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Regular infinite dimensional Lie groups

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Abstract. Regular Lie groups are infinite dimensional Lie groups with the property that smooth curves in the Lie algebra integrate to smooth curves in the group in a smooth way (an 'evolution operator' exists). Up to now all known smooth Lie groups are regular. We show in this paper that regular Lie groups allow us to push surprisingly far the geometry of principal bundles: parallel transport exists and flat connections integrate to horizontal foliations as in finite dimensions. As consequences we obtain that Lie algebra homomorphisms integrate to Lie group homomorphisms, if the source group is simply connected and the image group is regular.

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1. Introduction

On the one hand the theory of infinite dimensional Lie groups and Lie algebras is very rich: Kac-Moody algebras and the Virasoro algebra have a rich and important theory of representations and many applications, and subgroups of diffeomorphism groups play an extremely important role in differential topology, differential geometry, and general relativity. On the other hand classical Lie theory carries over to them only in rare pieces: There are (even Banach) Lie algebras without Lie groups, see [3] and [7], and the exponential mapping in general is not surjective onto any neighborhood of the identity. The most surprising result in this direction is [5], where it is shown that in the diffeomorphism group of any manifold of dimension at least 2 one can find a smooth curve of diffeomorphisms starting at the identity such that the points of this curve form a set of generators for a free subgroup of the diffeomorphism group which meets the image of the exponential mapping only in the identity.

In view of these difficulties the theory of infinite dimensional Lie groups and Lie algebras can be pushed surprisingly far: Exponential mappings are

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unique if they exist, and then one can give a formula for their derivatives, see [6] and 5.9 below.

In [14] and [15] the notion of a 'regular Fréchet Lie group' was introduced in an attempt to find conditions which ensure the existence of exponential mappings: certain product integrals were required to converge. Their main result was that the invertible Fourier integral operators form a regular Fréchet Lie group with the space of pseudo differential operators as Lie algebra, see [19], and also [1] for a more general group of Fourier integral operators, but without regularity. In [13] Milnor weakened this to the assumption that smooth curves in the Lie algebra integrate to smooth curves in the group in a smooth way (an 'evolution operator' exists), and it is this notion which we take up in this paper, except that we introduce it for general Lie groups modelled on locally convex spaces, where we use the convenient calculus from [4]. Up to now nobody has found a non-regular Lie group.

We show in this paper that for regular Lie groups one can push surprisingly far the geometry of principal bundles: parallel transport exists and flat connections integrate to horizontal foliations as in finite dimensions. As consequences we obtain that Lie algebra homomorphisms intergrate to Lie group homomorphisms, if the source group is simply connected and the image group is regular.

The actual development is quite involved. We start with general infinite dimensional Lie groups in Section 3. For a detailed study of the evolution operator of regular Lie groups (cf. 5.3) we need in 5.9 the Maurer-Cartan equation for right (or left) logarithmic derivatives (cf. 5.1) of mappings with values in the Lie group, and this we can get only by looking at principal connections. Thus Section 4 treats principal bundles, connections, and curvature as far as we shall need them. In Section 5 we then prove the above mentioned strong existence results and discuss regular Lie groups. Principal bundles with regular structure groups are treated in Section 6. The last section develops rudiments of Lie theory for regular Lie groups as sketched above.

These results were obtained in a systematic study of properties of regular Lie groups for the book in preparation [11], where also many of the known Lie groups are treated and shown to be regular.

2. Calculus of smooth mappings

The traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces a whole flock of different theories were developped, most of them rather complicated and not really convincing. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. We shall use in this paper the calculus in infinite dimensions as developed in [4].

2.1. Convenient vector spaces. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous

— this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, but only on its associated bornology (system of bounded sets).

E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(i) For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.

(ii) A curve $c : \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E'$, where E' is the dual consisting of all continuous linear functionals on E.

(iii) Any Mackey-Cauchy-sequence (i.e., $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a weak completeness requirement.

The final topology with respect to all smooth curves is called the c^∞-topology on E, which then is denoted by $c^\infty E$. For Fréchet spaces it coincides with the given locally convex topology, but on the space $\mathcal{D}$ of test functions with compact support on $\mathbb{R}$ it is strictly finer.

2.2. Smooth mappings. Let E and F be locally convex vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

2.3. Results. *The main properties of smooth calculus are the following.*

(i) *For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Even on $\mathbb{R}^2$ this is non-trivial, see* [2].

(ii) *Multilinear mappings are smooth if and only if they are bounded.*

(iii) *If $f : E \supseteq U \to F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.*

(iv) *The chain rule holds.*

(v) *The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection*

$$C^\infty(U, F) \to \prod_{c \in C^\infty(\mathbb{R}, U)} C^\infty(\mathbb{R}, F) \to \prod_{\substack{c \in C^\infty(\mathbb{R}, U)\\ \lambda \in F'}} C^\infty(\mathbb{R}, \mathbb{R}).$$

(vi) *The exponential law holds:*

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomeorphism of convenient vector spaces. Note that this is the main assumption of variational calculus.

(vii) *A linear mapping $f : E \to C^\infty(V, G)$ is smooth (bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. This is called the smooth uniform boundedness theorem and it is quite applicable.*

2.4 Counterexamples in infinite dimensions against common beliefs on ordinary differential equations. Let $E := s$ be the Fréchet space of rapidly decreasing sequences (Note that by the theory of Fourier series we have $s = C^\infty(S^1, \mathbb{R})$) and consider the continuous linear operator $T : E \to E$ given by $T(x_0, x_1, x_2, \dots) := (0, 1^2 x_1, 2^2 x_2, 3^2 x_3, \dots)$. The ordinary linear differential equation $x'(t) = T(x(t))$ with constant coefficients has no solution in s for certain initial values. By recursion one sees that the general solution should be given by

$$x_n(t) = \sum_{i=0}^{n} \left(\tfrac{n!}{i!}\right)^2 x_i(0) \tfrac{t^{n-i}}{(n-i)!}$$

If the initial value is a finite sequence, say $x_n(0) = 0$ for $n > N$ and $x_N(0) \neq 0$, then

$$\begin{aligned}
x_n(t) &= \sum_{i=0}^{N} \left(\tfrac{n!}{i!}\right)^2 x_i(0) \tfrac{t^{n-i}}{(n-i)!} \\
&= \tfrac{(n!)^2}{(n-N)!}\, t^{n-N} \textstyle\sum_{i=0}^{N} \left(\tfrac{1}{i!}\right)^2 x_i(0) \tfrac{(n-N)!}{(n-i)!}\, t^{N-i}; \\
|x_n(t)| &\geq \tfrac{(n!)^2}{(n-N)!}\, |t|^{n-N} \left(|x_N(0)| \left(\tfrac{1}{N!}\right)^2 - \textstyle\sum_{i=0}^{N-1} \left(\tfrac{1}{i!}\right)^2 |x_i(0)| \tfrac{(n-N)!}{(n-i)!} |t|^{N-i}\right) \\
&\geq \tfrac{(n!)^2}{(n-N)!}\, |t|^{n-N} \left(|x_N(0)| \left(\tfrac{1}{N!}\right)^2 - \textstyle\sum_{i=0}^{N-1} \left(\tfrac{1}{i!}\right)^2 |x_i(0)| |t|^{N-i}\right)
\end{aligned}$$

where the first factor does not lie in the space s of rapidly decreasing sequences and where the second factor is larger than $\varepsilon > 0$ for t small enough. So at least for a dense set of initial values this differential equation has no local solution.

This shows also, that the theorem of Frobenius is wrong, in the following sense: The vector field $x \mapsto T(x)$ generates a 1-dimensional subbundle E of the tangent bundle on the open subset $s \setminus 0$. It is involutive since it is 1-dimensional. But through points representing finite sequences there exist no local integral submanifolds (M with $TM = E|M$). Namely, if c were a smooth nonconstant curve with $c'(t) = f(t).T(c(t))$ for some smooth function f, then $x(t) := c(h(t))$ would satisfy $x'(t) = T(x(t))$, where h is a solution of $h'(t) = 1/f(h(t))$.

As next example consider $E := \mathbb{R}^{\mathbb{N}}$ and the continuous linear operator $T : E \to E$ given by $T(x_0, x_1, \dots) := (x_1, x_2, \dots)$. The corresponding differential equation has solutions for every initial value $x(0)$, since the coordinates must satisfy the recusive relations $x_{k+1}(t) = x_k'(t)$ and hence any smooth functions $x_0 : \mathbb{R} \to \mathbb{R}$ gives rise to a solution $x(t) := (x_0^{(k)}(t))_k$ with initial value $x(0) = (x_0^{(k)}(0))_k$. So by Borel's theorem there exist solutions to this equation for any initial value and the difference of any two functions with same initial value is an arbitray infinite flat function. Thus the solutions are far from being unique. Note that $\mathbb{R}^{\mathbb{N}}$ is a topological direct summand in $C^\infty(\mathbb{R}, \mathbb{R})$ via the projection $f \mapsto (f(n))_n$, and hence the same situation occurs in $C^\infty(\mathbb{R}, \mathbb{R})$.

Let now $E := C^\infty(\mathbb{R}, \mathbb{R})$ and consider the continuous linear operator $T : E \to E$ given by $T(x) := x'$. Let $x : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ be a solution of the equation $x'(t) = T(x(t))$. In terms of $\hat{x} : \mathbb{R}^2 \to \mathbb{R}$ this says $\frac{\partial}{\partial t}\hat{x}(t, s) = \frac{\partial}{\partial s}\hat{x}(t, s)$. Hence $r \mapsto \hat{x}(t - r, s + r)$ has vanishing derivative everywhere and so this function is constant, and in particular $x(t)(s) = \hat{x}(t, s) = \hat{x}(0, s + t) = x(0)(s + t)$. Thus we have a smooth solution x uniquely determined by the initial value $x(0) \in C^\infty(\mathbb{R}, \mathbb{R})$ which even describes a flow for the vector field T in the sense of 2.7 below. In general this solution is however not real-analytic, since for any $x(0) \in C^\infty(\mathbb{R},\mathbb{R})$, which is not real-analytic in a neighborhood of a point s the composite $\mathrm{ev}_s \circ x = x(s+\ \ )$ is not real-analytic around 0.

2.5. Manifolds. In the sequel we shall use smooth manifolds M modelled on c^∞-open subsets of convenient vector spaces. See [10] for an account of this. Since we shall need it we also include some results on vector fields and their flows.

2.6. Lemma. *Consider vector fields $X_i \in C^\infty(TM)$ and $Y_i \in C^\infty(TN)$ for $i = 1,2$, and a smooth mapping $f : M \to N$. If X_i and Y_i are f-related for $i = 1,2$, i.e., $Tf \circ X_i = Y_i \circ f$, then also $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.*

Proof. We choose $h \in C^\infty(N,\mathbb{R})$ and we view each vector field as a derivation. This is possible if we either have smooth partitions of unity or if we pass to sheaves of smooth functions. The converse is wrong in general, see [10] and [11]. Then by assumption we have $Tf \circ X_i = Y_i \circ f$, thus:

$$\begin{aligned}&(X_i(h\circ f))(x) = X_i(x)(h\circ f) = (T_xf.X_i(x))(h) =\\ &= (Tf\circ X_i)(x)(h) = (Y_i\circ f)(x)(h) = Y_i(f(x))(h) = (Y_i(h))(f(x)),\end{aligned}$$

so $X_i(h\circ f) = (Y_i(h))\circ f$, and we may continue:

$$\begin{aligned}[X_1,X_2](h\circ f) &= X_1(X_2(h\circ f)) - X_2(X_1(h\circ f)) =\\ &= X_1(Y_2(h)\circ f) - X_2(Y_1(h)\circ f) =\\ &= Y_1(Y_2(h))\circ f - Y_2(Y_1(h))\circ f = [Y_1,Y_2](h)\circ f.\end{aligned}$$

But this means $Tf \circ [X_1, X_2] = [Y_1, Y_2] \circ f$. ∎

In particular if $f : M \to N$ is a local diffeomorphism (so $(T_xf)^{-1}$ makes sense for each $x \in M$), then for $Y \in C^\infty(TN)$ a vector field $f^*Y \in C^\infty(TM)$ is defined by $(f^*Y)(x) = (T_xf)^{-1}.Y(f(x))$. The linear mapping $f^* : C^\infty(TN) \to C^\infty(TM)$ is then a Lie algebra homomorphism.

2.7. The flow of a vector field. Let $X \in C^\infty(TM)$ be a vector field. A *local flow* Fl^X for X is a smooth mapping $\mathrm{Fl}^X : M \times \mathbb{R} \supset U \to M$ defined on a c^∞-open neighborhood U of $M \times 0$ such that

(i) $\frac{d}{dt}\mathrm{Fl}^X_t(x) = X(\mathrm{Fl}^X_t(x))$.
(ii) $\mathrm{Fl}^X_0(x) = x$ for all $x \in M$.
(iii) $U \cap (\{x\} \times \mathbb{R})$ is a connected open interval.
(iv) $\mathrm{Fl}^X_{t+s} = \mathrm{Fl}^X_t \circ \mathrm{Fl}^X_s$ holds in the following sense. If the right hand side exists then also the left hand side exists and we have equality. Moreover: If Fl^X_s exists, then the existence of both sides is equivalent and they are equal.

2.8. Lemma. *Let $X \in C^\infty(TM)$ be a vector field which admits a local flow Fl^X_t. Then each for integral curve c of X we have $c(t) = \mathrm{Fl}^X_t(c(0))$, thus there exists a unique maximal flow. Furthermore X is Fl^X_t-related to itself, i.e., $T(\mathrm{Fl}^X_t)\circ X = X \circ \mathrm{Fl}^X_t$.*

Proof. We compute

$$\begin{aligned}\tfrac{d}{dt}\mathrm{Fl}^X(-t,c(t)) &= -\tfrac{d}{ds}|_{s=-t}\mathrm{Fl}^X(s,c(t)) + \tfrac{d}{ds}|_{s=t}\mathrm{Fl}^X(-t,c(s))\\ &= -\tfrac{d}{ds}|_{s=0}\mathrm{Fl}^X_{-t}\mathrm{Fl}^X(s,c(t)) + T(\mathrm{Fl}^X_{-t}).c'(t)\\ &= -T(\mathrm{Fl}^X_{-t}).X(c(t)) + T(\mathrm{Fl}^X_{-t}).X(c(t)) = 0,\end{aligned}$$

Thus $\mathrm{Fl}^X_{-t}(c(t)) = c(0)$ is constant, so $c(t) = \mathrm{Fl}^X_t(c(0))$. For the second assertion we have $X\circ\mathrm{Fl}^X_t = \frac{d}{dt}\mathrm{Fl}^X_t = \frac{d}{ds}|_0\mathrm{Fl}^X_{t+s} = \frac{d}{ds}|_0(\mathrm{Fl}^X_t\circ\mathrm{Fl}^X_s) = T(\mathrm{Fl}^X_t)\circ\frac{d}{ds}|_0\mathrm{Fl}^X_s = T(\mathrm{Fl}^X_t)\circ X$. ∎

2.9. Lemma. *Let $X\in C^\infty(TM)$ and $Y\in C^\infty(TN)$ be f-related vector fields for a smooth mapping $f:M\to N$ which have local flows Fl^X and Fl^Y. Then we have $f\circ\mathrm{Fl}^X_t = \mathrm{Fl}^Y_t\circ f$, whenever both sides are defined.*
Moreover, if f is a diffeomorphism we have $\mathrm{Fl}^{f^*Y}_t = f^{-1}\circ\mathrm{Fl}^Y_t\circ f$ in the following sense: If one side exists then also the other and they are equal.

For $f = Id_M$ this again implies that if there exists a flow then there exists a unique maximal flow Fl^X_t.

Proof. We have $Y\circ f = Tf\circ X$ and thus (using 2.7.3 and 2.8) for small t we get

$$\begin{aligned}\tfrac{d}{dt}(\mathrm{Fl}^Y_t\circ f\circ\mathrm{Fl}^X_{-t}) &= Y\circ\mathrm{Fl}^Y_t\circ f\circ\mathrm{Fl}^X_{-t} - T(\mathrm{Fl}^Y_t)\circ Tf\circ X\circ\mathrm{Fl}^X_{-t}\\ &= T(\mathrm{Fl}^Y_t)\circ Y\circ f\circ\mathrm{Fl}^X_{-t} - T(\mathrm{Fl}^Y_t)\circ Tf\circ X\circ\mathrm{Fl}^X_{-t} = 0.\end{aligned}$$

So $(\mathrm{Fl}^Y_t\circ f\circ\mathrm{Fl}^X_{-t})(x) = f(x)$ or $f(\mathrm{Fl}^X_t(x)) = \mathrm{Fl}^Y_t(f(x))$ for small t. By the flow properties (2.7.4) we get the result by a connectedness argument as follows: In the common interval of definition we consider the closed subset $J_x := \{t : f(\mathrm{Fl}^X_t(x)) = \mathrm{Fl}^Y_t(f(x))\}$. This set is also open since for $t\in J_x$ and small $|s|$ we have $f(\mathrm{Fl}^X_{t+s}(x)) = f(\mathrm{Fl}^X_s(\mathrm{Fl}^X_t(x))) = \mathrm{Fl}^Y_s(f(\mathrm{Fl}^X_t(x))) = \mathrm{Fl}^Y_s(\mathrm{Fl}^Y_t(f(x))) = \mathrm{Fl}^Y_{t+s}(f(x))$. ∎

2.10. The Lie derivative. We will meet situations (in 4.2) where we do not know that the flow of X exists but where we will be able to produce the following assumption: Suppose that $\varphi:\mathbb{R}\times M\supset U\to M$ is a smooth mapping such that $(t,x)\mapsto(t,\varphi(t,x)=\varphi_t(x))$ is a diffeomorphism $U\to V$, where U and V are open neighborhoods of $\{0\}\times M$ in $\mathbb{R}\times M$, and such that $\varphi_0 = \mathrm{Id}_M$ and $\frac{\partial}{\partial t}\big|_0\varphi_t = X\in C^\infty(TM)$. Then again $\frac{d}{dt}|_0(\varphi_t)^*f = \frac{d}{dt}|_0 f\circ\varphi_t = df\circ X = X(f)$.

