\frac{\sin 2\pi\alpha/3n}{2\pi\alpha/3n}\Big)^n.$$

We shall choose $n$ to be a large even integer and write $r$ for $n+2$, to be explicit we can take $r = 4k$. Then the kernel $K_1(\alpha)$ is a non-negative even function such that

$$K_1(\alpha) \ll \min(1, \mid \alpha \mid^{-r}).$$

Further

$$\int_{-\infty}^{\infty} K_1(\alpha) e(\alpha\xi)\, d\alpha = \psi(\xi)$$

where $\psi(\xi)$ is an even function such that $0 \le \psi(\xi) \le 1$, $\psi(\xi) = 0$ for $\xi \ge 1$ and $\psi(\xi) \ge \frac{1}{3}$ for $\xi \le \frac{1}{9}$. Finally we modify the kernel $K_1(\alpha)$ to count the solutions of our inequality by taking

$$K(\alpha) = \tau^r K_1(\tau\alpha).$$

Then

$$\int_{-\infty}^{\infty} K(\alpha) e(\alpha\xi)\, d\alpha = W(\xi) \tag{7}$$

provides the required non-negative weight $W$ with $0 \le W(\xi) \le \tau^{r-1}$ and

$$K(\alpha) \ll \min(\tau^r, \mid \alpha \mid^{-r}).$$

From (7) we have

$$\mathcal{N}_v = \int_{-\infty}^{\infty} S_1(\alpha) \dots S_s(\alpha) e(-v\alpha) K(\alpha) d\alpha \tag{8}$$

where

$$S(\alpha)=S(\alpha,X)=\sum_{\eta P<p\le P}(\log p)e(\alpha p^k)$$

and for any function $f(\alpha)$, $f_j(\alpha)$ denotes $f(\lambda_j\alpha)$.

The main contribution to the integral (8) should arise from the neighbourhood of the origin and we divide the real line into 3 parts

$$\mathbf{M}=\{\alpha :\mid \alpha\mid\le Y\}, \tag{9}$$

$$\mathbf{m}=\{\alpha : Y<\mid \alpha\mid< Z\}, \tag{10}$$

and

$$\mathbf{t}=\{\alpha :\mid \alpha\mid\ge Z\}, \tag{11}$$

with $Y=P^u$ and $Z=P^\mu$ where $u$ is less than, but close to, $-k+\frac{1}{3}$ and $\mu>k\delta$ will be chosen later (a suitable choice is $\mu=\frac{1}{5}$).

## 3. The major arc

The main contribution to the integral (8) should arise from the major arc $\mathbf{M}$. Our treatment of the major arc is based on the methods of Vaughan [19],[20]. We approximate the sum $S(\alpha)$ by the integral

$$I(\alpha)=\int_{\eta P}^{P}e(\alpha y^k)dy. \tag{12}$$

We use $\rho=\beta+i\gamma$ to denote a typical zero of the Riemann zeta function and $\sum^*$ to denote summation over all those $\rho$ with $\mid\gamma\mid\le T$ and $\beta\ge\frac{2}{3}$, where $T=P^{\frac{1}{3}}$. We then take

$$\Xi_\rho(\alpha)=\sum_{(\eta P)^k<n\le P^k}n^{-1+\frac{\rho}{k}}e(\alpha n), \tag{13}$$

$$J(\alpha)=\sum\nolimits^*\Xi_\rho(\alpha), \tag{14}$$

and

$$B(\alpha)=S(\alpha)-I(\alpha)+J(\alpha). \tag{15}$$

Our first Lemma is essentially Lemma 8 of Vaughan [20], which in turn depends on Lemma 8 of Vaughan [19]. There are two changes that we have made. The first is to weight the sums $S(\alpha)$, and integral $I(\alpha)$, with a logarithmic weight, this does not affect the argument significantly. The other change is to replace Vaughan's estimates $\exp(-2(\log P)^{\frac{1}{5}})$ by an estimate $\exp(-2(\log P)^{\theta})$ for any $\theta<\frac{1}{3}$. This can be justified on noting that the exponent $\frac{1}{5}$ arises in the last 5 lines of the proof of Lemma 8 of Vaughan

[19] and depends essentially on the zero free region for the Riemann zeta function. Using the estimate

$$\beta \leq 1 - C(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}},$$

with $t = 3+ \mid \gamma \mid$, for a zero $\rho = \beta + i\gamma$ the exponent $\frac{1}{5}$ can be replaced by any $\theta < \frac{1}{3}$.