Lemma. *In this situation we have for $Y\in C^\infty(TM)$, and for a k-form $\omega\in\Omega^k(M)$:*

$$\tfrac{d}{dt}|_0(\varphi_t)^*Y = [X,Y],\qquad \tfrac{\partial}{\partial t}|_0(\varphi_t)^*\omega = \mathcal{L}_X\omega$$

Proof. Let $f\in C^\infty(M,\mathbb{R})$ be a function and consider the mapping $\alpha(t,s) := Y(\varphi(t,x))(f\circ\varphi_s)$, which is locally defined near 0. It satisfies

$$\begin{aligned}&\alpha(t,0) = Y(\varphi(t,x))(f),\\ &\alpha(0,s) = Y(x)(f\circ\varphi_s),\\ &\tfrac{\partial}{\partial t}\alpha(0,0) = \tfrac{\partial}{\partial t}\big|_0 Y(\varphi(t,x))(f) = \tfrac{\partial}{\partial t}\big|_0(Yf)(\varphi(t,x)) = X(x)(Yf),\\ &\tfrac{\partial}{\partial s}\alpha(0,0) = \tfrac{\partial}{\partial s}|_0 Y(x)(f\circ\varphi_s) = Y(x)\tfrac{\partial}{\partial s}|_0(f\circ\varphi_s) = Y(x)(Xf).\end{aligned}$$

So $\frac{\partial}{\partial u}|_0 \alpha(u,-u) = [X,Y]_x(f)$. But on the other hand we have

$$\begin{aligned}
\tfrac{\partial}{\partial u}|_0 \alpha(u,-u) &= \tfrac{\partial}{\partial u}|_0 Y(\varphi(u,x))(f\circ\varphi_{-u}) = \\
&= \tfrac{\partial}{\partial u}|_0 \left(T(\varphi_{-u})\circ Y\circ\varphi_u\right)_x(f) \\
&= (\tfrac{d}{dt}|_0(\varphi_t)^*Y)_x(f).
\end{aligned}$$

We may identify k-forms on M with $C^\infty(M,\mathbb{R})$-multilinear mappings on vector fields (if smooth partitions of unity exist or if we pass to sheaves of vector fields). The converse is wrong in general, see [11]. For (local) vector fields $Y_i \in C^\infty(TM)$ we have

$$\begin{aligned}
(\tfrac{\partial}{\partial t}|_0(\varphi_t)^*\omega)(Y_1,\dots,Y_k) &= \tfrac{\partial}{\partial t}|_0(\omega((\varphi_{-t})^*Y_1,\dots,(\varphi_{-t})^*Y_k)\circ\varphi_t) \\
&= \sum_{j=1}^k \omega(Y_1,\dots,\tfrac{\partial}{\partial t}|_0(\varphi_{-t})^*Y_j,\dots,Y_k) + \tfrac{\partial}{\partial t}|_0(\varphi_t)^*(\omega(Y_1,\dots,Y_p) \\
&= X(\omega(Y_1,\dots,Y_k)) - \sum_{i=1}^k \omega(Y_1,\dots,[X,Y_i],\dots,Y_k) \\
&= \mathcal{L}_X\omega(Y_1,\dots,Y_k).
\end{aligned}$$

This is the usual formula for the Lie derivative. ■

3. Lie groups

3.1. Definition. A *Lie group* G is a smooth manifold modelled on c^∞-open subsets of a convenient vector space, and a group such that the multiplication $\mu : G\times G\to G$ and the inversion $\nu : G\to G$ are smooth. We shall use the following notation:
$\mu : G\times G\to G$, multiplication, $\mu(x,y) = x.y$.
$\mu_a : G\to G$, left translation, $\mu_a(x) = a.x$.
$\mu^a : G\to G$, right translation, $\mu^a(x) = x.a$.
$\nu : G\to G$, inversion, $\nu(x) = x^{-1}$.
$e\in G$, the unit element.

3.2. Lemma. *The tangent mapping $T_{(a,b)}\mu : T_aG\times T_bG\to T_{ab}G$ is given by*

$$T_{(a,b)}\mu.(X_a,Y_b) = T_a(\mu^b).X_a + T_b(\mu_a).Y_b.$$

and $T_a\nu : T_aG\to T_{a^{-1}}G$ is given by

$$T_a\nu = -T_e(\mu^{a^{-1}}).T_a(\mu_{a^{-1}}) = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).$$

Proof. Let $\mathrm{ins}_a : G\to G\times G$, $\mathrm{ins}_a(x) = (a,x)$ be the right insertion and let $\mathrm{ins}^b : G\to G\times G$, $\mathrm{ins}^b(x) = (x,b)$ be the left insertion. Then we have

$$\begin{aligned}
T_{(a,b)}\mu.(X_a,Y_b) &= T_{(a,b)}\mu.(T_a(\mathrm{ins}^b).X_a + T_b(\mathrm{ins}_a).Y_b) = \\
&= T_a(\mu\circ\mathrm{ins}^b).X_a + T_b(\mu\circ\mathrm{ins}_a).Y_b = T_a(\mu^b).X_a + T_b(\mu_a).Y_b.
\end{aligned}$$

Now we differentiate the equation $\mu(a,\nu(a)) = e$; this gives in turn

$$0_e = T_{(a,a^{-1})}\mu.(X_a, T_a\nu.X_a) = T_a(\mu^{a^{-1}}).X_a + T_{a^{-1}}(\mu_a).T_a\nu.X_a,$$
$$T_a\nu.X_a = -T_e(\mu_a)^{-1}.T_a(\mu^{a^{-1}}).X_a = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).X_a$$

and the proof is complete. ∎

3.3. Invariant vector fields and Lie algebras. Let G be a (real) Lie group. A vector field ξ on G is called *left invariant*, if $\mu_a^*\xi = \xi$ for all $a \in G$, where $\mu_a^*\xi = T(\mu_{a^{-1}}) \circ \xi \circ \mu_a$. Since we have $\mu_a^*[\xi,\eta] = [\mu_a^*\xi, \mu_a^*\eta]$, the space $\mathfrak{X}_L(G)$ of all left invariant vector fields on G is closed under the Lie bracket, so it is a sub Lie algebra of $\mathfrak{X}(G)$. Any left invariant vector field ξ is uniquely determined by $\xi(e) \in T_eG$, since $\xi(a) = T_e(\mu_a).\xi(e)$. Thus the Lie algebra $\mathfrak{X}_L(G)$ of left invariant vector fields is linearly isomorphic to T_eG, and on T_eG the Lie bracket on $\mathfrak{X}_L(G)$ induces a Lie algebra structure, whose bracket is again denoted by $[\ ,\ ]$. This Lie algebra will be denoted as usual by $\mathfrak{g}$, sometimes by Lie(G).

We will also give a name to the isomorphism with the space of left invariant vector fields: $L : \mathfrak{g} \to \mathfrak{X}_L(G)$, $X \mapsto L_X$, where $L_X(a) = T_e\mu_a.X$. Thus $[X,Y] = [L_X, L_Y](e)$.

Similarly a vector field η on G is called *right invariant*, if $(\mu^a)^*\eta = \eta$ for all $a \in G$. If ξ is left invariant, then $\nu^*\xi$ is right invariant. The right invariant vector fields form a sub Lie algebra $\mathfrak{X}_R(G)$ of $\mathfrak{X}(G)$, which is again linearly isomorphic to T_eG and induces the negative of the Lie algebra structure on T_eG. We will denote by $R : \mathfrak{g} = T_eG \to \mathfrak{X}_R(G)$ the isomorphism discussed, which is given by $R_X(a) = T_e(\mu^a).X$.

3.4. Lemma. *If L_X is a left invariant vector field and R_Y is a right invariant one, then $[L_X, R_Y] = 0$. So if the flows of L_X and R_Y exist, they commute.*

Proof. We consider the vector field $0 \times L_X \in \mathfrak{X}(G \times G)$, given by $(0 \times L_X)(a,b) = (0_a, L_X(b))$. Then $T_{(a,b)}\mu.(0_a, L_X(b)) = T_a\mu^b.0_a + T_b\mu_a.L_X(b) = L_X(ab)$, so $0 \times L_X$ is μ-related to L_X. Likewise $R_Y \times 0$ is μ-related to R_Y. But then $0 = [0 \times L_X, R_Y \times 0]$ is μ-related to $[L_X, R_Y]$ by 2.6. Since μ is surjective, $[L_X, R_Y] = 0$ follows. ∎

3.5. Lemma. *Let $\varphi : G \to H$ be a smooth homomorphism of Lie groups. Then $\varphi' := T_e\varphi : \mathfrak{g} = T_eG \to \mathfrak{h} = T_eH$ is a Lie algebra homomorphism.*

Proof. For $X \in \mathfrak{g}$ and $x \in G$ we have

$$\begin{aligned} T_x\varphi.L_X(x) &= T_x\varphi.T_e\mu_x.X = T_e(\varphi \circ \mu_x).X \\ &= T_e(\mu_{\varphi(x)} \circ \varphi).X = T_e(\mu_{\varphi(x)}).T_e\varphi.X = L_{\varphi'(X)}(\varphi(x)). \end{aligned}$$

So L_X is φ-related to $L_{\varphi'(X)}$. By 2.6 the field $[L_X, L_Y] = L_{[X,Y]}$ is φ-related to $[L_{\varphi'(X)}, L_{\varphi'(Y)}] = L_{[\varphi'(X),\varphi'(Y)]}$. So we have $T\varphi \circ L_{[X,Y]} = L_{[\varphi'(X),\varphi'(Y)]} \circ \varphi$. If we evaluate this at e the result follows. ∎

3.6. One parameter subgroups. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A *one parameter subgroup* of G is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$, i.e. a smooth curve α in G with $\alpha(s + t) = \alpha(s).\alpha(t)$, and hence $\alpha(0) = e$.

Note that a smooth mapping $\beta : (-\varepsilon, \varepsilon) \to G$ satisfying $\beta(t)\beta(s) = \beta(t + s)$ for $|t|$, $|s|$, $|t + s| < \varepsilon$ is the restriction of a one parameter subgroup. Namely, choose $0 < t_0 < \varepsilon/2$. Any $t \in \mathbb{R}$ can be uniquely written as $t = N.t_0 + t'$ for $0 \le t' < t_0$ and $N \in \mathbb{Z}$. Put $\alpha(t) = \beta(t_0)^N \beta(t')$. The required properties are easy to check.

Lemma. *Let $\alpha : \mathbb{R} \to G$ be a smooth curve with $\alpha(0) = e$. Let $X \in \mathfrak{g}$. Then the following assertions are equivalent.*

(i) *α is a one parameter subgroup with $X = \frac{\partial}{\partial t}\big|_0 \alpha(t)$.*

(ii) *$\alpha(t)$ is an integral curve of the left invariant vector field L_X, and also an integral curve of the right invariant vector field R_X.*

(iii) *$\mathrm{Fl}^{L_X}(t, x) := x.\alpha(t)$ (or $\mathrm{Fl}^{L_X}_t = \mu^{\alpha(t)}$) is the (unique by 2.9) global flow of L_X in the sense of 2.7.*

(iv) *$\mathrm{Fl}^{R_X}(t, x) := \alpha(t).x$ (or $\mathrm{Fl}^{R_X}_t = \mu_{\alpha(t)}$) is the (unique) global flow of R_X.*

Moreover each of these properties determines α uniquely.

Proof. (i) $\Longrightarrow$ (iii). We have $\frac{d}{dt} x.\alpha(t) = \frac{d}{ds}|_0 x.\alpha(t + s) = \frac{d}{ds}|_0 x.\alpha(t).\alpha(s) = \frac{d}{ds}|_0 \mu_{x.\alpha(t)} \alpha(s) = T_e(\mu_{x.\alpha(t)}).\frac{d}{ds}|_0 \alpha(s) = L_X(x.\alpha(t))$. Since it is obviously a flow, we have (iii).

(iii) $\Longleftrightarrow$ (iv). We have $\mathrm{Fl}^{\nu^* \xi}_t = \nu^{-1} \circ \mathrm{Fl}^{\xi}_t \circ \nu$ by 2.9. Therefore we have by 3.3

$$\begin{aligned}(\mathrm{Fl}^{R_X}_t(x^{-1}))^{-1} &= (\nu \circ \mathrm{Fl}^{R_X}_t \circ \nu)(x) = \mathrm{Fl}^{\nu^* R_X}_t(x) \\ &= \mathrm{Fl}^{-L_X}_t(x) = \mathrm{Fl}^{L_X}_{-t}(x) = x.\alpha(-t).\end{aligned}$$

So $\mathrm{Fl}^{R_X}_t(x^{-1}) = \alpha(t).x^{-1}$, and $\mathrm{Fl}^{R_X}_t(y) = \alpha(t).y$.

(iii) and (iv) together clearly imply (ii).

(ii) $\Longrightarrow$ (i). This is a consequence of the following result.

Claim. *Consider two smooth curves $\alpha, \beta : \mathbb{R} \to G$ with $\alpha(0) = e = \beta(0)$ which satify the two differential equations*

$$\begin{aligned}\tfrac{d}{dt}\alpha(t) &= L_X(\alpha(t)) \\ \tfrac{d}{dt}\beta(t) &= R_X(\beta(t)).\end{aligned}$$

Then $\alpha = \beta$ and it is a 1-parameter subgroup.

We have $\alpha = \beta$ since

$$\begin{aligned}\tfrac{d}{dt}(\alpha(t)\beta(-t)) &= T\mu^{\beta(-t)}.L_X(\alpha(t)) - T\mu_{\alpha(t)}.R_X(\beta(-t)) \\ &= T\mu^{\beta(-t)}.T\mu_{\alpha(t)}.X - T\mu_{\alpha(t)}.T\mu^{\beta(-t)}.X = 0.\end{aligned}$$

Next we calculate for fixed s

$$\tfrac{d}{dt}(\beta(t - s)\beta(s)) = T\mu^{\beta(s)}.R_X(\beta(t - s)) = R_X(\beta(t - s)\beta(s)).$$

Hence by the first part of the proof $\beta(t - s)\beta(s) = \alpha(t) = \beta(t)$.

The statement about uniqueness follows from 2.9, or from the claim. ∎

3.7. Definition. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We say that G admits an *exponential mapping* if there exists a smooth mapping $\exp : \mathfrak{g} \to G$ such that $t \mapsto \exp(tX)$ is the (unique by 3.6) 1-parameter subgroup with tangent vector X at 0. Then we have by 3.6

(i) $\mathrm{Fl}^{L_X}(t,x) = x.\exp(tX)$.

(ii) $\mathrm{Fl}^{R_X}(t,x) = \exp(tX).x$.

(iii) $\exp(0) = e$ and $T_0\exp = Id : T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$ since $T_0\exp.X = \frac{d}{dt}|_0 \exp(0 + t.X) = \frac{d}{dt}|_0 \mathrm{Fl}^{L_X}(t,e) = X$.

(iv) Let $\varphi : G \to H$ be a smooth homomorphism of between Lie groups admitting exponential mappings. Then the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi'"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes, since $t \mapsto \varphi(\exp^G(tX))$ is a one parameter subgroup of H and $\frac{d}{dt}|_0 \varphi(\exp^G tX) = \varphi'(X)$, so $\varphi(\exp^G tX) = \exp^H(t\varphi'(X))$.

3.8. Remarks. In [14], [15] Omori, Maeda and Yoshioka gave conditions under which a smooth Lie group modelled on Fréchet spaces admits exponential mappings. We shall elaborate on this notion in 5.3 below. They called this 'regular Fréchet Lie groups'. We do not know of any smooth Fréchet Lie group which does not admit an exponential mapping.

If G admits an exponential mapping, it follows from 3.7.(iii) that exp is a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of e in G, if a suitable inverse function theorem is applicable. This is true for example for smooth Banach Lie groups, also for gauge groups, but it is wrong for diffeomorphism groups, see [5].

If E is a Banach space, then in the Banach Lie group $GL(E)$ of all bounded linear automorphisms of E the exponential mapping is given by the von Neumann series $\exp(X) = \sum_{i=0}^{\infty} \frac{1}{i!} X^i$.

If G is connected with exponential mapping and $U \subset \mathfrak{g}$ is open with $0 \in U$, then one may ask whether the group generated by $\exp(U)$ equals G. Note that this is a normal subgroup. So if G is simple, the answer is yes. This is true for connected components of diffeomorphism groups and many of their important subgroups.

Results on weakened versions of the Baker-Campbell-Hausdorff formula can be found in [18].

3.9. The adjoint representation. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $a \in G$ we define $\mathrm{conj}_a : G \to G$ by $\mathrm{conj}_a(x) = axa^{-1}$. It is called the *conjugation* or the *inner automorphism* by $a \in G$. This defines a smooth action of G on itself by automorphisms.

The adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g}) \subset L(\mathfrak{g}, \mathfrak{g})$ is given by $\mathrm{Ad}(a) = (\mathrm{conj}_a)' = T_e(\mathrm{conj}_a) : \mathfrak{g} \to \mathfrak{g}$ for $a \in G$. By 3.5 $\mathrm{Ad}(a)$ is a Lie algebra homomorphism. By 3.2 we have $\mathrm{Ad}(a) = T_e(\mathrm{conj}_a) = T_a(\mu^{a^{-1}}).T_e(\mu_a) = T_{a^{-1}}(\mu_a).T_e(\mu^{a^{-1}})$.

Finally we define the (lower case) *adjoint representation* of the Lie algebra $\mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) := L(\mathfrak{g},\mathfrak{g})$, by $\mathrm{ad} := \mathrm{Ad}' = T_e\mathrm{Ad}$.

We shall also use the *right Maurer-Cartan form* $\kappa^r \in \Omega^1(G,\mathfrak{g})$, given by $\kappa^r_g = T_g(\mu^{g^{-1}}) : T_gG \to \mathfrak{g}$; similarly the *left Maurer-Cartan form* $\kappa^l \in \Omega^1(G,\mathfrak{g})$ is given by $\kappa^l_g = T_g(\mu_{g^{-1}}) : T_gG \to \mathfrak{g}$.

Lemma.
(i) $L_X(a) = R_{\mathrm{Ad}(a)X}(a)$ *for* $X \in \mathfrak{g}$ *and* $a \in G$.
(ii) $\mathrm{ad}(X)Y = [X,Y]$ *for* $X,Y \in \mathfrak{g}$.
(iii) $d\mathrm{Ad} = (\mathrm{ad} \circ \kappa^r).\mathrm{Ad} = \mathrm{Ad}.(\mathrm{ad} \circ \kappa^l) : TG \to L(\mathfrak{g},\mathfrak{g})$.

Proof. (i) $L_X(a) = T_e(\mu_a).X = T_e(\mu^a).T_e(\mu^{a^{-1}} \circ \mu_a).X = R_{\mathrm{Ad}(a)X}(a)$.

(ii) We need some preparations. Let V be a convenient vector space. For $f \in C^\infty(G,V)$ we define the *left trivialized derivative* $D_lf \in C^\infty(G,L(\mathfrak{g},V))$ by

$$D_lf(x).X := df(x).T\mu_x.X = (L_Xf)(x). \tag{1}$$

For $f \in C^\infty(G,\mathbb{R})$ and $g \in C^\infty(G,V)$ we have

$$\begin{aligned} D_l(f.g)(x).X &= d(f.g)(T_e\mu_x.X) \\ &= df(T_e\mu_x.X).g(x) + f(x).dg(T_e\mu_x.X) \\ &= (f.D_lg + D_lf \otimes g)(x).X. \end{aligned} \tag{2}$$

From the fomula

$$\begin{aligned} D_lD_lf(x)(X)(Y) &= D_l(D_lf(\quad).Y)(x).X \\ &= D_l(L_Yf)(x).X = L_XL_Yf(x). \end{aligned}$$

follows

$$D_lD_lf(x)(X)(Y) - D_lD_lf(x)(Y)(X) = L_{[X,Y]}f(x) = D_lf(x).[X,Y]. \tag{3}$$

We consider now the linear isomorphism $L : C^\infty(G,\mathfrak{g}) \to \mathfrak{X}(G)$ given by $L_f(x) = T_e\mu_x.f(x)$ for $f \in C^\infty(G,\mathfrak{g})$. If $h \in C^\infty(G,V)$ we get $(L_fh)(x) = D_lh(x).f(x)$. For $f,g \in C^\infty(G,\mathfrak{g})$ and $h \in C^\infty(G,\mathbb{R})$ we get in turn, using (2), generalized to the bilinear pairing $L(\mathfrak{g},\mathbb{R}) \times \mathfrak{g} \to \mathbb{R}$,

$$\begin{aligned} (L_fL_gh)(x) &= D_l(D_lh(\quad).g(\quad))(x).f(x) \\ &= D_lD_lh(x)(f(x))(g(x)) + D_lh(x).D_lg(x).f(x) \\ ([L_f,L_g]h)(x) &= D_l^2h(x).(f(x),g(x)) + D_lh(x).D_lg(x).f(x) - \\ &\quad - D_l^2h(x).(g(x),f(x)) - D_lh(x).D_lf(x).g(x) \\ &= D_lh(x).\Big([f(x),g(x)]_{\mathfrak{g}} + D_lg(x).f(x) - D_lf(x).g(x)\Big) \end{aligned}$$

$$[L_f,L_g] = L\Big([f,g]_{\mathfrak{g}} + D_lg.f - D_lf.g\Big) \tag{4}$$

Now we are able to prove the second assertion of the lemma. For $X,Y \in \mathfrak{g}$ we will apply (4) to $f(x) = X$ and $g(x) = \mathrm{Ad}(x^{-1}).Y$. We have $L_g = R_Y$ by (i), and $[L_f, L_g] = [L_X, R_Y] = 0$ by 3.4. So

$$\begin{aligned} 0 = [L_X, R_Y](x) &= [L_f, L_g](x) \\ &= L([X, (\mathrm{Ad}\circ\nu)Y]_{\mathfrak{g}} + D_l((\mathrm{Ad}\circ\nu)(\quad).X).Y - 0)(x) \\ [X,Y] &= [X, \mathrm{Ad}(e)Y] = -D_l((\mathrm{Ad}\circ\nu)(\quad).X)(e).Y \\ &= d(\mathrm{Ad}(\quad).X)(e).Y = \mathrm{ad}(X)Y. \end{aligned}$$

(iii) Let $X, Y \in \mathfrak{g}$ and $g \in G$ and let $c : \mathbb{R} \to G$ be a smooth curve with $c(0) = e$ and $c'(0) = X$. Then we have

$$\begin{aligned} (d\mathrm{Ad}(R_X(g))).Y &= \tfrac{\partial}{\partial t}|_0 \mathrm{Ad}(c(t).g).Y = \tfrac{\partial}{\partial t}|_0 \mathrm{Ad}(c(t)).\mathrm{Ad}(g).Y \\ &= \mathrm{ad}(X)\mathrm{Ad}(g)Y = (\mathrm{ad}\circ\kappa^r)(R_X(g)).\mathrm{Ad}(g).Y, \end{aligned}$$

and similarly for the second formula. ∎

3.10. Let $r : M \times G \to M$ be a right action, so $\check{r} : G \to \mathrm{Diff}(M)$ is a group anti homomorphism. We will use the following notation: $r^a : M \to M$ and $r_x : G \to M$, given by $r_x(a) = r^a(x) = r(x,a) = x.a$.

For any $X \in \mathfrak{g}$ we define the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ by $\zeta_X(x) = T_e(r_x).X = T_{(x,e)}r.(0_x, X)$.

Lemma. *In this situation the following assertions hold:*

(i) $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ *is a Lie algebra homomorphism.*
(ii) $T_x(r^a).\zeta_X(x) = \zeta_{Ad(a^{-1})X}(x.a)$.
(iii) $0_M \times L_X \in \mathfrak{X}(M \times G)$ *is* r*-related to* $\zeta_X \in \mathfrak{X}(M)$. ∎

4. Bundles and connections

4.1. Definition. A *principal (fiber) bundle* (P, p, M, G) is a smooth mapping $p : P \to M$ such that there exist an open cover (U_α) of M and fiber respecting diffeomorphisms $\varphi_\alpha : P|U_\alpha := p^{-1}(U_\alpha) \to U_\alpha \times G$ with $(\varphi_\alpha \circ \varphi_\beta^{-1})(x, g) = (x, \varphi_{\alpha\beta}(x).g)$ for a smooth cocycle of transition functions $(\varphi_{\alpha\beta} : U_{\alpha\beta} := U_\alpha \cap U_\beta \to G)$. This is called a *principal bundle atlas*.

Each principal bundle admits a unique right action $r : P \times G \to P$, called the *principal right action*, given by $\varphi_\alpha(r(\varphi_\alpha^{-1}(x,a), g)) = (x, ag)$. Since left and right translation on G commute, this is well defined. We write $r(u, g) = u.g$ when the meaning is clear. The principal right action is visibly free and for any $u_x \in P_x$ the partial mapping $r_{u_x} = r(u_x, \quad) : G \to P_x$ is a diffeomorphism onto the fiber through u_x, whose inverse is denoted by $\tau_{u_x} : P_x \to G$. These inverses together give a smooth mapping $\tau : P \times_M P \to G$, whose local expression is $\tau(\varphi_\alpha^{-1}(x,a), \varphi_\alpha^{-1}(x,b)) = a^{-1}.b$. This mapping is uniquely determined by the implicit equation $r(u_x, \tau(u_x, v_x)) = v_x$, thus we also have $\tau(u_x.g, u'_x.g') = g^{-1}.\tau(u_x, u'_x).g'$ and $\tau(u_x, u_x) = e$.

4.2. Principal connections. Let (P, p, M, G) be a principal fiber bundle. Let $VP := (Tp)^{-1}(0_M) \to P$ be the vertical bundle. A (general) connection on P

is a smooth fiber projection $\Phi : TP \to VP$, viewed as a 1-form in $\Omega^1(P;VP) \subset \Omega^1(P;TP)$, which is called a *principal connection* if it is G-equivariant for the principal right action $r : P \times G \to P$, so that $T(r^g).\Phi = \Phi.T(r^g)$ and Φ is r^g-related to itself, or $(r^g)^*\Phi = \Phi$, for all $g \in G$. Then the kernel of Φ is called the *horizontal subbundle*, a splitting vector subbundle of $TP \to P$ complementary to VP.

The vertical bundle of P is trivialized as a vector bundle over P by the principal action. So $\omega(X_u) := T_e(r_u)^{-1}.\Phi(X_u) \in \mathfrak{g}$ is well defined, and in this way we get a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P;\mathfrak{g})$, which is called the (*Lie algebra valued*) *connection form* of the connection Φ. Recall from 3.10. the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(P)$ for the principal right action, which trivializes the vertical bundle $P \times \mathfrak{g} \cong VP$. The defining equation for ω can be written also as $\Phi(X_u) = \zeta_{\omega(X_u)}(u)$.

Lemma. *If $\Phi \in \Omega^1(P;VP)$ is a principal connection on the principal fiber bundle (P,p,M,G) then the connection form has the following two properties:*

(i) *ω reproduces the generators of fundamental vector fields, so that we have $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$.*

(ii) *ω is G-equivariant, i.e., for all $g \in G$ and $X_u \in T_uP$ we have $((r^g)^*\omega)(X_u) := \omega(T_u(r^g).X_u) = \mathrm{Ad}(g^{-1}).\omega(X_u)$.*

(iii) *For the Lie derivative we have $\mathcal{L}_{\zeta_X}\omega = -\mathrm{ad}(X).\omega$.*

Conversely, a 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ satisfying (i) defines a connection Φ on P by $\Phi(X_u) = T_e(r_u).\omega(X_u)$, which is a principal connection if and only if (ii) *is satisfied.*

Proof. (i) $T_e(r_u).\omega(\zeta_X(u)) = \Phi(\zeta_X(u)) = \zeta_X(u) = T_e(r_u).X$. Since $T_e(r_u) : \mathfrak{g} \to V_uP$ is an isomorphism, the result follows.

(ii) Both directions follow from

$$\begin{aligned}
T_e(r_{ug}).\omega(T_u(r^g).X_u) &= \zeta_{\omega(T_u(r^g).X_u)}(ug) = \Phi(T_u(r^g).X_u) \\
T_e(r_{ug}).\mathrm{Ad}(g^{-1}).\omega(X_u) &= \zeta_{\mathrm{Ad}(g^{-1}).\omega(X_u)}(ug) = T_u(r^g).\zeta_{\omega(X_u)}(u) \\
&= T_u(r^g).\Phi(X_u).
\end{aligned}$$

(iii) Let $g(t)$ be a smooth curve in G with $g(0) = e$ and $\frac{\partial}{\partial t}|_0 g(t) = X$. Then $\varphi_t := r^{g(t)}$ is a smooth curve of diffeomorphisms on P with $\frac{\partial}{\partial t}|_0\varphi_t = \zeta_X$, and by Lemma 2.10 we have

$$\mathcal{L}_{\zeta_X}\omega = \tfrac{\partial}{\partial t}|_0(r^{g(t)})^*\omega = \tfrac{\partial}{\partial t}|_0 Ad(g(t)^{-1})\omega = -ad(X)\omega. \qquad \blacksquare$$

4.3. Curvature. Let Φ be a principal connection on the principal fiber bundle (P,p,M,G) with connection form $\omega \in \Omega^1(P;\mathfrak{g})$.

Let us now define the curvature as the obstruction against integrability of the horizontal subbundle, i.e., $\mathcal{R}(X,Y) := \Phi[X - \Phi X, Y - \Phi Y]$ for vector fields X, Y on P. One can check easily that $\mathcal{R}$ is a skew-symmetric bilinear $C^\infty(P,\mathbb{R})$-module homomorphism, and that $(r^{g^{-1}})^*.\mathcal{R}(X,Y) = \mathcal{R}((r^g)^*X, (r^g)^*Y)$ holds, i.e., $(r^g)^*\mathcal{R} = \mathcal{R}$ for all $g \in G$. Since $\mathcal{R}$ has vertical values we may again define a $\mathfrak{g}$-valued 2-form by $\Omega(X,Y)(u) := -T_e(r_u)^{-1}.\mathcal{R}(X,Y)(u)$, which is called the (*Lie algebra-valued*) *curvature form* of the connection. We also have $\mathcal{R}(X,Y)(u) = -\zeta_{\Omega(X,Y)(u)}(u)$. We take the negative sign here to get in finite dimensions the usual curvature form as in [8].

We equip the space $\Omega(P;\mathfrak{g})$ of all $\mathfrak{g}$-valued forms on P in a canonical way with the structure of a graded Lie algebra by

$$[\Psi,\Theta]_\wedge(X_1,\dots,X_{p+q}) =$$
$$= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}\sigma\,[\Psi(X_{\sigma 1},\dots,X_{\sigma p}),\Theta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]_{\mathfrak{g}}$$

or equivalently by $[\psi\otimes X,\theta\otimes Y]_\wedge := \psi\wedge\theta\otimes[X,Y]_{\mathfrak{g}}$. From the latter description it is clear that $d[\Psi,\Theta]_\wedge = [d\Psi,\Theta]_\wedge + (-1)^{\deg\Psi}[\Psi,d\Theta]_\wedge$. In particular for $\omega\in\Omega^1(P;\mathfrak{g})$ we have $[\omega,\omega]_\wedge(X,Y) = 2[\omega(X),\omega(Y)]_{\mathfrak{g}}$.

Theorem. *The curvature form Ω of a principal connection with connection form ω has the following properties:*

(i) *Ω is horizontal, i.e. it kills vertical vector fields.*
(ii) *The Maurer-Cartan formula holds: $\Omega = d\omega + \frac{1}{2}[\omega,\omega]_\wedge \in \Omega^2(P;\mathfrak{g})$.*
(iii) *Ω is G-equivariant in the following sense: $(r^g)^*\Omega = \operatorname{Ad}(g^{-1}).\Omega$. Consequently $\mathcal{L}_{\zeta_X}\Omega = -\operatorname{ad}(X).\Omega$.*

Proof. (i) is true for $\mathcal{R}$ by definition. For (ii) we show that the formula holds if at least one vector field is vertical, or if both are horizontal. For $X\in\mathfrak{g}$ we have $i_{\zeta_X}\mathcal{R} = 0$ by (i), and using 4.2.(i) and (iii) we get

$$i_{\zeta_X}(d\omega + \tfrac{1}{2}[\omega,\omega]_\wedge) = i_{\zeta_X}d\omega + \tfrac{1}{2}[i_{\zeta_X}\omega,\omega]_\wedge - \tfrac{1}{2}[\omega, i_{\zeta_X}\omega]_\wedge =$$
$$= \mathcal{L}_{\zeta_X}\omega + [X,\omega]_\wedge = -\operatorname{ad}(X)\omega + \operatorname{ad}(X)\omega = 0.$$

So the formula holds for vertical vectors, and for horizontal vector fields X,Y we have

$$\mathcal{R}(X,Y) = \Phi[X-\Phi X, Y-\Phi Y] = \Phi[X,Y] = \zeta_{\omega([X,Y])}$$
$$(d\omega + \tfrac{1}{2}[\omega,\omega])(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + 0 = -\omega([X,Y]).$$

That Ω is really a 'tensorial' 2-form follows either from (ii) or from 4.4.(iv) below. ■

4.4. Local descriptions of principal connections. We consider a principal fiber bundle (P,p,M,G) together with some principal fiber bundle atlas $(U_\alpha,\varphi_\alpha : P|U_\alpha \to U_\alpha\times G)$ and corresponding cocycle $(\varphi_{\alpha\beta} : U_{\alpha\beta}\to G)$ of transition functions. Let $\Phi = \zeta\circ\omega \in \Omega^1(P;VP)$ be a principal connection with connection form $\omega\in\Omega^1(P;\mathfrak{g})$. We consider the sections $s_\alpha\in C^\infty(P|U_\alpha)$ which are given by $\varphi_\alpha(s_\alpha(x)) = (x,e)$ and satisfy $s_\alpha.\varphi_{\alpha\beta} = s_\beta$. Then we may associate to the connection the collection of the $\omega_\alpha := {s_\alpha}^*\omega \in \Omega^1(U_\alpha;\mathfrak{g})$, the physicists version of the connection.

Lemma. *These local data have the following properties and are related by the following formulas.*

(i) *The forms $\omega_\alpha\in\Omega^1(U_\alpha;\mathfrak{g})$ satisfy the transition formulas*

$$\omega_\alpha = \operatorname{Ad}(\varphi_{\beta\alpha}^{-1})\omega_\beta + (\varphi_{\beta\alpha})\kappa^l,$$

where $\kappa^l \in \Omega^1(G;\mathfrak{g})$ is the left Maurer-Cartan form from 3.9.

(ii) *The local expression of ω is given by*

$$(\varphi_\alpha^{-1})^*\omega(\xi_x, T\mu_g.X) = (\varphi_\alpha^{-1})^*\omega(\xi_x, 0_g) + X = \mathrm{Ad}(g^{-1})\omega_\alpha(\xi_x) + X.$$

(iii) *The local expression of Φ is given by*

$$((\varphi_\alpha)^{-1})^*\Phi(\xi_x, \eta_g) = T_e(\mu^g).\omega_\alpha(\xi_x) + \eta_g = R_{\omega_\alpha(\xi_x)}(g) + \eta_g$$

for $\xi_x \in T_xU_\alpha$ and $\eta_g \in T_gG$.

(iv) *The local expression of the curvature $\mathcal{R}$ is given by*

$$((\varphi_\alpha)^{-1})^*\mathcal{R} = -R_{d\omega_\alpha + \frac{1}{2}[\omega_\alpha,\omega_\alpha]^\wedge_\mathfrak{g}}$$

so that $\mathcal{R}$ and Ω are indeed 'tensorial' 2-forms.