**Lemma 1.** *Let $\theta$ be a positive number with $\theta < \frac{1}{3}$. Let $Y = P^u$ with $u = \theta - k$. Then*

$$I(\alpha) \ll P \min(1, P^{-k} \mid \alpha \mid^{-1}), \tag{16}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mid I(\alpha) \mid^2 d\alpha \ll P^{2-k}, \tag{17}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mid J(\alpha) \mid^2 d\alpha \ll P^{2-k} \exp(-2(\log P)^{\theta}), \tag{18}$$

$$\int_{-Y}^{Y} \mid B_j(\alpha) \mid^2 d\alpha \ll P^{2-k} \exp(-2(\log P)^{\theta}), \tag{19}$$

*and*

$$\int_{-Y}^{Y} \mid S_j(\alpha) \mid^2 d\alpha \ll P^{2-k}. \tag{20}$$

On $\mathbf{M}$ we replace the term $S_1 \dots S_s$ in the integrand by $I_1 \dots I_s$. Replacing the sums one at a time we can estimate the difference

$$\begin{aligned}
&\int_{\mathbf{M}} \Big|(S_1 \dots S_s - I_1 \dots I_s)e(-v\alpha)\Big| K(\alpha) d\alpha \\
&\quad \ll \quad \tau^r P^{s-2} \int_{\mathbf{M}} \Big|B(\alpha) - J(\alpha)\Big|(\mid S(\alpha) \mid + \mid I(\alpha) \mid) d\alpha \\
&\quad \ll \quad \tau^r P^{s-k} \exp(-(\log P)^{\theta}),
\end{aligned}$$

using Cauchy's inequality and the previous lemma.

Next we replace the integral over $\mathbf{M}$ by an integral over the real line, the difference is

$$\ll \tau^r P^{-s(k-1)} \int_{\alpha > Y} \alpha^{-s} d\alpha \ll \tau^r Y^{1-s} P^{-s(k-1)} \ll \tau^r P^{s-k-(s-1)\theta}.$$

The final step in estimating the major arc is to show that

$$\int_{-\infty}^{\infty} I_1(\alpha) \dots I_s(\alpha) e(-v\alpha) K(\alpha) d\alpha \gg \tau^r P^{s-k}. \tag{21}$$

**Lemma 2.** *Let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_s \geq 1$. Then, uniformly for all $0 < \tau < 1$ and all $v \in [\frac{1}{2}X, X]$ we have*

$$\int_{-\infty}^{\infty} I_1(\alpha) \ldots I_s(\alpha) e(-v\alpha) K(\alpha)\, d\alpha \gg \tau^r P^{s-k}.$$

This is now straightforward, using the methods of Lemma 51 of Davenport [5] or Lemma 5 of Brüdern, Cook and Perelli [2], since the conditions we have imposed ensure that there is a non-trivial real solution of the inequalities in some suitable $s$-dimensional box. We shall see later how the conditions on $v$ may be removed. Clearly we may renumber the variables to order the coefficients $\lambda_i$, since the algebraic irrationality of $\lambda_1/\lambda_2$ plays no part in the major arc estimate.

## 4. Lemmas of Hua and Heath-Brown

In this section we extend a lemma of Heath-Brown to Diophantine inequalities. We take the exponential sum

$$S_0(\alpha) = \sum_{1 \leq n \leq P} e(\alpha n^k).$$

A well-known lemma of Hua [13] provides the bound

**Lemma 3** (Hua's Inequality). *For any $\epsilon > 0$ and for any integer $l$ in the range $1 \leq l \leq k$, we have*

$$\int_0^1 |S_0(\alpha)|^{2^l}\, d\,\alpha \ll P^{2^l - l + \epsilon}. \tag{22}$$

When $k \geq 6$ Heath-Brown [11] sharpened the result in the case $l = k$.