Proof. For (i) to (iii) plug in the definitions. For (iv) note that the right trivialization or framing $(\kappa^r, \pi_G) : TG \to \mathfrak{g} \times G$ induces the isomorphism $R : C^\infty(G,\mathfrak{g}) \to \mathfrak{X}(G)$, given by $R_X(x) = T_e(\mu^x).X(x)$. For the Lie bracket we then have

$$\begin{aligned}
[R_X, R_Y] &= R_{-[X,Y]_\mathfrak{g} + dY.R_X - dX.R_Y},\\
R^{-1}[R_X, R_Y] &= -[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X).
\end{aligned}$$

We write a vector field on $U_\alpha \times G$ as (ξ, R_X) where $\xi : G \to \mathfrak{X}(U_\alpha)$ and $X \in C^\infty(U_\alpha \times G, \mathfrak{g})$. Then the local expression of the curvature is given by

$$\begin{aligned}
&({\varphi_\alpha}^{-1})^*\mathcal{R}((\xi, R_X),(\eta, R_Y)) = (\varphi_\alpha^{-1})^*(\mathcal{R}((\varphi_\alpha)^*(\xi, R_X),(\varphi_\alpha)^*(\eta, R_Y)))\\
&\quad= (\varphi_\alpha^{-1})^*(\Phi[(\varphi_\alpha)^*(\xi, R_X) - \Phi(\varphi_\alpha)^*(\xi, R_X), \dots])\\
&\quad= (\varphi_\alpha^{-1})^*(\Phi[(\varphi_\alpha)^*(\xi, R_X) - (\varphi_\alpha)^*(R_{\omega_\alpha(\xi)} + R_X), \dots])\\
&\quad= (\varphi_\alpha^{-1})^*(\Phi(\varphi_\alpha)^*[(\xi, -R_{\omega_\alpha(\xi)}),(\eta, -R_{\omega_\alpha(\eta)})])\\
&\quad= ((\varphi_\alpha^{-1})^*\Phi)([\xi,\eta]_{\mathfrak{X}(U_\alpha)} - R_{\omega_\alpha(\xi)}(\eta) + R_{\omega_\alpha(\eta)}(\xi),\\
&\qquad\quad - \xi(R_{\omega_\alpha(\eta)}) + \eta(R_{\omega_\alpha(\xi)}) + R_{-[\omega_\alpha(\xi),\omega_\alpha(\eta)] + R_{\omega_\alpha(\xi)}(\omega_\alpha(\eta)) - R_{\omega_\alpha(\xi)}(\omega_\alpha(\eta))})\\
&\quad= R_{\omega_\alpha([\xi,\eta]_{\mathfrak{X}(U_\alpha)} - R_{\omega_\alpha(\xi)}(\eta) + R_{\omega_\alpha(\eta)}(\xi))} - R_{\xi(\omega_\alpha(\eta))} + R_{\eta(\omega_\alpha(\xi))}\\
&\qquad + R_{-[\omega_\alpha(\xi),\omega_\alpha(\eta)] + R_{\omega_\alpha(\xi)}(\omega_\alpha(\eta)) - R_{\omega_\alpha(\xi)}(\omega_\alpha(\eta))}\\
&\quad= -R_{(d\omega_\alpha + \frac{1}{2}[\omega_\alpha,\omega_\alpha]^\wedge_\mathfrak{g})(\xi,\eta)}.
\end{aligned}$$

This finishes the proof of (iv). ∎

5. Regular Lie groups

5.1. The right and left logarithmic derivatives. Let M be a manifold and let $f : M \to G$ be a smooth mapping into a Lie group G with Lie algebra $\mathfrak{g}$. We define the mapping $\delta^r f : TM \to \mathfrak{g}$ by the formula

$$\delta^r f(\xi_x) := T_{f(x)}(\mu^{f(x)^{-1}}).T_xf.\xi_x \text{ for } \xi_x \in T_xM.$$

Then $\delta^r f$ is a $\mathfrak{g}$-valued 1-form on M, $\delta^r f \in \Omega^1(M;\mathfrak{g})$. We call $\delta^r f$ the *right logarithmic derivative* of f, since for $f:\mathbb{R}\to(\mathbb{R}^+,\cdot)$ we have $\delta^r f(x).1 = \frac{f'(x)}{f(x)} = (\log\circ f)'(x)$.

Similarly the *left logarithmic derivative* $\delta^l f \in \Omega^1(M,\mathfrak{g})$ of a smooth mapping $f:M\to G$ is given by

$$\delta^l f.\xi_x = T_{f(x)}(\mu_{f(x)^{-1}}).T_x f.\xi_x.$$

Lemma. *Let $f,g:M\to G$ be smooth. Then the Leibniz rule holds:*

$$\delta^r(f.g)(x) = \delta^r f(x) + \mathrm{Ad}(f(x)).\delta^r g(x).$$

Moreover, the differential form $\delta^r f\in\Omega^1(M;\mathfrak{g})$ satisfies the 'left Maurer-Cartan equation' (left because it stems from the left action of G on itself)

$$\begin{aligned} &d\delta^r f(\xi,\eta) - [\delta^r f(\xi),\delta^r f(\eta)]^{\mathfrak{g}} = 0,\\ \textit{or}\quad &d\delta^r f - \tfrac{1}{2}[\delta^r f,\delta^r f]^{\mathfrak{g}}_\wedge = 0,\end{aligned}$$

where $\xi,\eta\in T_xM$, and where for $\varphi\in\Omega^p(M;\mathfrak{g}),\psi\in\Omega^q(M;\mathfrak{g})$ one puts

$$[\varphi,\psi]^{\mathfrak{g}}_\wedge(\xi_1,\dots,\xi_{p+q}) := \tfrac{1}{p!q!}\textstyle\sum_\sigma \mathrm{sign}(\sigma)[\varphi(\xi_{\sigma 1},\dots),\psi_{\sigma(p+1)},\dots)]^{\mathfrak{g}}.$$

For the left logarithmic derivative the corresponding Leibniz rule is uglier, and it satisfies the 'right Maurer Cartan equation':

$$\begin{aligned} \delta^l(fg)(x) &= \delta^l g(x) + Ad(g(x)^{-1})\delta^l f(x),\\ d\delta^l f &+ \tfrac{1}{2}[\delta^l f,\delta^l f]^{\mathfrak{g}}_\wedge = 0.\end{aligned}$$

For 'regular Lie groups' we will prove a converse to this statement later in 7.2.

Proof. We treat only the right logarithmic derivative, the proof for the left one is similar.

$$\begin{aligned}\delta^r(f.g)(x) &= T(\mu^{g(x)^{-1}.f(x)^{-1}}).T_x(f.g)\\ &= T(\mu^{f(x)^{-1}}).T(\mu^{g(x)^{-1}}).T_{(f(x),g(x))}\mu.(T_xf,T_xg)\\ &= T(\mu^{f(x)^{-1}}).T(\mu^{g(x)^{-1}}).\Big(T(\mu^{g(x)}).T_xf + T(\mu_{f(x)}).T_xg\Big)\\ &= \delta^r f(x) + \mathrm{Ad}(f(x)).\delta^r g(x),\end{aligned}$$

We shall use now principal bundle geometry from Section 3. We consider the trivial principal bundle $\mathrm{pr}_1: M\times G\to M$ with right principal action. Then the submanifolds $\{(x,f(x).g): x\in M\}$ for $g\in G$ form a foliation of $M\times G$ whose tangent distribution is complementary to the vertical bundle $M\times TG\subseteq T(M\times G)$ and is invariant under the principal right G-action. So it is the horizontal distribution of a principal connection on $M\times G\to G$. For a tangent vector $(\xi_x,Y_g)\in T_xM\times T_gG$ the horizontal part is the right translate to the foot point (x,g) of $(\xi_x, T_xf.\xi_x)$, so the decomposition in horizontal and vertical parts according to this distribution is

$$(\xi_x, Y_g) = (\xi_x, T(\mu^g).T(\mu^{f(x)^{-1}}).T_xf.\xi_x) + (0_x, Y_g - T(\mu^g).T(\mu^{f(x)^{-1}}).T_xf.\xi_x).$$

Since the fundamental vector fields for the right action on G are the left invariant vector fields, the corresponding connection form is given by

$$\begin{aligned} \omega^r(\xi_x, Y_g) &= T(\mu_{g^{-1}}).(Y_g - T(\mu^g).T(\mu^{f(x)^{-1}}).T_xf.\xi_x),\\ \omega^r_{(x,g)} &= T(\mu_{g^{-1}}) - \mathrm{Ad}(g^{-1}).\delta^r f_x,\\ \omega^r &= \kappa^l - (\mathrm{Ad}\circ\nu).\delta^r f, \end{aligned} \tag{1}$$

where $\kappa^l : TG \to \mathfrak{g}$ is the left Maurer-Cartan form on G (the left trivialization), given by $\kappa^l_g = T(\mu_{g^{-1}})$. Note that κ^l is the principal connection form for the (unique) principal connection $p : G \to$ point with right principal action, which is flat so that the right (from right action) Maurer-Cartan equation equation holds in the form

$$d\kappa^l + \tfrac{1}{2}[\kappa^l, \kappa^l]_\wedge = 0. \tag{2}$$

The principal connection ω^r is flat since we got it via the horizontal leaves, so the principal connection form vanishes:

$$\begin{aligned} 0 &= d\omega^r + \tfrac{1}{2}[\omega^r, \omega^r]_\wedge \\ &= d\kappa^l + \tfrac{1}{2}[\kappa^l, \kappa^l]_\wedge - d(\mathrm{Ad}\circ\nu)\wedge\delta^r f - (\mathrm{Ad}\circ\nu).d\delta^r f \\ &\quad - [\kappa^l, (\mathrm{Ad}\circ\nu).\delta^r f]_\wedge + \tfrac{1}{2}[(\mathrm{Ad}\circ\nu).\delta^r f, (\mathrm{Ad}\circ\nu).\delta^r f]_\wedge \\ &= -(\mathrm{Ad}\circ\nu).(d\delta^r f - \tfrac{1}{2}[\delta^r f, \delta^r f]_\wedge), \end{aligned} \tag{3}$$

where we used (2), and since for $\xi\in\mathfrak{g}$ and a smooth curve $c:\mathbb{R}\to G$ with $c(0)=e$ and $c'(0)=\xi$ we have:

$$\begin{aligned} d(\mathrm{Ad}\circ\nu)(T(\mu_g)\xi) &= \tfrac{\partial}{\partial t}\big|_0 \mathrm{Ad}(c(t)^{-1}.g^{-1}) = -\mathrm{ad}(\xi)\mathrm{Ad}(g^{-1}) \\ &= -\mathrm{ad}\Big(\kappa^l(T(\mu_g)\xi)\Big)(\mathrm{Ad}\circ\nu)(g), \\ d(\mathrm{Ad}\circ\nu) &= -(\mathrm{ad}\circ\kappa^l).(\mathrm{Ad}\circ\nu). \end{aligned} \tag{4}$$

So we have $d\delta^r f - \frac{1}{2}[\delta^r f, \delta^r f]_\wedge$, as asserted.

For the left logarithmic derivative $\delta^l f$ the proof is similar, and we discuss only the essential deviations. First note that on the trivial principal bundle $\mathrm{pr}_1 : M\times G\to M$ with left principal action of G the fundamental vector fields are the right invariant vector fields on G, and that for a principal connection form ω^l the curvature form is given by $d\omega^l - \frac{1}{2}[\omega^l, \omega^l]_\wedge$. Look at the proof of Theorem 4.3 to see this. The connection form is then given by

$$\omega^l = \kappa^r - \mathrm{Ad}.\delta^l f, \tag{1$'$}$$

where the right Maurer-Cartan form $(\kappa^r)_g = T(\mu^{g^{-1}}) : T_gG \to \mathfrak{g}$ now satifies the left Maurer-Cartan equation

$$(2)' \qquad d\kappa^r - \tfrac{1}{2}[\kappa^r,\kappa^r]_\wedge = 0.$$

Flatness of ω^l now leads to the computation

$$(3') \qquad \begin{aligned} 0 &= d\omega^l - \tfrac{1}{2}[\omega^l,\omega^l]_\wedge \\ &= d\kappa^r - \tfrac{1}{2}[\kappa^r,\kappa^r]_\wedge - d\mathrm{Ad}\wedge\delta^l f - \mathrm{Ad}.d\delta^l f \\ &\quad + [\kappa^r, \mathrm{Ad}.\delta^l f]_\wedge - \tfrac{1}{2}[\mathrm{Ad}.\delta^l f, \mathrm{Ad}.\delta^l f]_\wedge \\ &= -\mathrm{Ad}.(d\delta^l f + \tfrac{1}{2}[\delta^l f, \delta^l f]_\wedge), \end{aligned}$$

where we used $d\mathrm{Ad} = (\mathrm{ad}\circ\kappa^r)\mathrm{Ad}$ from 3.9.(3) directly. ∎

5.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For a closed interval $I\subset\mathbb{R}$ and for $X\in C^\infty(I,\mathfrak{g})$ we consider the ordinary differential equation

$$(1) \qquad \begin{cases} g(t_0) = e \\ \frac{\partial}{\partial t}g(t) = T_e(\mu^{g(t)})X(t) = R_{X(t)}(g(t)), \qquad \text{or } \kappa^r(\frac{\partial}{\partial t}g(t)) = X(t), \end{cases}$$

for local smooth curves g in G, where $t_0\in I$.

Lemma.

(i) *Local solution curves g of the differential equation (1) are uniquely determined.*

(ii) *If for fixed X the differential equation (1) has a local solution near each $t_0\in I$, then it has also a global solution $g\in C^\infty(I,G)$.*

(iii) *If for all $X\in C^\infty(I,\mathfrak{g})$ the differential equation (1) has a local solution near one fixed $t_0\in I$, then it has also a global solution $g\in C^\infty(I,G)$ for each X. Moreover, if the local solutions near t_0 depend smoothly on the vector fields X (see the proof for the exact formulation), then so does the global solution.*

(iv) *The curve $t\mapsto g(t)^{-1}$ is the unique local smooth curve h in G which satifies*

$$\begin{cases} h(t_0) = e \\ \frac{\partial}{\partial t}h(t) = T_e(\mu_{h(t)})(-X(t)) = L_{-X(t)}(h(t)), \qquad \text{or } \kappa^l(\frac{\partial}{\partial t}h(t)) = -X(t). \end{cases}$$

Proof. (i) Suppose that $g(t)$ and $g_1(t)$ both satisfy (1). Then on the intersection of their intervals of definition we have

$$\begin{aligned} \tfrac{\partial}{\partial t}(g(t)^{-1}g_1(t)) &= -T(\mu^{g_1(t)}).T(\mu_{g(t)^{-1}}).T(\mu^{g(t)^{-1}}).T(\mu^{g(t)}).X(t) \\ &\quad + T(\mu_{g(t)^{-1}}).T(\mu^{g_1(t)}).X(t) = 0, \end{aligned}$$

so that $g = g_1$.

(ii) It suffices to prove the claim for every compact subintervall of I, so let I be compact. If g is a local solution of (1) then $t\mapsto g(t).x$ is a local solution of the same differential equation with initial value x. By assumption for each $s \in I$ there is a unique solution g_s of the differential equation with $g_s(s) = e$; so there exists $\delta_s > 0$ such that $g_s(s+t)$ is defined for $|t| < \delta_s$. Since I is compact there exist $s_0 < s_1 < \ldots < s_k$ such that $I = [s_0, s_k]$ and $s_{i+1} - s_i < \delta_{s_i}$. Then we put

$$g(t) := \begin{cases} g_{s_0}(t) & \text{for } s_0 \le t \le s_1 \\ g_{s_1}(t).g_{s_0}(s_1) & \text{for } s_1 \le t \le s_2 \\ \ldots & \ldots \\ g_{s_i}(t).g_{s_{i-1}}(s_i)\ldots g_{s_0}(s_1) & \text{for } s_i \le t \le s_{i+1} \\ \ldots & \ldots \end{cases}$$

which is smooth by the first case and solves the problem.

(iii) Given $X : I \to \mathfrak{g}$ we first extend X to a smooth curve $\mathbb{R} \to \mathfrak{g}$, using the formula of [17]. For $t_1 \in I$, by assumption there exists a local solution g near t_0 of the translated vector field $t \mapsto X(t_1 - t_0 + t)$, thus $t \mapsto g(t_0 - t_1 + t)$ is a solution near t_1 of X. So by assertion (iii) the differential equation has a global solution for X on I.

Now we assume that the local solutions near t_0 depend smoothly in the vector field: So for any smooth curve $X : \mathbb{R} \to C^\infty(I, \mathfrak{g})$ we have:

For each compact intervall $K \subset \mathbb{R}$ there is a neighborhood $U_{X,K}$ of t_0 in I and a smooth mapping $g : K \times U_{X,K} \to G$ with

$$\begin{cases} g(k, t_0) = e \\ \frac{\partial}{\partial t} g(k,t) = T_e(\mu^{g(k,t)}).X(k)(t) \qquad \text{for all } k \in K, t \in U_{X,K}. \end{cases}$$

Given a smooth curve $X : \mathbb{R} \to C^\infty(I, \mathfrak{g})$ we extend (or lift) it smoothly to $X : \mathbb{R} \to C^\infty(\mathbb{R}, \mathfrak{g})$ by using the formula of [17]. Then the smooth parameter k from the compact intervall K passes smoothly through the proofs given above to give a smooth global solution $g : K \times I \to G$. So the 'solving operation' respects smooth curves and thus is 'smooth'.

(iv) One can show in a similar way that h is the unique solution of the equation in (iv) by differentiating $h_1(t).h(t)^{-1}$. Moreover the curve $t \mapsto g(t)^{-1} = h(t)$ satisfies the equation of (iv), since

$$\tfrac{\partial}{\partial t}(g(t)^{-1}) = -T(\mu_{g(t)^{-1}}).T(\mu^{g(t)^{-1}}).T(\mu^{g(t)}).X(t) = T(\mu_{g(t)^{-1}}).(-X(t)). \quad \blacksquare$$

5.3. Definition. Regular Lie groups. If for each $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists $g \in C^\infty(\mathbb{R}, G)$ satisfying

$$(1) \qquad \begin{cases} g(0) = e \\ \frac{\partial}{\partial t} g(t) = T_e(\mu^{g(t)})X(t) = R_{X(t)}(g(t)), \\ \quad \text{or } \kappa^r(\frac{\partial}{\partial t} g(t)) = \delta^r g(\partial_t) = X(t). \end{cases}$$

then we write

$$\operatorname{evol}^r_G(X) = \operatorname{evol}_G(X) := g(1),$$
$$\operatorname{Evol}^r_G(X)(t) := \operatorname{evol}_G(s \mapsto tX(ts)) = g(t),$$

and call it the *right evolution* of the curve X in G. By Lemma 5.2 the solution of the differential equation (1) is unique, and for global existence it is sufficient that it has a local solution. Then

$$\mathrm{Evol}_G^r : C^\infty(\mathbb{R},\mathfrak{g}) \to \{g \in C^\infty(\mathbb{R},G) : g(0)=e\}$$

is bijective with inverse the right logarithmic derivative δ^r.