**Lemma 4** (Heath-Brown). *Let $k \geq 6$ and $\epsilon > 0$. Then*

$$\int_0^1 |S_0(\alpha)|^{2(s-1)}\, d\,\alpha \ll P^{2(s-1)-k+\epsilon}. \tag{23}$$

The analogous result for Diophantine inequalities is

**Lemma 5.** *Let $k \geq 6$ and $\epsilon > 0$. Then for any fixed non-zero real $\lambda$*

$$\int_{-\infty}^{\infty} |S(\lambda\alpha)|^{2(s-1)} K(\alpha)\, d\,\alpha \ll \tau^{r-1} P^{2(s-1)-k+\epsilon}. \tag{24}$$

*Proof.* We begin by observing that the lemmas of Hua and Heath-Brown bound the number of integer solutions of a certain equation and *eo ipso* bound the number of solutions in primes. The exponential sum $S(\alpha)$ introduces a logarithmic weighting but any particular solution is counted with a multiplicity bounded by $(\log P)^s$ and this weighting may be absorbed in the factor $P^\epsilon$. Thus for any $m$

$$\int_m^{m+1} |S(\alpha)|^{2(s-1)}\, d\,\alpha \ll P^{2(s-1)-k+\epsilon}.$$

Since the integrand in (24) is an even function, changing the variable of integration and using the bounds for the kernel $K(\alpha)$ we can bound this integral by

$$2\sum_{m=0}^{\infty}\int_m^{m+1} |S(\beta)|^{2(s-1)}K(\beta/\lambda)\lambda^{-1}\, d\,\beta \ll \sum_{m=0}^{\infty} P^{2(s-1)-k+\epsilon}\min(\tau^r, m^{-r})$$
$$\ll \tau^{r-1}P^{2(s-1)-k+\epsilon}$$

as required. □

We shall also need the corresponding analogue of Hua's Lemma, which may be obtained in the same way.

## 5. The tail

The simplest contribution to estimate is that coming from the tail $\mathbf{t}$. We give the details for $k \geq 6$, the case $k < 6$ is similar. We write $S_j(\alpha)$ for $S(\lambda_j\alpha)$.

**Lemma 6.** *For $Z = P^\mu$ with any fixed $\mu > k\delta$*

$$\int_{\mathbf{t}} \Big|S_1(\alpha)\dots S_s(\alpha)\Big| K(\alpha)d\alpha \ll Z^{1-r}P^{s-k/2+\epsilon}. \tag{25}$$

*Proof.* For each $j$ we have

$$\int_{\mathbf{t}} \mid S_j(\alpha)\mid^{s-1} K(\alpha)d\alpha \ll \Big(\int_{\mathbf{t}} K(\alpha)d\alpha\Big)^{\frac{1}{2}}\Big(\int_{\mathbf{t}} \mid S_j(\alpha)\mid^{2(s-1)} K(\alpha)d\alpha\Big)^{\frac{1}{2}}$$
$$\ll Z^{\frac{1-r}{2}}\Big(\int_{\mathbf{t}} \mid S_j(\alpha)\mid^{2(s-1)} K(\alpha)d\alpha\Big)^{\frac{1}{2}}$$

Since the integrand is an even function, changing the variable of integration to $\beta = \lambda\alpha$ and using the bounds for the kernel $K(\alpha)$, we can bound this integral by

$$2\sum_{m\geq\lambda(Z-1)}\int_m^{m+1} |S(\beta)|^{2(s-1)}K(\beta/\lambda)\lambda^{-1}\, d\,\beta$$
$$\ll \sum_{m\geq Z} P^{2(s-1)-k+\epsilon}\min(\tau^r, m^{-r})$$
$$\ll Z^{1-r}P^{2(s-1)-k+\epsilon}.$$

The lemma now follows using Hölder's inequality and the trivial estimate $S(\alpha) \ll P$. □

We need the contribution (25) from the tail to be small compared to the contribution from the major arc, that is

$$P^{\mu(1-r)}P^{s-k/2} = o\Big(P^{-rk\delta}P^{s-k}\Big)$$

or

$$\mu(r-1) > \frac{k}{2} + rk\delta.$$

Now $k \geq 3$, $k\delta < \frac{\gamma}{3} \leq \frac{1}{48}$ and $r = 4k$ so this will be satisfied if $\mu > \frac{7}{44}$. We choose a fixed value $\mu$ in the interval $\frac{2}{9} > \mu > \frac{7}{44}$. For example, we could specify $\mu = \frac{1}{5}$.

## 6. The minor arc m

Now it only remains to discuss the contribution of the pair of intervals

$$\mathbf{m} = \{\alpha : Y <\mid \alpha \mid\leq Z\}$$

to the integral (8).

An essential part is played by an estimate of Harman [9] for weighted exponential sums in place of the usual Weyl estimate. (The estimate is an extension to higher powers of an earlier result for quadratics by Ghosh [8]). We replace $S(\alpha)$ by the weighted sum

$$T(\alpha) = \sum_{\eta P < n \leq P} \Lambda(n)e(\alpha n^k)$$

which has a more powerful estimate available than that known for unweighted sums.

**Lemma 7** (Harman). *Suppose that $\alpha$ has a rational approximation $a/q$ with $(a,q) = 1$ and satisfying*

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}.$$

*Let*

$$\gamma = 4^{1-k}.$$

*Then given any real number $\epsilon > 0$ the sum $T(\alpha)$ satisfies*

$$T(\alpha) \ll P^{1+\epsilon}(q^{-1} + P^{-\frac{1}{2}} + qP^{-k})^{\gamma}, \tag{26}$$

*where the implicit constant depends only on $\epsilon$.*

Although Harman proved this estimate for sums of the form $\sum_{n\leq P}$ the estimate will also hold for sums over the range $\sum_{n\leq \eta P}$ and hence for the difference $T(\alpha)$. Using elementary estimates for the number of prime powers $q \leq P$ we see that

$$S(\alpha) - T(\alpha) \ll P^{\frac{1}{2}},$$

and therefore the bound (26) also holds for $S(\alpha)$.

**Lemma 8.** *Let $\alpha \in \mathbf{m}$ and suppose that*

$$\mid S(\alpha) \mid = A^{-1}P$$

*with*

$$A \leq P^{\frac{\gamma}{3}-2\epsilon}.$$

*Then there are coprime integers $a, q$, with $a \neq 0$, satisfying*

$$(i) \quad 1 \leq q \ll P^{\frac{1}{3}}$$
$$(ii) \quad \mid q\alpha - a \mid \ll P^{-k+\frac{1}{3}-10\epsilon}.$$

*Proof.* Let $Q = P^{k-\frac{1}{3}+10\epsilon}$, then by Dirichlet's Theorem there exist coprime integers $a, q$ with $1 \leq q \leq Q$ such that

$$\mid q\alpha - a \mid \ll P^{-k+\frac{1}{3}-10\epsilon}.$$

Choosing $u < -k + \frac{1}{3}$ close to the upper bound, for $\alpha \in \mathbf{m}$ we have $a \neq 0$.

Our hypothesis $\mid S(\alpha) \mid \geq P^{1-\frac{\gamma}{3}+2\epsilon}$ ensures that the term $P^{-\frac{1}{2}}$ in Harman's Lemma may be neglected. The condition $q \leq Q$ then gives

$$A^{-1}P = \mid S(\alpha) \mid \ll P^{1+\epsilon}(q^{-1} + P^{-\frac{1}{3}+10\epsilon})^{\gamma}.$$

Hence

$$P^{1-\frac{\gamma}{3}+2\epsilon} \ll P^{1+\epsilon}(q^{-1} + P^{-\frac{1}{3}+10\epsilon})^{\gamma}.$$

Therefore the dominant term in the bracket is $q^{-1}$ and hence

$$A^{-1}P = \mid S(\alpha) \mid \ll q^{-\gamma}P^{1+\epsilon}.$$

Thus

$$q^{\gamma} \ll AP^{\epsilon} \ll P^{\frac{\gamma}{3}}$$

and hence $q \ll P^{\frac{1}{3}}$. □

The next step is to show that on $\mathbf{m}$ one of the exponential sums $S_1$ and $S_2$ is small.

**Lemma 9.** *Let $\alpha \in \mathbf{m}$. Let $\lambda_1, \lambda_2$ be positive real numbers such that $\lambda_1/\lambda_2$ is an algebraic irrational. Then*

$$\min(|S_1(\alpha)|, |S_2(\alpha)|) \ll P^{1-\frac{\gamma}{3}+2\epsilon}. \tag{27}$$

*Proof.* If the Lemma is false then there is an infinite sequence of values $P$ for which $|S_1(\alpha)|, |S_2(\alpha)|$ are both large. By the previous lemma this implies that $\lambda_1\alpha, \lambda_2\alpha$ both have good rational approximations. More specifically, there are integers $a_1, a_2, q_1, q_2$ with $(a_i, q_i) = 1$, $a_1 a_2 \neq 0$,

$$(i) \quad 1 \leq q_i \ll P^{\frac{1}{3}}$$

and

$$(ii) \quad | q_i \lambda_i \alpha - a_i | = q_i \psi_i \ll P^{-k+\frac{1}{3}+10\epsilon}$$

for $i = 1, 2$.