The Lie group G is called a *regular Lie group* if $\mathrm{evol}^r : C^\infty(\mathbb{R},\mathfrak{g}) \to G$ exists and is smooth.

We also write

$$\begin{aligned}\mathrm{evol}_G^l(X) = \mathrm{evol}_G(X) &:= h(1),\\ \mathrm{Evol}_G^l(X)(t) &:= \mathrm{evol}_G^l(s \mapsto tX(ts)) = h(t),\end{aligned}$$

if h is the (unique) solution of

$$(2)\qquad \begin{cases} h(0)=e \\ \frac{\partial}{\partial t}h(t) = T_e(\mu_{h(t)})(X(t)) = L_{X(t)}(h(t)), \\ \quad \text{or } \kappa^l(\frac{\partial}{\partial t}h(t)) = \delta^l h(\partial_t) = X(t). \end{cases}$$

Clearly $\mathrm{evol}^l : C^\infty(\mathbb{R},\mathfrak{g}) \to G$ exists and is also smooth if evol^r does, since we have $\mathrm{evol}^l(X) = \mathrm{evol}^r(-X)^{-1}$ by Lemma 5.2.

Let us collect some easily seen properties of the evolution mappings. If $f \in C^\infty(\mathbb{R},\mathbb{R})$, then we have

$$\begin{aligned}\mathrm{Evol}^r(X)(f(t)) &= \mathrm{Evol}^r(f'.(X\circ f))(t).\mathrm{Evol}^r(X)(f(0)),\\ \mathrm{Evol}^l(X)(f(t)) &= \mathrm{Evol}^l(X)(f(0)).\mathrm{Evol}^l(f'.(X\circ f))(t).\end{aligned}$$

If $\varphi : G \to H$ is a smooth homomorphism between regular Lie groups then the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R},\mathfrak{g}) \arrow[r, "\varphi'_*"] \arrow[d, "\mathrm{evol}_G"'] & C^\infty(\mathbb{R},\mathfrak{h}) \arrow[d, "\mathrm{evol}_H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes, since $\frac{\partial}{\partial t}\varphi(g(t)) = T\varphi.T(\mu^{g(t)}).X(t) = T(\mu^{\varphi(g(t))}).\varphi'.X(t)$.

Note that each regular Lie group admits an exponential mapping, namely the restriction of evol^r to the constant curves $\mathbb{R} \to \mathfrak{g}$. A Lie group is regular if and only if its universal covering group is regular.

This notion of regularity is a weakening of the same notion of [14], [15], who considered a sort of product integration property on a smooth Lie group modelled on Fréchet spaces. Our notion here is due to [13]. Up to now the following statement holds:

All known Lie groups are regular.

Any Banach Lie group is regular since we may consider the time dependent right invariant vector field $R_{X(t)}$ on G and its integral curve $g(t)$ starting at e, which exists and depends smoothly on (a further parameter in) X. In particular finite dimensional Lie groups are regular.

For diffeomorphism groups the evolution operator is just integration of time dependent vector fields with compact support.

5.4. Some abelian regular Lie groups. For $(E,+)$, where E is a convenient vector space, we have $\operatorname{evol}(X) = \int_0^1 X(t)dt$, so convenient vector spaces are regular abelian Lie groups. We shall need 'discrete' subgroups, which is not an obvious notion since $(E,+)$ is not a topological group: the addition is continuous only $c^\infty(E\times E)\to c^\infty E$, and not for the cartesian product of the c^∞-topologies.

Next let Z be a 'discrete' subgroup of a convenient vector space E in the sense that there exists a c^∞-open neighborhood U of zero in E such that $U\cap(z+U)=\emptyset$ for all $0\neq z\in Z$ (equivalently $(U-U)\cap(Z\setminus 0)=\emptyset$). For that it suffices eg that Z is discrete in the bornological topology on E. Then E/Z is an abelian but possibly non Hausdorff Lie group. It does not suffice to take Z discrete in the c^∞-topology: Take as Z the subgroup generated by A in $\mathbb{R}^{\mathbb{N}\times c_0}$ in the proof of [4], 6.2.8.(iv).

Let us assume that Z fulfills the stronger condition: there exists a symmetric c^∞-open neighborhood W of 0 such that $(W+W)\cap(z+W+W)=\emptyset$ for all $0\neq z\in Z$ (equivalently $(W+W+W+W)\cap(Z\setminus 0)=\emptyset$). Then E/Z is Hausdorff and thus an abelian regular Lie group, since its universal cover E is regular. Namely, for $x\notin Z$, we have to find neighborhoods U and V of 0 such that $(Z+U)\cap(x+Z+V)=\emptyset$. There are two cases. If $x\in Z+W+W$ then there is a unique $z\in Z$ with $x\in z+W+W$ and we may choose $U,V\subset W$ such that $(z+U)\cap(x+V)=\emptyset$; then $(Z+U)\cap(x+Z+V)=\emptyset$. In the other case, if $x\notin Z+W+W$, then we have $(Z+W)\cap(x+Z+W)=\emptyset$.

Notice that the two conditions above and their consequences also hold for general (non-abelian) (regular) Lie groups instead of E, and their 'discrete' normal subgroups (which turn out to be central if G is connected).

It would be nice if any regular abelian Lie group were of the form E/Z described above. A first result in this direction is that for an abelian Lie group G with Lie algebra $\mathfrak{g}$ which admits a smooth exponential mapping $\exp:\mathfrak{g}\to G$ one can check easily by using 5.10 that $\frac{\partial}{\partial t}(\exp(-tX).\exp(tX+Y))=0$ so that $\exp$ is a smooth homomorphism of Lie groups.

Let us consider some examples. For the first one we consider a discrete subgroup $Z\subset\mathbb{R}^{\mathbb{N}}$. There exists a neighborhood of 0, without loss of the form $U\times\mathbb{R}^{\mathbb{N}\setminus n}$ for $U\subset\mathbb{R}^n$, with $U\cap(Z\setminus 0)=\emptyset$. Then we consider the following diagram of Lie group homomorphisms

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{R}^{\mathbb{N}\setminus n} \arrow[r] \arrow[d] & \mathbb{R}^{\mathbb{N}\setminus n} \arrow[d] & \\
Z \arrow[r] \arrow[d, "\cong"'] & \mathbb{R}^{\mathbb{N}} \arrow[r] \arrow[d, "\pi"'] & \mathbb{R}^{\mathbb{N}}/Z \arrow[d] \arrow[r, phantom, "="] & (S^1)^k\times\mathbb{R}^{\mathbb{N}\setminus(n-k)} \arrow[d] \\
\pi(Z) \arrow[r] & \mathbb{R}^n \arrow[r] & \mathbb{R}^n/\pi(Z) \arrow[r, phantom, "="] & (S^1)^k\times\mathbb{R}^{n-k}
\end{tikzcd}$$

which has exact lines and columns. For the right hand column we use a diagram chase to see this. Choose a global linear section of π inverting $\pi|Z$. This factors to a global homomorphism of the right hand side column.

As next example we consider $\mathbb{Z}^{(\mathbb{N})} \subset \mathbb{R}^{(\mathbb{N})}$. Then obviously $\mathbb{R}^{(\mathbb{N})}/\mathbb{Z}^{(\mathbb{N})} = (S^1)^{\mathbb{N}}$, which is a smooth (even real analytic, see [10]) manifold modeled on $\mathbb{R}^{(\mathbb{N})}$. The reader may convince himself that the general Lie group covered by $\mathbb{R}^{(\mathbb{N})}$ is isomorphic to $(S^1)^{(A)} \times \mathbb{R}^{(\mathbb{N}\setminus A)}$ for $A \subseteq \mathbb{N}$.

As another example one may check easily that $\ell^\infty/(\mathbb{Z}^{\mathbb{N}} \cap \ell^\infty) = (S^1)^{\mathbb{N}}$, equipped with the 'uniform box topology'.

5.5. Extensions of Lie groups. Let H and K be Lie groups. A Lie group G is called an smooth *extension of groups* of H with kernel K if we have a short exact sequence of groups

$$\{e\} \to K \xrightarrow{i} G \xrightarrow{p} H \to \{e\}, \tag{1}$$

such that i and p are smooth and one of the following two equivalent conditions is satified:

(ii) p admits a local smooth section s near e (equivalently near any point), and i is initial (i.e., any f into K is smooth if and only if $i \circ f$ is smooth).

(iii) i admits a local smooth retraction r near e (equivalently near any point), and p is final (i.e., f from H is smooth if and only if $f \circ p$ is smooth).

Of course by $s(p(x))i(r(x)) = x$ the two conditions are equivalent, and then G is locally diffeomorphic to $K \times H$ via (r,p) with local inverse $(i \circ \mathrm{pr}_1).(s \circ \mathrm{pr}_2)$.

Not every smooth exact sequence of Lie groups admits local sections as required in (ii). Let for example K be a closed linear subspace in a convenient vector space G which is not a direct summand, and let H be G/K. Then the tangent mapping at 0 of a local smooth splitting would make K a direct summand.

Theorem. *Let $\{e\} \to K \xrightarrow{i} G \xrightarrow{p} H \to \{e\}$ be a smooth extension of Lie groups. Then G is regular if and only if both K and H are regular.*

Proof. Clearly the induced sequence of Lie algebras is also exact,

$$0 \to \mathfrak{k} \xrightarrow{i'} \mathfrak{g} \xrightarrow{p'} \mathfrak{h} \to 0,$$

with a bounded linear section T_es of p', so $\mathfrak{g}$ is isomorphic to $\mathfrak{k} \times \mathfrak{h}$ as convenient vector space.

Let us suppose that K and H are regular. Given $X \in C^\infty(\mathbb{R}, \mathfrak{g})$, we consider $Y(t) := p'(X(t)) \in \mathfrak{h}$ with evolution curve h satisfying $\frac{\partial}{\partial t}h(t) = T(\mu^{h(t)}).Y(t)$ and $h(0) = e$. By Lemma 5.2 it suffices to find smooth local solutions g near 0 of $\frac{\partial}{\partial t}g(t) = T(\mu^{g(t)}).X(t)$ with $g(0) = e$, depending smoothly on X. We look for solutions of the form $g(t) = s(h(t)).i(k(t))$, where k is a local evolution curve in K of a suitable curve $t \mapsto Z(t)$ in $\mathfrak{k}$, i.e., $\frac{\partial}{\partial t}k(t) = T(\mu^{k(t)}).Z(t)$ and $k(0) = e$. For this ansatz we have

$$\begin{aligned}
\tfrac{\partial}{\partial t}g(t) = \tfrac{\partial}{\partial t}\Big(s(h(t)).i(k(t))\Big) &= T(\mu_{s(h(t))}).Ti.\tfrac{\partial}{\partial t}k(t) + T(\mu^{i(k(t))}).Ts.\tfrac{\partial}{\partial t}h(t) \\
&= T(\mu_{s(h(t))}).Ti.T(\mu^{k(t)}).Z(t) + T(\mu^{i(k(t))}).Ts.T(\mu^{h(t)}).Y(t),
\end{aligned}$$

and we want this to be

$$T(\mu^{g(t)}).X(t) = T(\mu^{s(h(t)).i(k(t))}).X(t) = T(\mu^{i(k(t))}).T(\mu^{s(h(t))}).X(t).$$

Using $i \circ \mu^k = \mu^{i(k)} \circ i$ one quickly sees that

$$i'.Z(t) := \operatorname{Ad}\Big(s(h(t))^{-1}\Big).\Big(X(t) - T(\mu^{s(h(t))^{-1}}).Ts.T(\mu^{h(t)}).Y(t)\Big) \in \ker p'$$

solves the problem, so G is regular.

Let now G be regular. If $Y \in C^\infty(\mathbb{R},\mathfrak{h})$, then $p \circ \operatorname{Evol}^r_G(s' \circ Y) = \operatorname{Evol}_H(Y)$, since for $g := \operatorname{Evol}^r_G(s' \circ Y)$ we have

$$\tfrac{\partial}{\partial t} p(g(t)) = Tp.\tfrac{\partial}{\partial t} g(t) = Tp.T(\mu^{g(t)}).T_e s.Y(t) = T(\mu^{p(g(t))}).Y(t).$$

If $U \in C^\infty(\mathbb{R},\mathfrak{k})$, then $p\circ\operatorname{Evol}_G(i'\circ U) = \operatorname{Evol}_H(0) = e$ so that $\operatorname{Evol}_G(i'\circ U)(t) \in i(K)$ for all t and thus equals $i(\operatorname{Evol}_K(U)(t))$. ∎

5.6. Subgroups of regular Lie groups. Let G and K be Lie groups, let G be regular and let $i : K \to G$ be a smooth homomorphism which is initial (see 5.5) with $T_e i = i' : \mathfrak{k} \to \mathfrak{g}$ injective. We suspect that K is then regular, but we are only able to prove this under the following assumption:

> There is an open neighborhood $U \subset G$ of e and a smooth mapping $p : U \to E$ into a convenient vector space E such that $p^{-1}(0) = K \cap U$ and p constant on left cosets $Kg \cap U$.

Proof. For $Z \in C^\infty(\mathbb{R},\mathfrak{k})$ we consider $g(t) = \operatorname{Evol}_G(i' \circ Z)(t) \in G$. Then we have $\frac{\partial}{\partial t}(p(g(t))) = Tp.T(\mu^{g(t)}).i'(Z(t)) = 0$ by the assumption, so $p(g(t))$ is constant $p(e) = 0$, thus $g(t) = i(h(t))$ for a smooth curve h in H, since i is initial. Then $h = \operatorname{Evol}_H(Y)$ since i is an immersion, and h depends smoothly on Z since i is initial. ∎

5.7. Abelian and central extensions. From theorem 5.5 it is clear that any smooth extension G of a regular Lie group H with an abelian regular Lie group $(K,+)$ is again regular. We shall describe Evol_G in terms of Evol_G, Evol_K, and in terms of the action of H on K and the cocycle $c : H \times H \to K$ if the latter exists.

Let us first recall these notions. If we have a smooth extension with abelian normal subgroup K,

$$\{e\} \to K \xrightarrow{i} G \xrightarrow{p} H \to \{e\}$$

then a unique smooth action $\alpha : H \times K \to K$ by automorphisms is given by $i(\alpha_h(k)) = s(h)i(k)s(h)^{-1}$, where s is any smooth local section of p defined near h. If moreover p admits a global smooth section $s : H \to G$, which we assume without loss to satisfy $s(e) = e$, then we consider the smooth mapping $c : H \times H \to K$ given by $ic(h_1,h_2) := s(h_1).s(h_2).s(h_1.h_2)^{-1}$. Via the diffeomorphism

$K \times H \to G$ given by $(k,h) \mapsto i(k).s(h)$ the identity corresponds to $(0,e)$, the multiplication and the inverse in G look as follows:

$$\text{(1)} \qquad \begin{cases} (k_1,h_1).(k_2,h_2) = (k_1 + \alpha_{h_1} k_2 + c(h_1,h_2), h_1 h_2), \\ (k,h)^{-1} = (-\alpha_{h^{-1}}(k) - c(h^{-1},h), h^{-1}). \end{cases}$$

Associativity and $(0,e)^2 = (0,e)$ correspond to the fact that c satisfies the following *cocycle condition* and normalization

$$\text{(2)} \qquad \begin{cases} \alpha_{h_1}(c(h_2,h_3)) - c(h_1 h_2, h_3) + c(h_1, h_2 h_3) - c(h_1,h_2) = 0 \\ c(e,e) = 0. \end{cases}$$

These imply that $c(e,h) = 0 = c(h,e)$ and $\alpha_h(c(h^{-1},h)) = c(h,h^{-1})$. For a central extension the action is trivial, $\alpha_h = \mathrm{Id}_K$ for all $h \in H$.

If conversely H acts smoothly by automorphisms on an abelian Lie group K and if $c : H \times H \to K$ satisfies (2), then (1) describes a smooth Lie group structure on $K \times H$, which is a smooth extension of H over K with a global smooth section.

For later purposes let us compute

$$\begin{aligned} &(0,h_1).(0,h_2)^{-1} = (-\alpha_{h_1}(c(h_2^{-1},h_2)) + c(h_1,h_2^{-1}), h_1 h_2^{-1}), \\ &T_{(0,h_1)}(\mu^{(0,h_2)^{-1}}).(0,Y_{h_1}) = \\ &\qquad = (-T(\alpha^{c(h_2^{-1},h_2)}).Y_{h_1} + T(c(\quad, h_2^{-1})).Y_{h_1}, T(\mu^{h_2^{-1}}).Y_{h_1}). \end{aligned}$$

Let us now assume that K and H are moreover regular Lie groups. We consider a curve $t \mapsto X(t) = (U(t), Y(t))$ in the Lie algebra $\mathfrak{g}$ which as convenient vector space equals $\mathfrak{k} \times \mathfrak{h}$. From the proof of 5.5 we get that

$$\begin{aligned} g(t) &:= \mathrm{Evol}_G(U,Y)(t) = (0,h(t)).(k(t),e) = (\alpha_{h(t)}(k(t)), h(t)), \text{ where} \\ h(t) &:= \mathrm{Evol}_H(Y)(t) \in H, \\ (Z(t),0) &:= \mathrm{Ad}_G(0,h(t))^{-1}\Big((U(t),Y(t)) - T\mu^{(0,h(t))^{-1}}.(0, \tfrac{\partial}{\partial t} h(t))\Big) \\ Z(t) &= T_0(\alpha_{h(t)^{-1}}).\Big(U(t) + \big(T(\alpha^{c(h(t)^{-1},h(t))}) - T(c(\quad, h(t)^{-1}))\big).\tfrac{\partial}{\partial t} h(t)\Big), \\ k(t) &:= \mathrm{Evol}_K(Z)(t) \in K. \end{aligned}$$

5.8. Semidirect products. From theorem 5.5 we see immediately that the semidirect product of regular Lie groups is again regular. Since we shall need explicit formulas later we specialize the proof of 5.5 to this case.