Recall that $Z = P^{\mu}$, so

$$a_i \ll Z q_i \ll P^{\frac{1}{3}+\mu}.$$

Then

$$\begin{aligned} \left| \frac{\lambda_1}{\lambda_2} - \frac{a_1 q_2}{a_2 q_1} \right| &= \frac{q_2}{q_1} \left| \frac{a_1 + q_1 \psi_1}{a_2 + q_2 \psi_2} - \frac{a_1}{a_2} \right| \\ &= \frac{q_2}{q_1} \left| \frac{a_2 q_1 \psi_1 - a_1 q_2 \psi_2}{a_2 (a_2 + q_2 \psi_2)} \right| \\ &\ll P^{1-k+\mu+10\epsilon}. \end{aligned}$$

Further

$$a_2 q_1 \ll P^{\frac{2}{3}+\mu}.$$

Since $k \geq 3$ and $\mu < \frac{2}{9}$ the ratio

$$\frac{k - 1 - \mu - 10\epsilon}{\frac{2}{3} + \mu} > 2.$$

Thus the algebraic irrational $\frac{\lambda_1}{\lambda_2}$ would be approximable of order greater than 2, contradicting Roth's Theorem. This contradiction establishes the lemma. □

On the minor arc $\mathbf{m}$ reasonable control is possible only on average over $v$.

**Lemma 10.** *Let $0 < \tau < 1$. Suppose that $\mathcal{V}$ is a set of real numbers such that $| v_1 - v_2 | > 2\tau$ whenever $v_1, v_2 \in \mathcal{V}$ are distinct. Let $\lambda_1, \lambda_2$ be non-zero real numbers with $\lambda_1/\lambda_2$ algebraic and irrational. Then, in the notation introduced above, for any $\epsilon > 0$*

$$(28) \qquad \sum_{v \in \mathcal{V}} \left| \int_{\mathbf{m}} S_1(\alpha) \ldots S_s(\alpha) e(-v\alpha) K(\alpha) \, d\alpha \right|^2 \ll \tau^{2r-2} P^{2s-k-\frac{2\gamma}{3}+\epsilon}.$$

*Proof.* We work in $L_2(\mathbb{R})$ and denote the standard inner product by

$$(f, g) = \int f \bar{g} \, d\alpha.$$

For $v \in \mathbb{R}$ define

$$k_v(\alpha) = \Big(\frac{2}{3}\Big)^{\frac{1}{2}} \tau^{\frac{r}{2}} \Big(\frac{\sin(2\pi\alpha\tau/3)}{2\pi\alpha\tau/3}\Big)\Big(\frac{\sin(2\pi\alpha\tau/3n)}{2\pi\alpha\tau/3n}\Big)^{\frac{n}{2}} e(v\alpha).$$

Then $k_v \in L_2(\mathbb{R})$, and by (7) we have

$$(k_v, k_{v'}) = W(v - v').$$

Hence the set of functions $k_v$ with $v \in \mathcal{V}$ is an orthogonal family which can be normalized on multiplying by a suitable constant multiple of $\tau^{\frac{1-r}{2}}$. Bessel's inequality yields

$$\sum_{v\in\mathcal{V}} \Big|(\phi, k_v)\Big|^2 \ll \tau^{r-1}(\phi,\phi)$$

for any function $\phi \in L_2(\mathbb{R})$. We take

$$\phi(\alpha) = \begin{cases} S_1(\alpha)\dots S_s(\alpha)k_0(\alpha) & \text{if } \alpha \in \mathbf{m}, \\ 0 & \text{otherwise} \end{cases}$$

to deduce that the left hand side of the inequality proposed in the lemma

$$\ll \tau^{r-1} \int_{\mathbf{m}} \Big|S_1(\alpha)\dots S_s(\alpha)\Big|^2 K(\alpha)d\alpha. \tag{29}$$

We estimate this integral by separating $\mathbf{m}$ into two subsets. Let $\chi$ be chosen to satisfy $1 - \frac{\gamma}{3} < \chi < 1$ and take

$$\mathbf{m}_i = \{\alpha \in \mathbf{m} : \ \min(| S_1(\alpha) |, | S_2(\alpha) |) =| S_i(\alpha) |\}.$$

Then, by Lemma 5,

$$\int_{\mathbf{m}_1} \Big|S_1(\alpha)\dots S_s(\alpha)\Big|^2 K(\alpha)d\alpha \le P^{2\chi} \int_{-\infty}^{\infty} \Big|S_2(\alpha)\dots S_s(\alpha)\Big|^2 K(\alpha)d\alpha,$$

$$\ll P^{2\chi}\tau^{r-1}P^{2s-2-k+\epsilon} \ll \tau^{r-1}P^{2s-k-2\frac{\gamma}{3}+2\epsilon}$$

on choosing $\chi$ close to $1 - \frac{\gamma}{3}$.

By symmetry the same bound holds for $\mathbf{m}_2$ in place of $\mathbf{m}_1$, and hence for $\mathbf{m}$ itself. $\square$

## 7. Conclusion

It is now easy to deduce the following result which implies Theorem 3.

**Theorem 4.** *Let $k$ be a positive integer and let $s$ be chosen as in Theorem 3. Let $0 < \tau < 1$. Suppose that $\mathcal{V}(X)$ is a set of real numbers contained in $[\frac{1}{2}X, X]$ such that $| v_1 - v_2 |> 2\tau$ whenever $v_1, v_2 \in \mathcal{V}$ are distinct. Let $\lambda_1, \dots, \lambda_s \ge 1$ be real numbers and $\lambda_1/\lambda_2$ be algebraic and irrational. Then the number of $v \in \mathcal{V}$ for which $| \lambda_1 p_1^k + \ldots + \lambda_s p_s^k - v |\le \tau$ has no solution in primes $p_1, \dots, p_s$ does not exceed $O(\tau^{-2}X^{1-\frac{2\gamma}{3k}+\epsilon})$.*

*Proof.* We use the notation from the earlier sections. Suppose that $\mathcal{N}_v = 0$ for some $v \in \mathcal{V}$. Then we must have

$$\left| \int_{\mathbf{m}} S_1(\alpha) \ldots S_s(\alpha) e(-v\alpha) K(\alpha) d\alpha \right| \gg \tau^r P^{s-k}. \tag{30}$$

Hence

$$\#\{v \in \mathcal{V} : \mathcal{N}_v = 0\}$$
$$\ll \tau^{-2r} P^{2(k-s)} \sum_{v \in \mathcal{V}} \left| \int_{\mathbf{m}} S_1(\alpha) \ldots S_s(\alpha) e(-v\alpha) K(\alpha) d\alpha \right|^2,$$

and this is in turn bounded by

$$\tau^{-2r} P^{2(k-s)} \tau^{2r-2} P^{2s-k-2\frac{\gamma}{3}+\epsilon} \ll \tau^{-2} X^{1-\frac{2\gamma}{3k}+\epsilon}.$$

□

To deduce Theorem 3 we may suppose that $\delta < \frac{\gamma}{3k}$ for otherwise the estimate in Theorem 3 is trivial. Take $\tau = X^{-\delta}$ in Theorem 2. Then

$$\#\{v \in \mathcal{V} : \frac{1}{2}X < v \leq X, \mid \lambda_1 p_1^k + \ldots + \lambda_s p_s^k - v \mid < X^{-\delta} \text{not soluble}\}$$
$$\ll X^{1-\frac{2\gamma}{3k}+2\delta+\epsilon}.$$

The condition $\lambda_i \geq 1$ is easily removed. We may assume $\lambda_1 \geq \ldots \geq \lambda_s$ by renumbering the variables. If $\lambda_s < 1$ consider the inequality

$$\mid (\lambda_1/\lambda_s) p_1^2 + \ldots + p_s^k - (v/\lambda_s) \mid \leq \lambda_s^{-1} v^{-\delta}$$

and note that the numbers $v/\lambda_s$ are still well-spaced. To obtain Theorem 2, we now replace $X$ by $2^{-l}X$ and sum over $1 \leq l \ll \log X$.

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