Let H and K be regular Lie groups with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively. Let $\alpha : H \times K \to K$ be smooth such that $\check{\alpha} : H \to \mathrm{Aut}(K)$ is a group homomorphism. Then the semidirect product $K \rtimes H$ is the Lie group $K \times H$ with multiplication $(k,h).(k',h') = (k.\alpha_h(k'), h.h')$ and inverse $(k,h)^{-1} = (\alpha_{h^{-1}}(k)^{-1}, h^{-1})$. We have then $T_{(e,e)}(\mu^{(k',h')}).(U,Y) = (T(\mu^{k'}).U + T(\alpha^{k'}).Y, T(\mu^{h'}).Y)$.

Now we consider a curve $t \mapsto X(t) = (U(t), Y(t))$ in the Lie algebra $\mathfrak{k} \rtimes \mathfrak{h}$. Since $s : h \mapsto (e, h)$ is a smooth homomorphism of Lie groups, from the proof of 5.5 we get that

$$\begin{aligned}
g(t) :&= \operatorname{Evol}_{K \rtimes H}(U, Y)(t) = (e, h(t)).(k(t), e) = (\alpha_{h(t)}(k(t)), h(t)), \text{ where} \\
h(t) :&= \operatorname{Evol}_H(Y)(t) \in H, \\
(Z(t), 0) :&= \operatorname{Ad}_{K \rtimes H}(e, h(t)^{-1})(U(t), 0) = (T_e(\alpha_{h(t)^{-1}}).U(t), 0), \\
k(t) :&= \operatorname{Evol}_K(Z)(t) \in K.
\end{aligned}$$

5.9. Corollary. *Let G be a Lie group. Then via right trivialization $(\kappa^r, \pi_G) : TG \to \mathfrak{g} \times G$ the tangent group TG is isomorphic to the semidirect product $\mathfrak{g} \rtimes G$, where G acts by $\operatorname{Ad} : G \to \operatorname{Aut}(\mathfrak{g})$.*

So if G is a regular Lie group, then $TG \cong \mathfrak{g} \rtimes G$ is also regular, and $T\operatorname{evol}^r_G$ corresponds to $\operatorname{evol}^r_{TG}$. In particular for $(Y, X) \in C^\infty(\mathbb{R}, \mathfrak{g} \times \mathfrak{g}) = TC^\infty(\mathbb{R}, \mathfrak{g})$, where X is the footpoint, and we have

$$\begin{aligned}
\operatorname{evol}^r_{\mathfrak{g} \rtimes G}(Y, X) &= \Big(\operatorname{Ad}(\operatorname{evol}^r_G(X)) \int_0^1 \operatorname{Ad}(\operatorname{Evol}^r_G(X)(s)^{-1}).Y(s)\, ds,\ \operatorname{evol}^r_G(X)\Big) \\
T_X \operatorname{evol}^r_G.Y &= T(\mu_{\operatorname{evol}^r_G(X)}). \int_0^1 \operatorname{Ad}(\operatorname{Evol}^r_G(X)(s)^{-1}).Y(s)\, ds, \\
T_X(\operatorname{Evol}^r_G(\quad)(t)).Y &= T(\mu_{\operatorname{Evol}^r_G(X)(t)}). \int_0^t \operatorname{Ad}(\operatorname{Evol}^r_G(X)(s)^{-1}).Y(s)\, ds.
\end{aligned}$$

Note that in the semidirect product representation $TG \cong \mathfrak{g} \rtimes G$ the footpoint appears in the right factor G, contrary to the usual convention. We followed this also in $T\mathfrak{g} = \mathfrak{g} \rtimes \mathfrak{g}$.

Proof. Via right trivialization the tangent group TG is the semidirect product $\mathfrak{g} \rtimes G$, where G acts on the Lie algebra $\mathfrak{g}$ by $\operatorname{Ad} : G \to \operatorname{Aut}(\mathfrak{g})$, because by 3.2 we have for $g, h \in G$ and $X, Y \in \mathfrak{g}$, where $\mu = \mu_G$ is the multiplication on G:

$$\begin{aligned}
T_{(g,h)}\mu.(R_X(g), R_Y(h)) &= T(\mu^h).R_X(g) + T(\mu_g).R_Y(h) \\
&= T(\mu^h).T(\mu^g).X + T(\mu_g).T(\mu^h).Y \\
&= R_X(gh) + R_{\operatorname{Ad}(g)Y}(h), \\
T_g\nu.R_X(g) &= -T(\mu^{g^{-1}}).T(\mu_{g^{-1}}).T(\mu^g).X \\
&= -R_{\operatorname{Ad}(g^{-1})X}(g^{-1}),
\end{aligned}$$

so that we have

$$\begin{aligned}
\text{(1)} \qquad \mu_{\mathfrak{g} \rtimes G}((X, g), (Y, h)) &= (X + \operatorname{Ad}(g)Y, gh) \\
\nu_{\mathfrak{g} \rtimes G}(X, g) &= (-\operatorname{Ad}(g^{-1})X, g^{-1}).
\end{aligned}$$

Now we shall prove that the following diagram commutes and that the equations of the corollary follow. The lower triangle commutes by definition.

$$\begin{tikzcd}
TC^\infty(\mathbb{R},\mathfrak{g}) \arrow[r, "\cong"] \arrow[d, "T\mathrm{evol}_G"'] & C^\infty(\mathbb{R},\mathfrak{g}\rtimes\mathfrak{g}) \arrow[d, "\mathrm{evol}_{\mathfrak{g}\rtimes G}"] \arrow[dl, "\mathrm{evol}_{TG}"] \\
TG \arrow[r, "\cong"'] & \mathfrak{g}\rtimes G
\end{tikzcd}$$

For that we choose $X, Y \in C^\infty(\mathbb{R},\mathfrak{g})$. Let us first consider the evolution operator of the tangent group TG in the picture $\mathfrak{g}\rtimes G$. On $(\mathfrak{g},+)$ the evolution mapping is the definite integral, so going through the prescription 5.8 for $\mathrm{evol}_{\mathfrak{g}\rtimes G}$ we have in turn the following data:

$$\begin{aligned}
(2)\qquad & \mathrm{evol}_{\mathfrak{g}\rtimes G}(Y,X) = (h(1), g(1)), \qquad \text{where}\\
& g(t) := \mathrm{Evol}_G(X)(t) \in G,\\
& Z(t) := \mathrm{Ad}(g(t)^{-1}).Y(t) \in \mathfrak{g},\\
& h_0(t) := \mathrm{Evol}_{(\mathfrak{g},+)}(Z)(t) = \int_0^t \mathrm{Ad}(g(u)^{-1}).Y(u)\,du \in \mathfrak{g},\\
& h(t) := \mathrm{Ad}(g(t))h_0(t) = \mathrm{Ad}(g(t))\int_0^t \mathrm{Ad}(g(u)^{-1}).Y(u)\,du \in \mathfrak{g}.
\end{aligned}$$

This shows the first equation in the corollary. The differential equation for the curve $(h(t), g(t))$, which by Lemma 5.2 has a unique solution starting at $(0,e)$, looks as follows, using (1):

$$\begin{aligned}
\Big((h'(t), h(t)), g'(t)\Big) &= T_{(0,e)}(\mu^{(h(t),g(t))}_{\mathfrak{g}\rtimes G}).\Big((Y(t),0), X(t)\Big)\\
&= \Big(Y(t) + \Big(d\mathrm{Ad}(X(t)).h(t), 0 + \mathrm{Ad}(e).h(t)\Big), T(\mu_G^{g(t)}).X(t)\Big)\\
(3)\qquad h'(t) &= Y(t) + \mathrm{ad}(X(t))h(t)\\
g'(t) &= T(\mu_G^{g(t)})X(t).
\end{aligned}$$

For the computation of $T\mathrm{evol}_G$ we let

$$\begin{gathered}
g(t,s) := \mathrm{evol}_G\Big(u \mapsto t(X(tu) + sY(tu))\Big) = \mathrm{Evol}_G(X+sY)(t),\\
\text{satisfying } \delta^r g(\partial_t(t,s)) = X(t) + sY(t).
\end{gathered}$$

Then $T\mathrm{evol}_G(Y,X) = \partial_s|_0 g(1,s)$, and the derivative $\partial_s|_0 g(t,s)$ in TG corresponds to the element

$$(T(\mu^{g(t,0)^{-1}}).\partial_s|_0 g(t,s), g(t,0)) = (\delta^r g(\partial_s(t,0)), g(t,0)) \in \mathfrak{g}\rtimes G$$

via right trivialization. For the right hand side we have $g(t,0) = g(t)$, so it remains to show that $\delta^r g(\partial_s(t,0)) = h(t)$. We will show that $\delta^r g(\partial_s(t,0))$ is the unique solution of the differential equation (3) for $h(t)$. Using the Maurer Cartan equation $d\delta^r g - \frac{1}{2}[\delta^r g, \delta^r g] = 0$ from Lemma 5.1 we get

$$\begin{aligned}
\partial_t \delta^r g(\partial_s) &= \partial_s \delta^r g(\partial_t) + d(\delta^r g)(\partial_t, \partial_s) + \delta^r g([\partial_t, \partial_s]) \\
&= \partial_s \delta^r g(\partial_t) + [\delta^r g(\partial_t), \delta^r g(\partial_s)]_{\mathfrak{g}} + 0 \\
&= \partial_s (X(t) + sY(t)) + [X(t) + sY(t), \delta^r g(\partial_s)]_{\mathfrak{g}}
\end{aligned}$$

so that for $s = 0$ we get

$$\begin{aligned}
\partial_t \delta^r g(\partial_s(t,0)) &= Y(t) + [X(t), \delta^r g(\partial_s(t,0))]_{\mathfrak{g}} \\
&= Y(t) + \operatorname{ad}(X(t))\delta^r g(\partial_s(t,0)).
\end{aligned}$$

Thus $\delta^r g(\partial_s(t,0))$ is a solution of the inhomogeneous linear ordinary differential equation (3) as required.

It remains to check the last formula. Note that $X \mapsto tX(t\quad)$ is a bounded linear operator. So we have

$$\begin{aligned}
\operatorname{Evol}^r(X)(t) &= \operatorname{evol}(s \mapsto tX(ts)), \\
T_X(\operatorname{Evol}^r_G(\quad)(t)).Y &= T_{tX(t\quad)} \operatorname{evol}^r_G.(tY(t\quad)) \\
&= T(\mu_{\operatorname{evol}^r_G(tX(t\quad))}).\int_0^1 \operatorname{Ad}_G\Big(\operatorname{Evol}^r_G(tX(t\quad))(s)^{-1}\Big).tY(ts)\,ds \\
&= T(\mu_{\operatorname{Evol}^r_G(X)(t)}).\int_0^1 \operatorname{Ad}_G\Big(\operatorname{evol}^r_G(stX(st\quad))^{-1}\Big).tY(ts)\,ds \\
&= T(\mu_{\operatorname{Evol}^r_G(X)(t)}).\int_0^t \operatorname{Ad}_G\Big(\operatorname{Evol}^r_G(X)(s)^{-1}\Big).Y(s)\,ds.
\end{aligned}$$

This finishes the proof. ■

5.10. Corollary. *For a regular Lie group G the tangent mapping of the exponential mapping $\exp : \mathfrak{g} \to G$ is given by:*

$$\begin{aligned}
T_X \exp .Y &= T_e \mu_{\exp X}.\int_0^1 \operatorname{Ad}(\exp(-tX))Y\,dt \\
&= T_e \mu^{\exp X}.\int_0^1 \operatorname{Ad}(\exp(tX))Y\,dt
\end{aligned}$$

Remark. This formula was first proved by [6] for Lie groups with smooth exponential mapping. If G is a Banach Lie group then we have from 3.7.(iv) and 3.8 the series $\operatorname{Ad}(\exp(tX)) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\operatorname{ad}(X)^i$, so that we get the usual formula

$$T_X \exp = T_e \mu^{\exp X}.\sum_{i=0}^{\infty} \tfrac{1}{(i+1)!}\operatorname{ad}(X)^i.$$

Proof. Just apply 5.9 to constant curves $X, Y \in \mathfrak{g}$. ■

6. Bundles with regular structure groups

6.1. Theorem. *Let (P,p,M,G) be a smooth (locally trivial) principal fiber bundle with a regular Lie group as structure group. Let $\omega \in \Omega^1(P,\mathfrak{g})$ be a principal connection form.*

Then the parallel transport for the principal connection exists, is globally defined, and is G-equivariant. In detail: For each smooth curve $c:\mathbb{R}\to M$ there is a unique smooth mapping $\mathrm{Pt}_c:\mathbb{R}\times P_{c(0)}\to P$ such that the following holds:

(i) $\mathrm{Pt}(c,t,u)\in P_{c(t)}$, $\mathrm{Pt}(c,0)=Id_{P_{c(0)}}$, *and* $\omega(\frac{d}{dt}\mathrm{Pt}(c,t,u))=0$.
It has the following further properties:

(ii) $\mathrm{Pt}(c,t):P_{c(0)}\to P_{c(t)}$ *is G-equivariant, i.e., for all $g\in G$ and $u\in P$ we have* $\mathrm{Pt}(c,t,u.g)=\mathrm{Pt}(c,t,u).g$. *Moreover,* $\mathrm{Pt}(c,t)^*(\zeta_X|P_{c(t)})=\zeta_X|P_{c(0)}$ *for all* $X\in\mathfrak{g}$.

(iii) *For any smooth function $f:\mathbb{R}\to\mathbb{R}$ we have*
$\mathrm{Pt}(c,f(t),u)=\mathrm{Pt}(c\circ f,t,\mathrm{Pt}(c,f(0),u))$.

(iv) *The parallel transport is smooth as a mapping*

$$\mathrm{Pt}:C^\infty(\mathbb{R},M)\times_{(\mathrm{ev}_0,M,p\circ\mathrm{pr}_2)}(\mathbb{R}\times P)\to P,$$

where $C^\infty(\mathbb{R},M)$ is considered as a smooth space, see [4], 1.4.1.

Proof. For a principal bundle chart $(U_\alpha,\varphi_\alpha)$ we have the data from 4.4

$$\begin{aligned}
s_\alpha(x) &:= \varphi_\alpha^{-1}(x,e),\\
\omega_\alpha &:= s_\alpha^*\omega,\\
\omega\circ T(\varphi_\alpha^{-1}) &= (\varphi_\alpha^{-1})^*\omega\in\Omega^1(U_\alpha\times G;\mathfrak{g})\\
(\varphi_\alpha^{-1})^*\omega(\xi_x,T\mu_g.X) &= (\varphi_\alpha^{-1})^*\omega(\xi_x,0_g)+X=\mathrm{Ad}(g^{-1})\omega_\alpha(\xi_x)+X.
\end{aligned}$$

For a smooth curve $c:\mathbb{R}\to M$ the horizontal lift $\mathrm{Pt}(c,\quad,u)$ through $u\in P_{c(0)}$ is given by the ordinary differential equation $\omega(\frac{d}{dt}\mathrm{Pt}(c,t,u))=0$ with initial condition $\mathrm{Pt}(c,0,u)=u$, among all smooth lifts of c. Locally we have

$$\varphi_\alpha(\mathrm{Pt}(c,t,u))=(c(t),\gamma(t)),$$

so that

$$\begin{aligned}
0&=\mathrm{Ad}(\gamma(t))\omega(\tfrac{d}{dt}\mathrm{Pt}(c,t,u))=\mathrm{Ad}(\gamma(t))(\omega\circ T(\varphi_\alpha^{-1}))(c'(t),\gamma'(t))\\
&=\mathrm{Ad}(\gamma(t))((\varphi_\alpha^{-1})^*\omega)(c'(t),\gamma'(t))=\omega_\alpha(c'(t))+T(\mu^{\gamma(t)^{-1}})\gamma'(t),
\end{aligned}$$

i.e., $\gamma'(t)=-T(\mu^{\gamma(t)}).\omega_\alpha(c'(t))$, thus $\gamma(t)$ is given by

$$\gamma(t)=\mathrm{Evol}_G(-\omega_\alpha(c'))(t).\gamma(0)=\mathrm{evol}_G(s\mapsto -t\omega_\alpha(c'(ts))).\gamma(0).$$

By Lemma 5.2 we may glue the local solutions over different bundle charts U_α, so Pt exists globally.

Properties (i) and (iii) are now clear, and (ii) can be checked as follows: The condition $\omega(\frac{d}{dt}\mathrm{Pt}(c,t,u).g) = \mathrm{Ad}(g^{-1})\omega(\frac{d}{dt}\mathrm{Pt}(c,t,u)) = 0$ implies $\mathrm{Pt}(c,t,u).g = \mathrm{Pt}(c,t,u.g)$. For the second assertion we compute for $u \in P_{c(0)}$:

$$\begin{aligned}\mathrm{Pt}(c,t)^*(\zeta_X|P_{c(t)})(u) &= T\mathrm{Pt}(c,t)^{-1}\zeta_X(\mathrm{Pt}(c,t,u))\\ &= T\mathrm{Pt}(c,t)^{-1}\tfrac{d}{ds}|_0\mathrm{Pt}(c,t,u).\exp(sX)\\ &= T\mathrm{Pt}(c,t)^{-1}\tfrac{d}{ds}|_0\mathrm{Pt}(c,t,u.\exp(sX))\\ &= \tfrac{d}{ds}|_0\mathrm{Pt}(c,t)^{-1}\mathrm{Pt}(c,t,u.\exp(sX))\\ &= \tfrac{d}{ds}|_0 u.\exp(sX) = \zeta_X(u).\end{aligned}$$

(iv) It suffices to check that Pt respects smooth curves. So let $(f,g) : \mathbb{R} \to C^\infty(\mathbb{R},M) \times_M P \subset C^\infty(\mathbb{R},M) \times P$ be a smooth curve. By cartesian closedness of smooth spaces (see [4], 1.4.3) the smooth curve $f : \mathbb{R} \to C^\infty(\mathbb{R},M)$ corresponds to a smooth mapping $\hat{f} \in C^\infty(\mathbb{R}^2,M)$. For a principal bundle chart $(U_\alpha,\varphi_\alpha)$ as above we have $\varphi_\alpha(\mathrm{Pt}(f(s),t,g(s))) = (f(s)(t),\gamma(s,t))$, where γ is the evolution curve

$$\gamma(s,t) = \mathrm{Evol}_G\Big(-\omega_\alpha(\tfrac{\partial}{\partial t}\hat{f}(s,\quad))\Big)(t).\varphi_\alpha(g(s)),$$

which is clearly smooth in (s,t). ∎

6.2. Theorem. *Let (P,p,M,G) be a smooth principal bundle with a regular Lie group as structure group. Let $\omega \in \Omega^1(P,\mathfrak{g})$ be a principal connection form. If the connection is flat, then the horizontal subbundle $H^\omega(P) := \ker(\omega) \subset TP$ is integrable and defines a foliation.*

If M is connected then each leaf of this horizontal foliation is a covering of M. All leaves are isomorphic.

By standard arguments it follows that the principal bundle P is associated to the universal covering of M viewed as a principal fiber bundle with structure group the (discrete) fundamental group $\pi_1(M)$.

Proof. Let $(U_\alpha, u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subset E_\alpha)$ be a smooth chart of the manifold M and let $x_\alpha \in U_\alpha$ be such that $u_\alpha(x_\alpha) = 0$ and the c^∞-open subset $u_\alpha(U_\alpha)$ is disked in E_α. Let us also suppose that we have a principal fiber bundle chart $(U_\alpha, \varphi_\alpha : P|U_\alpha \to U_\alpha \times G)$. We may cover M by such U_α.

We shall now construct for each $w_\alpha \in P_{x_\alpha}$ a smooth section $\psi_\alpha : U_\alpha \to P$ whose image is an integral submanifold for the horizontal subbundle $\ker(\omega)$. Namely, for $x \in U_\alpha$ let $c_x(t) := u_\alpha^{-1}(tu_\alpha(x))$ for $t \in [0,1]$. Then we put

$$\psi_\alpha(x) := \mathrm{Pt}(c_x, 1, w_\alpha).$$

We have to show that the image of $T\psi_\alpha$ is contained in the horizontal bundle $\ker(\omega)$. Then we get $T_x\psi_\alpha = Tp|H^\omega(p)^{-1}_{\psi_\alpha(x)}$. This is a consequence of the following notationally more suitable claim.

Let $h : \mathbb{R}^2 \to U_\alpha$ be smooth with $h(0,s) = x_\alpha$ for all s.

Claim. $\frac{\partial}{\partial s}|Pt(h(.,s),1,w_\alpha)$ is horizontal.

Let $\varphi_\alpha(w_\alpha) = (x_\alpha, g_\alpha) \in U_\alpha \times G$. Then from the proof of Theorem 6.1 we know that

$$\begin{aligned}
\varphi_\alpha \mathrm{Pt}(h(\quad,s),1,w_\alpha) &= (h(1,s),\gamma(1,s)), \qquad \text{where}\\
\gamma(t,s) &= \tilde\gamma(t,s).g_\alpha\\
\tilde\gamma(t,s) &= \mathrm{evol}_G\Big(u\mapsto -t\omega_\alpha(\tfrac{\partial}{\partial t}h(tu,s))\Big)\\
&= \mathrm{Evol}_G\Big(-(h^*\omega_\alpha)(\partial_t(\quad,s))\Big)(t),\\
\omega_\alpha &= s_\alpha^*\omega, \qquad s_\alpha(x) = \varphi_\alpha^{-1}(x,e).
\end{aligned}$$

Since the curvature $\Omega = d\omega + \frac12[\omega,\omega]_\wedge = 0$ we have

$$\begin{aligned}
\partial_s(h^*\omega_\alpha)(\partial_t) &= \partial_t(h^*\omega_\alpha)(\partial_s) - d(h^*\omega_\alpha)(\partial_t,\partial_s) - (h^*\omega_\alpha)([\partial_t,\partial_s])\\
&= \partial_t(h^*\omega_\alpha)(\partial_s) + [(h^*\omega_\alpha)(\partial_t),(h^*\omega_\alpha)(\partial_s)]_{\mathfrak g} - 0.
\end{aligned}$$

Using this and the expression for $T\mathrm{evol}_G$ from 5.9 we have then:

$$\begin{aligned}
\tfrac{\partial}{\partial s}\tilde\gamma(1,s) &= T_{-(h^*\omega_\alpha)(\partial_t)(\quad,s)}\mathrm{evol}_G.\Big(-\partial_s(h^*\omega_\alpha)(\partial_t)(\quad,s)\Big)\\
&= -T(\mu_{\tilde\gamma(1,s)}).\int_0^1 \mathrm{Ad}(\tilde\gamma(t,s)^{-1})\partial_s(h^*\omega_\alpha)(\partial_t)\,dt\\
&= -T(\mu_{\tilde\gamma(1,s)}).\Big(\int_0^1 \mathrm{Ad}(\tilde\gamma(t,s)^{-1})\partial_t(h^*\omega_\alpha)(\partial_s)\,dt +\\
&\qquad + \int_0^1 \mathrm{Ad}(\tilde\gamma(t,s)^{-1}).\mathrm{ad}((h^*\omega_\alpha)(\partial_t)).(h^*\omega_\alpha)(\partial_s)\,dt\Big).
\end{aligned}$$

Next we integrate by parts, use 3.9.(3), and $\kappa^l(\partial_t\tilde\gamma(t,s)^{-1}) = (h^*\omega_\alpha)(\partial_t)(t,s)$ which follows from 5.2.

$$\begin{aligned}
&\int_0^1 \mathrm{Ad}(\tilde\gamma(t,s)^{-1})\partial_t(h^*\omega_\alpha)(\partial_s)\,dt =\\
&\quad= -\int_0^1 \Big(\partial_t\mathrm{Ad}(\tilde\gamma(t,s)^{-1})\Big)(h^*\omega_\alpha)(\partial_s)\,dt + \mathrm{Ad}(\tilde\gamma(t,s)^{-1})(h^*\omega_\alpha)(\partial_s)\Big|_{t=0}^{t=1}\\
&\quad= -\int_0^1 \mathrm{Ad}(\tilde\gamma(t,s)^{-1}).\mathrm{ad}\Big(\kappa^l\partial_t(\tilde\gamma(t,s)^{-1})\Big).(h^*\omega_\alpha)(\partial_s)\,dt\\
&\qquad + \mathrm{Ad}(\tilde\gamma(1,s)^{-1})(h^*\omega_\alpha)(\partial_s)(1,s) - 0\\
&\quad= -\int_0^1 \mathrm{Ad}(\tilde\gamma(t,s)^{-1}).\mathrm{ad}\Big((h^*\omega_\alpha)(\partial_t)\Big).(h^*\omega_\alpha)(\partial_s)\,dt\\
&\qquad + \mathrm{Ad}(\tilde\gamma(1,s)^{-1})(h^*\omega_\alpha)(\partial_s)(1,s),
\end{aligned}$$

so that finally

$$\begin{aligned}
\tfrac{\partial}{\partial s}\tilde\gamma(1,s) &= -T(\mu_{\tilde\gamma(1,s)}).\mathrm{Ad}(\tilde\gamma(1,s)^{-1})(h^*\omega_\alpha)(\partial_s)(1,s)\\
&= -T(\mu^{\tilde\gamma(1,s)}).(h^*\omega_\alpha)(\partial_s)(1,s),\\
\tfrac{\partial}{\partial s}\gamma(1,s) &= T(\mu^{g_\alpha}).\tfrac{\partial}{\partial s}\tilde\gamma(1,s)\\
&= -T(\mu_{\gamma(1,s)}).\mathrm{Ad}(\gamma(1,s)^{-1})(h^*\omega_\alpha)(\partial_s)(1,s)\\
\omega(\tfrac{\partial}{\partial s}\mathrm{Pt}(h(\quad,s),1,w_\alpha)) &= ((\varphi_\alpha^{-1})^*\omega)\Big(\tfrac{\partial}{\partial s}h(1,s),\tfrac{\partial}{\partial s}\gamma(1,s)\Big)\\
&= \mathrm{Ad}(\gamma(1,s)^{-1})\omega_\alpha(\tfrac{\partial}{\partial s}h(1,s)) - \mathrm{Ad}(\gamma(1,s)^{-1})(h^*\omega_\alpha)(\partial_s)(1,s) = 0,
\end{aligned}$$

where in the end we used 4.4.(6). So the claim follows.

By the claim and by the uniqueness of parallel transport 6.1.(i) we conclude that for any smooth curve c in U_α the horizontal curve $\psi_\alpha(c(t))$ coincides with $\mathrm{Pt}(c,t,\psi_\alpha(c(0)))$. Moreover $U_\alpha \times G$ is G-equivariantly diffeomorphic to $p^{-1}(U_\alpha)$ via $(x,g)\mapsto \psi_\alpha(x).g$.

To finish the proof we may now glue overlapping right translations of $\psi_\alpha(U_\alpha)$ to maximal integral manifolds of the horizontal subbundle. As a subset such an integral manifold consists of all endpoints of parallel transports of a fixed point. These are diffeomorphic covering spaces of M via right translations. ■

It is not clear, however, that the integral submanifolds of the theorem are initial submanifolds of P, or that they intersect each fiber in a totally disconnected subset, since M might have uncountable fundamental group.

6.3. Holonomy groups. Let (P,p,M,G) be a principal fiber bundle with regular structure group G so that all parallel transports exist by Theorem 6.1. Let $\Phi = \zeta\circ\omega$ be a principal connection. We assume that M is connected and we fix $x_0\in M$.

Now let us fix $u_0\in P_{x_0}$. Consider the subgroup $\mathrm{Hol}(\omega,u_0)$ of the structure group G which consists of all elements $\tau(u_0,\mathrm{Pt}(c,t,u_0))\in G$ for c any piecewise smooth closed loop through x_0. Reparametrizing c by a function which is flat at each corner of c we may assume that any c is smooth. We call $\mathrm{Hol}(\omega,u_0)$ the *holonomy group* of the connection. If we consider only those curves c which are nullhomotopic, we obtain the *restricted holonomy group* $\mathrm{Hol}_0(\omega,x_0)$, a normal subgroup in $\mathrm{Hol}(\omega,u_0)$.

Theorem.

(i) *We have* $\mathrm{Hol}(\omega,u_0.g)=\mathrm{conj}(g^{-1})\mathrm{Hol}(\omega,u_0)$ *and* $\mathrm{Hol}_0(\omega,u_0.g)=\mathrm{conj}(g^{-1})\mathrm{Hol}_0(\omega,u_0)$.

(ii) *For each curve c in M with $c(0)=x_0$ we have* $\mathrm{Hol}(\omega,\mathrm{Pt}(c,t,u_0))=\mathrm{Hol}(\omega,u_0)$ *and* $\mathrm{Hol}_0(\omega,\mathrm{Pt}(c,t,u_0))=\mathrm{Hol}_0(\omega,u_0)$.

Proof. (i) This follows from the properties of the mapping τ from 4.1 and from the from the G-equivariance of the parallel transport:

$$\tau(u_0.g,\mathrm{Pt}(c,1,u_0.g)) = \tau(u_0,\mathrm{Pt}(c,1,u_0).g) = g^{-1}.\tau(u_0,\mathrm{Pt}(c,1,u_0)).g.$$

(ii) By reparametrizing the curve c we may assume that $t=1$, and we put $\mathrm{Pt}(c,1,u_0)=:u_1$. Then by definition for an element $g\in G$ we have

$g \in \mathrm{Hol}(\omega, u_1)$ if and only if $g = \tau(u_1, \mathrm{Pt}(e, 1, u_1))$ for some closed smooth loop e through $x_1 := c(1) = p(u_1)$, i.e.,

$$\mathrm{Pt}(c,1)(r^g(u_0)) = r^g(\mathrm{Pt}(c,1)(u_0)) = u_1 g = \mathrm{Pt}(e,1)(\mathrm{Pt}(c,1)(u_0))$$
$$u_0 g = \mathrm{Pt}(c,1)^{-1}\mathrm{Pt}(e,1)\mathrm{Pt}(c,1)(u_0) = \mathrm{Pt}(c.e.c^{-1}, 3)(u_0),$$

where $c.e.c^{-1}$ is the curve travelling along $c(t)$ for $0 \le t \le 1$, along $e(t-1)$ for $1 \le t \le 3$, and along $c(3-t)$ for $2 \le t \le 3$. This is equivalent to $g \in \mathrm{Hol}(\omega, u_0)$. Furthermore e is nullhomotopic if and only if $c.e.c^{-1}$ is nullhomotopic, so we also have $\mathrm{Hol}_0(\omega, u_1) = \mathrm{Hol}_0(\omega, u_0)$. ∎

7. Rudiments of Lie theory for regular Lie groups

7.1. From Lie algebras to Lie groups. It is not true in general that every convenient Lie agebra is the Lie algebra of a convenient Lie group. This is wrong for Banach Lie algebras and Banach Lie groups, one of the first examples is from [3], see also [7].

To Lie subalgebras in the Lie algebra of a Lie group do not correspond Lie subgroups in general, see the following easy example:

Let $\mathfrak{g} \subset \mathfrak{X}_c(\mathbb{R}^2)$ be the closed Lie subalgebra of all vector fields with compact support on $\mathbb{R}^2$ of the form $X(x,y) = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$ where g vanishes on the strip $0 \le x \le 1$.

Claim. There is no Lie subgroup G of $\mathrm{Diff}(\mathbb{R}^2)$ corresponding to $\mathfrak{g}$.

If G exists there is a smooth curve $t \mapsto f_t \in G \subset \mathrm{Diff}_c(\mathbb{R}^2)$ such that the smooth curve $X_t := (\frac{\partial}{\partial t} f_t) \circ f_t^{-1}$ in $\mathfrak{g}$ has the property that $X_0 = f\frac{\partial}{\partial x}$ where $f = 1$ near 0. But then f_t moves the strip to the right for small t, so $\mathfrak{g}$ is not invariant under $\mathrm{Ad}^G(f_t) = f_t^*$, a contradiction.

So we see that on any manifold of dimension greater that 2 there are closed Lie subalgebras of the Lie algebra of vector fields with compact support, which do not admit Lie subgroups.

Note that this example does not work for the Lie group of real analytic diffeomorphisms on a compact manifold, see [9].

7.2. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. For a smooth mapping $f : M \to G$ we considered in 5.1 the right logarithmic derivative $\delta^r f \in \Omega^1(M;\mathfrak{g})$ which is given by $\delta^r f_x := T(\mu^{f(x)^{-1}}) \circ T_x f : T_x M \to T_{f(x)}G \to \mathfrak{g}$ and which satisfies the left (from the left action) Maurer-Cartan equation

$$d\delta^r f - \tfrac{1}{2}[\delta^r f, \delta^r f]^{\mathfrak{g}}_{\wedge} = 0.$$

Similarly the left logarithmic derivative $\delta^l f \in \Omega^1(M;\mathfrak{g})$ of $f \in C^\infty(M,G)$ was given by $\delta^l f_x := T(\mu_{f(x)^{-1}}) \circ T_x f : T_x M \to T_{f(x)}G \to \mathfrak{g}$ and satisfies the right (from the right action) Maurer Cartan equation

$$d\delta^l f + \tfrac{1}{2}[\delta^l f, \delta^l f]^{\mathfrak{g}}_{\wedge} = 0.$$

For regular Lie groups we have the following converse:

Theorem. *Let G be a connected regular Lie group with Lie algebra $\mathfrak{g}$.*

If a 1-form $\varphi \in \Omega^1(M;\mathfrak{g})$ satisfies $d\varphi - \frac{1}{2}[\varphi,\varphi]_\wedge = 0$ then for each simply connected subset $U \subset M$ there exists a smooth mapping $f : U \to G$ with $\delta^r f = \varphi|U$, and f is uniquely detemined up to a right translation in G.

If a 1-form $\psi \in \Omega^1(M;\mathfrak{g})$ satisfies $d\psi + \frac{1}{2}[\psi,\psi]_\wedge = 0$ then for each simply connected subset $U \subset M$ there exists a smooth mapping $f : U \to G$ with $\delta^l f = \psi|U$, and f is uniquely determined up to a left translation in G.

The mapping f is called the *left developping* of φ, or the *right developping* of ψ, respectively.

Proof. Let us treat the right logarithmic derivative since it leads to a principal connection for a bundle with right principal action. For the left logarithmic derivative the proof is similar, with the changes described in the second part of the proof of 5.1.

We put ourselves into the situation of the proof of 5.1. If we are given a 1-form $\varphi \in \Omega^1(M;\mathfrak{g})$ with $d\varphi - \frac{1}{2}[\varphi,\varphi]_\wedge = 0$ then we consider the 1-form $\omega^r \in \Omega^1(M \times G;\mathfrak{g})$, given by the analogon of 5.1(1),

$$\omega^r = \kappa^l - (\mathrm{Ad} \circ \mathrm{Inv}).\varphi \tag{1}$$

Then ω^r is a principal connection form on $M \times G$, since it reproduces the generators in $\mathfrak{g}$ of the fundamental vector fields for the principal right action, i.e., the left invariant vector fields, and ω^r is G-equivariant:

$$\begin{aligned}((\mu^g)^*\omega^r)_h &= \omega^r_{hg} \circ (Id \times T(\mu^g)) = T(\mu_{g^{-1}.h^{-1}}).T(\mu^g) - \mathrm{Ad}(g^{-1}.h^{-1}).\varphi \\ &= \mathrm{Ad}(g^{-1}).\omega^r_h.\end{aligned}$$

The computation in 5.1(3) for φ instead of $\delta^r f$ shows that this connection is flat. Since the structure group G is regular, by Theorem 6.2 the horizontal bundle is integrable, and $\mathrm{pr}_1 : M \times G \to M$, restricted to each horizontal leaf, is a covering. Thus it may be inverted over each simply connected subset $U \subset M$, and the inverse $(\mathrm{Id}, f) : U \to M\times G$ is unique up to the choice of the branch of the covering, and the choice of the leaf, i.e., f is unique up to a right translation by an element of G. The beginning of the proof of 5.1 then shows that $\delta^r f = \varphi|U$.∎

7.3. Theorem. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $f : \mathfrak{g} \to \mathfrak{h}$ be a bounded homomorphism of Lie algebras. If H is regular and if G is simply connected then there exists a unique homomorphism $F : G \to H$ of Lie groups with $T_eF = f$.*

This theorem is due to Milnor ([13]) and Yoshioka et al. ([19]).

Proof. We consider the 1-form

$$\psi \in \Omega^1(G;\mathfrak{h}), \quad \psi := f \circ \kappa^r, \quad \psi_g(\xi_g) = f(T(\mu^{g^{-1}}).\xi_g),$$

where κ^r is the right Maurer Cartan form from 5.1. It satisfies the left Maurer Cartan equation

$$\begin{aligned}d\psi - \tfrac{1}{2}[\psi,\psi]^{\mathfrak{h}}_\wedge &= d(f\circ\kappa^r) - \tfrac{1}{2}[f\circ\kappa^r, f\circ\kappa^r]^{\mathfrak{h}}_\wedge \\ &= f\circ(d\kappa^r - \tfrac{1}{2}[\kappa^r,\kappa^r]^{\mathfrak{g}}_\wedge) = 0,\end{aligned}$$

by 5.1.(2′). But then we can use Theorem 7.2 to conclude that there exists a unique smooth mapping $F : G \to H$ with $F(e) = e$ and whose right logarithmic derivative satisfies $\delta^r F = \psi$. For $g \in G$ we have $(\mu^g)^*\psi = \psi$, thus also

$$\delta^r(F \circ \mu^g) = \delta^r F \circ T(\mu^g) = (\mu^g)^*\psi = \psi.$$

By uniqueness in Theorem 7.2 again the mappings $F \circ \mu^g, F : G \to H$ differ only by right translation in H by $(F \circ \mu^g)(e) = F(g)$, so that $F \circ \mu^g = \mu^{F(g)} \circ F$, or $F(g.g_1) = F(g).F(g_1)$. This also implies $F(g).F(g^{-1}) = F(g.g^{-1}) = F(e) = e$, so that F is the unique homomorphism of Lie groups we looked for. ∎

7.4. Theorem. *For a regular Lie group G we have*

$$\begin{aligned}\operatorname{evol}^r(X).\operatorname{evol}^r(Y) &= \operatorname{evol}^r\Big(t \mapsto X(t) + \operatorname{Ad}_G(\operatorname{Evol}^r(X)(t)).Y(t)\Big),\\ \operatorname{evol}^r(X)^{-1} &= \operatorname{evol}^r\Big(t \mapsto -\operatorname{Ad}_G(\operatorname{Evol}^r(X)(t)^{-1}).X(t)\Big),\end{aligned}$$

so that $\operatorname{evol}^r : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is a surjective smooth homomorphism of Lie groups, where on $C^\infty(\mathbb{R}, \mathfrak{g})$ we consider the operations

$$\begin{aligned}(X * Y)(t) &= X(t) + \operatorname{Ad}_G(\operatorname{Evol}^r(X)(t)).Y(t),\\ X^{-1}(t) &= -\operatorname{Ad}_G(\operatorname{Evol}^r(X)(t)^{-1}).X(t).\end{aligned}$$

With this operations and with 0 as unit element $(C^\infty(\mathbb{R}, \mathfrak{g}), *)$ becomes again a regular Lie group. Its Lie algebra is $C^\infty(\mathbb{R}, \mathfrak{g})$ with bracket

$$\begin{aligned}[X, Y]_{C^\infty(\mathbb{R},\mathfrak{g})}(t) &= \Big[\int_0^t X(s)\,ds, Y(t)\Big]_{\mathfrak{g}} + \Big[X(t), \int_0^t Y(s)\,ds\Big]_{\mathfrak{g}}\\ &= \tfrac{\partial}{\partial t}\Big[\textstyle\int_0^t X(s)\,ds, \int_0^t Y(s)\,ds\Big]_{\mathfrak{g}}.\end{aligned}$$

Its evolution operator is given by

$$\begin{aligned}\operatorname{evol}_{(C^\infty(\mathbb{R},\mathfrak{g}),*)}(X) &:= \operatorname{Ad}_G(\operatorname{evol}_G(Y^s)).\int_0^1 \operatorname{Ad}_G(\operatorname{Evol}_G(Y^s)(v)^{-1}).X(v)(s)\,dv,\\ Y^s(t) &:= \int_0^s X(t)(u)du.\end{aligned}$$

Proof. For $X, Y \in C^\infty(\mathbb{R}, \mathfrak{g})$ we compute

$$\begin{aligned}&\tfrac{\partial}{\partial t}\Big(\operatorname{Evol}^r(X)(t).\operatorname{Evol}^r(Y)(t)\Big) =\\ &\quad = T(\mu^{\operatorname{Evol}^r(Y)(t)}).T(\mu^{\operatorname{Evol}^r(X)(t)}).X(t) + T(\mu_{\operatorname{Evol}^r(X)(t)}).T(\mu^{\operatorname{Evol}^r(Y)(t)}).Y(t)\\ &\quad = T(\mu^{\operatorname{Evol}^r(X)(t).\operatorname{Evol}^r(Y)(t)}).(X(t) + \operatorname{Ad}_G(\operatorname{Evol}^r(X)(t))Y(t)),\end{aligned}$$

which implies also

$$\operatorname{Evol}^r(X).\operatorname{Evol}^r(Y) = \operatorname{Evol}^r(X * Y), \quad \operatorname{Evol}(X)^{-1} = \operatorname{Evol}(X^{-1}).$$

Thus $\mathrm{Evol} : C^\infty(\mathbb{R},\mathfrak{g}) \to C^\infty(\mathbb{R},G)$ is a group isomorphism onto the subgroup $\{c \in C^\infty(\mathbb{R},G) : c(0) = e\}$ of $C^\infty(\mathbb{R},G)$ with the pointwise product, which, however, is only a smooth space, see [4], 1.4.1. Nevertheless it follows that the product on $C^\infty(\mathbb{R},\mathfrak{g})$ is associative. It is clear that these operations are smooth, so that the convenient vector space $C^\infty(\mathbb{R},\mathfrak{g})$ becomes a Lie group; and $C^\infty(\mathbb{R},G)$ becomes a manifold.

Now we aim for the Lie bracket. We have

$$\begin{aligned}
(X * Y * X^{-1})(t) &= \Big(\big(X + \mathrm{Ad}(\mathrm{Evol}^r(X)).Y\big) * \big(-\mathrm{Ad}(\mathrm{Evol}^r(X)^{-1}).X\big)\Big)(t)\\
&= X(t) + \mathrm{Ad}(\mathrm{Evol}^r(X)(t)).Y(t) -\\
&\quad - \mathrm{Ad}\Big(\mathrm{Evol}^r(X * Y)(t)\Big).\mathrm{Ad}\Big(\mathrm{Evol}^r(X)(t)^{-1}\Big).X(t)\\
&= X(t) + \mathrm{Ad}\Big(\mathrm{Evol}^r(X)(t)\Big).Y(t) -\\
&\quad - \mathrm{Ad}\Big(\mathrm{Evol}^r(X)(t)\Big).\mathrm{Ad}\Big(\mathrm{Evol}^r(Y)(t)\Big).\mathrm{Ad}\Big(\mathrm{Evol}^r(X)(t)^{-1}\Big).X(t).
\end{aligned}$$

We shall need

$$\begin{aligned}
T_0\Big(\mathrm{Ad}_G(\mathrm{Evol}^r(\quad)(t))\Big).Y &= T_e\mathrm{Ad}_G.T_0(\mathrm{Evol}^r(\quad)(t)).Y\\
&= \mathrm{ad}_{\mathfrak{g}}\Big(\int_0^t Y(s)\,ds\Big), \qquad \text{by 5.9.}
\end{aligned}$$

Using this we can differentiate the conjugation,

$$\begin{aligned}
(\mathrm{Ad}_{C^\infty(\mathbb{R},\mathfrak{g})}(X).Y)(t) &= (T_0(X * (\quad) * X^{-1}).Y)(t)\\
&= 0 + \mathrm{Ad}(\mathrm{Evol}^r(X)(t)).Y(t) -\\
&\quad - \mathrm{Ad}(\mathrm{Evol}^r(X)(t)).\Big(T_0(\mathrm{Ad}(\mathrm{Evol}^r(\quad)(t))).Y\Big).\mathrm{Ad}(\mathrm{Evol}^r(X)(t)^{-1}).X(t)\\
&= \mathrm{Ad}(\mathrm{Evol}^r(X)(t)).Y(t) -\\
&\quad - \mathrm{Ad}(\mathrm{Evol}^r(X)(t)).\mathrm{ad}_{\mathfrak{g}}\Big(\int_0^t Y(s)\,ds\Big).\mathrm{Ad}(\mathrm{Evol}^r(X)(t)^{-1}).X(t)\\
&= \mathrm{Ad}(\mathrm{Evol}^r(X)(t)).Y(t) - \mathrm{ad}_{\mathfrak{g}}.\Big(\mathrm{Ad}(\mathrm{Evol}^r(X)(t)).\int_0^t Y(s)\,ds\Big).X(t).
\end{aligned}$$

Now we can compute the Lie bracket

$$\begin{aligned}
[X,Y]_{C^\infty(\mathbb{R},\mathfrak{g})}(t) &= \Big(T_0(\mathrm{Ad}_{C^\infty(\mathbb{R},\mathfrak{g})}(\quad).Y).X\Big)(t)\\
&= T_0\Big(\mathrm{Ad}(\mathrm{Evol}^r(\quad)(t)).X\Big).Y(t) - 0 - \Big[\mathrm{Ad}(\mathrm{Evol}^r(0)(t)).\int_0^t Y(s)\,ds, X(t)\Big]_{\mathfrak{g}}\\
&= \Big[\int_0^t X(s)\,ds, Y(t)\Big]_{\mathfrak{g}} - \Big[\int_0^t Y(s)\,ds, X(t)\Big]_{\mathfrak{g}}\\
&= \Big[\int_0^t X(s)\,ds, Y(t)\Big]_{\mathfrak{g}} + \Big[X(t), \int_0^t Y(s)\,ds\Big]_{\mathfrak{g}}\\
&= \tfrac{\partial}{\partial t}\Big[\textstyle\int_0^t X(s)\,ds, \int_0^t Y(s)\,ds\Big]_{\mathfrak{g}}.
\end{aligned}$$

Now we show that the Lie group $(C^\infty(\mathbb{R},\mathfrak{g}),*)$ is regular. For this let $\check{X}\in C^\infty(\mathbb{R},C^\infty(\mathbb{R},\mathfrak{g}))$ correspond to $X\in C^\infty(\mathbb{R}^2,\mathfrak{g})$. We look for $g\in C^\infty(\mathbb{R}^2,\mathfrak{g})$ which satisfies the equation 5.3.(1):

$$\begin{aligned}
\mu^{g(t,\quad)}(Y)(s) &= (Y*g(t,\quad))(s)=Y(s)+\mathrm{Ad}_G(\mathrm{Evol}_G(Y)(s)).g(t,s)\\
\tfrac{\partial}{\partial t}g(t,s) &= \Big(T_0(\mu^{g(t,\quad)}).X(t,\quad)\Big)(s)\\
&= X(t,s)+\Big(T_0\Big(\mathrm{Ad}_G(\mathrm{Evol}_G(\quad)(s))\Big).X(t,\quad)\Big).g(t,s)\\
&= X(t,s)+\mathrm{ad}_{\mathfrak{g}}\Big(\int_0^s X(t,u)du\Big).g(t,s)\\
&= X(t,s)+\Big[\int_0^s X(t,u)du),g(t,s)\Big]_{\mathfrak{g}}.
\end{aligned}$$

This is the differential equation 5.9.(3), depending smoothly on a further parameter s, which has the following unique solution which is given by 5.9.(2)

$$\begin{aligned}
g(t,s) &:= \mathrm{Ad}_G(\mathrm{Evol}_G(Y^s)(t)).\int_0^t \mathrm{Ad}_G(\mathrm{Evol}_G(Y^s)(v)^{-1}).X(v,s)\,dv\\
Y^s(t) &:= \int_0^s X(t,u)du.
\end{aligned}$$

Since this solution is visibly smooth in X, the Lie group $C^\infty(\mathbb{R},\mathfrak{g})$ is regular. For convenience (yours, not ours) we show now (once more) that this is a solution. Putting $Y^s(t):=\int_0^s X(t,u)du$ we have by 3.9.(iii)

$$\begin{aligned}
&\tfrac{\partial}{\partial t}g(t,s)=\\
&\quad= d\mathrm{Ad}(\tfrac{\partial}{\partial t}\mathrm{Evol}(Y^s)(t)).\textstyle\int_0^t \mathrm{Ad}(\mathrm{Evol}(Y^s)(v)^{-1}).X(v,s)\,dv\\
&\qquad+\mathrm{Ad}(\mathrm{Evol}(Y^s)(t)).\mathrm{Ad}(\mathrm{Evol}(Y^s)(t)^{-1}).X(t,s)\\
&\quad= ((\mathrm{ad}\circ\kappa^r).\mathrm{Ad})\Big(T(\mu^{\mathrm{Evol}(Y^s)(t)}).Y^s(t)\Big).\int_0^t \mathrm{Ad}(\mathrm{Evol}(Y^s)(v)^{-1}).X(v,s)\,dv\\
&\qquad+X(t,s)\\
&\quad= \mathrm{ad}(Y^s(t)).\mathrm{Ad}(\mathrm{Evol}(Y^s)(t)).\int_0^t \mathrm{Ad}(\mathrm{Evol}(Y^s)(v)^{-1}).X(v,s)\,dv+X(t,s)\\
&\quad= \Big[\int_0^s X(t,u)du),g(t,s)\Big]_{\mathfrak{g}}+X(t,s).
\end{aligned}$$

∎

7.5. Corollary. *Let G be a regular Lie group. Then as smooth spaces and groups we have the following isomorphims*

$$(C^\infty(\mathbb{R},\mathfrak{g}),*)\rtimes G\cong\{f\in C^\infty(\mathbb{R},G):f(0)=e\}\rtimes G\cong C^\infty(\mathbb{R},G),$$

where $g\in G$ acts on f by $(\alpha_g(f))(t)=g.f(t).g^{-1}$, and on $X\in C^\infty(\mathbb{R},\mathfrak{g})$ by $\alpha_g(X)(t)=\mathrm{Ad}_G(g)(X(t))$. The leftmost space is a smooth manifold, thus all spaces are regular Lie groups.

For the Lie algebras we have an isomorphism

$$C^\infty(\mathbb{R},\mathfrak{g}) \rtimes \mathfrak{g} \cong C^\infty(\mathbb{R},\mathfrak{g}),$$
$$(X,\eta) \mapsto \Big(t \mapsto \eta + \int_0^t X(s)ds\Big)$$
$$(Y',Y(0)) \leftarrow Y$$

where on the left hand side the Lie bracket is given by

$$[(X_1,\eta_1),(X_2,\eta_2)] =$$
$$= \Big(t \mapsto [\textstyle\int_0^t X_1(s)\,ds, X_2(t)]_{\mathfrak{g}} + [X_1(t),\int_0^t X_2(s)\,ds]_{\mathfrak{g}} + [\eta_1,X_2(t)]_{\mathfrak{g}} - [\eta_2,X_1]_{\mathfrak{g}},$$
$$[\eta_1,\eta_2]_{\mathfrak{g}}\Big),$$

and where on the right hand side the bracket is given by

$$[X,Y](t) = [X(t),Y(t)]_{\mathfrak{g}}.$$

On the right hand sides the evolution operator is given by

$$\operatorname{Evol}^r_{C^\infty(\mathbb{R},G)} = C^\infty(\mathbb{R},\operatorname{Evol}^r_G).$$

7.6. Remarks . Let G be a connected regular Lie group. The smooth homomorphism $\operatorname{evol}^r_G : C^\infty(\mathbb{R},\mathfrak{g}) \to G$ admits local smooth sections. Namely using a smooth chart near e of G we can choose a smooth curve $c_g : \mathbb{R} \to G$ with $c_g(0) = e$ and $c_g(1) = g$, depending smoothly on g, for g near e. Then $s(g) := \delta^r c_g$ is a local smooth section. We have an extension of groups

$$0 \to K \to C^\infty(\mathbb{R},\mathfrak{g}) \overset{\operatorname{evol}^r_G}{\to} G \to \{e\}$$

where $K = \ker(\operatorname{evol}^r_G)$ is isomorphic to the smooth group $\{f \in C^\infty(\mathbb{R},G) : f(0) = e, f(1) = e\}$ via the mapping Evol^r_G. We do not know whether K is a submanifold.

Next we consider the smooth group $C^\infty((S^1,1),(G,e))$ of all smooth mappings $f : S^1 \to G$ with $f(1) = e$. With pointwise multiplication this is a splitting closed normal subgroup of the regular Lie group $C^\infty(S^1,G)$ with the manifold structure described in [10] and [12]. Moreover $C^\infty(S^1,G)$ is the semidirect product $C^\infty((S^1,1),(G,e)) \rtimes G$, where G acts by conjugation on $C^\infty((S^1,1),(G,e))$. So by Theorem 5.5 the subgroup $C^\infty((S^1,1),(G,e))$ is also regular.

The right logarithmic derivative $\delta^r : C^\infty(S^1,G) \to C^\infty(S^1,\mathfrak{g})$ restricts to a diffeomorphism $C^\infty((S^1,1),(G,e)) \to \ker(\operatorname{evol}_G) \subset C^\infty(S^1,\mathfrak{g})$, thus the group $\ker(\operatorname{evol}_G : C^\infty(S^1,\mathfrak{g}) \to G)$ is a regular Lie group which is isomorphic to $C^\infty((S^1,1),(G,e))$. It is also a subgroup (via pullback by the covering mapping $e^{2\pi i t} : \mathbb{R} \to S^1$) of the regular Lie group $(C^\infty(\mathbb{R},\mathfrak{g}),*)$. Note that $C^\infty(S^1,\mathfrak{g})$ is not a subgroup, it is not closed under the product $*$, if G is not abelian.

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