Journal of Lie Theory
Volume **7** (1997) 29–60

The spherical transform for homogeneous vector bundles over Riemannian symmetric spaces

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Communicated by G. Mauceri

Abstract. Let G/K be a Riemannian symmetric space of the noncompact type. For $\tau \in \hat{K}$, let E^τ be the homogeneous vector bundle over G/K associated with τ, and let $C_0^\infty(G,\tau,\tau)$ be the related convolution algebra of *radial systems of sections* of E^τ. Assuming $C_0^\infty(G,\tau,\tau)$ commutative, we use the theory of spherical functions of type τ on G to define a spherical transform for $F \in C_0^\infty(G,\tau,\tau)$. The corresponding inversion formula is obtained by using the Plancherel formula on G. The example of real hyperbolic spaces $H^N(\mathbb{R})$ is discussed. The Plancherel measure is written down explicitly in this case, and the vector bundles of Dirac spinors, symmetric traceless tensors, and p-forms on $H^N(\mathbb{R})$ are considered in detail.

1. Introduction

Let G be a connected noncompact semisimple Lie group with finite center, K a maximal compact subgroup, and G/K the corresponding Riemannian symmetric space of the noncompact type. Scalar harmonic analysis on G/K is by now well understood (see, e.g., Helgason's books [12, 13]). In this paper we investigate the case of homogeneous vector bundles over G/K.

For a given irreducible unitary representation τ of K, we can let the induced bundle $L^2(G,\tau)$ sit in $L^2(G)$ in a natural way. Indeed the left regular representation of G in $L^2(G)$ is unitarily equivalent to the direct sum over $\hat{K}$ of the induced representations $\pi_\tau = \mathrm{ind}_K^G(\tau)$, each π_τ occurring d_τ times (d_τ the dimension of τ). The direct sum $\sum_1^{d_\tau} L^2(G,\tau)$ may be identified with the subspace $L^2(G) * d_\tau \bar{\chi}_\tau$ of $L^2(G)$ (χ_τ is the character of τ), see [19] Lemma 5.1.

By the generalized Frobenius Reciprocity principle, if we have the Plancherel formula for G we also have the direct integral decomposition of any induced representation π_τ (K being compact) [18]. The Plancherel formula for $f \in L^2(G)$ will produce a Plancherel formula for $f \in L^2(G,\tau)$.

Compared to the scalar case (when τ is the trivial representation of K), several new features arise in the case of bundles. For example the discrete series of G

ISSN 0949–5932 / \$2.50

(when they exist), and the generalized principal series representations (induced from nonminimal cuspidal parabolic subgroups of G) will enter the Plancherel formula for a generic vector bundle over G/K.

In this paper we use the theory of spherical functions of type τ on G, as developed by Godement, Harish Chandra and Warner [11, 28], to discuss the analysis of *radial systems of sections* of a homogeneous vector bundle E^τ over G/K.

Radial systems of sections of E^τ and the related convolution algebra $C_0^\infty(G,\tau,\tau)$ are defined in Section 2. This algebra may be identified with a certain subalgebra of $C_0^\infty(G)$, denoted $I_{0,\tau}(G)$. More precisely, $I_{0,\tau}(G)$ consists of those $f \in C_0^\infty(G)$ which are K-central and are invariant under convolution with $d_\tau \bar{\chi}_\tau$. The algebras $C_0^\infty(G,\tau,\tau)$ and $I_{0,\tau}(G)$ play the same role, for vector bundles, as the convolution algebra $C_0^\#(G)$ in the scalar case. (This is the algebra of compactly supported smooth functions on G which are biinvariant under K.)

In Section 3 we first define spherical functions of type τ on G, and discuss their functional and differential properties in the commutative case, namely when the convolution algebra $C_0^\infty(G,\tau,\tau)$ is commutative (for the given τ). The algebra $\mathbf{D}(G,\tau)$ of invariant differential operators on E^τ is then also commutative [8]. The analysis of radial sections in this case is as close as possible to scalar harmonic analysis on Riemannian symmetric spaces G/K, or on the homogeneous spaces G/K with (G,K) a Gelfand pair.

Then we define (in the commutative case) a spherical transform for radial systems of sections of E^τ, and derive an inversion formula from the (abstract) Plancherel formula on G. In this part of Section 3, we do not use the structure theory of semisimple Lie groups, and keep the notations as general as possible. In fact the results obtained here apply (in the commutative case) to any pair (G,K) of a locally compact unimodular Lie group G and a compact subgroup K, provided that (i) K is sufficiently large in G so that every irreducible unitary representation U of G is K-finite; (ii) there is a well defined theory of global characters on G, i.e., for every $U \in \hat{G}$ the operator $U(f) = \int_G f(x)U(x)dx$ is of trace class for all $f \in C_0^\infty(G)$, and the mapping $\Theta_U : f \to \operatorname{Tr} U(f)$ is a distribution on G. For example if G admits a *uniformly large* compact subgroup K (see the definition in [28] vol.I p. 305), then the conditions (i) and (ii) are satisfied. This includes all reductive pairs and all motion groups [28].

Another example is given by a pair (K,M) of a compact Lie group K and a closed subgroup $M \subset K$. It is clear that for each $\sigma \in \hat{M}$ we can formulate a theory of spherical functions of type σ on K, and define a spherical transform for radial sections of the bundle E^σ over K/M, even if (K,M) is not a symmetric pair. This theory will be analogous (in the commutative case) to that developed in Section 3.

At the end of Section 3 we consider the semisimple case in more detail. The spherical transform and the inversion formula on $C_0^\infty(G,\tau,\tau)$ are written down explicitly in this case, using Harish Chandra's Plancherel formula and Subquotient Theorem.

An important example of commutative algebras $I_{0,\tau}(G)$ is provided by the pairs $(G,K) = (SO_0(N,1), SO(N))$ or $(G,K) = (SU(N,1), S(U(N) \times U(1)))$, i.e., G/K is either a real or a complex hyperbolic space. In section 4 we discuss the example of real hyperbolic spaces $H^N(\mathbb{R})$. We make explicit the Plancherel measure and the inversion formula for the double cover $\mathrm{Spin}(N,1)$ of $SO_0(N,1)$. This allows us to discuss also the spinor bundles over $H^N(\mathbb{R})$. The homogeneous vector bundles of Dirac spinors, symmetric traceless tensors, and p-forms over $H^N(\mathbb{R})$ are considered in detail. The spherical transform reduces to the Jacobi transform in this case. The approach using Jacobi functions analysis agrees with the representation theoretic approach presented here.

Acknowledgements

The author would like to thank E. Badertscher, A. Higuchi, E. Pedon, M. Reimann, and F. Ricci for interesting conversations on the subject of this paper.

2. Radial systems of sections

Let G be a connected noncompact semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and G/K the corresponding Riemannian symmetric space of the noncompact type.

Let τ be an irreducible unitary representation of K on V_τ, and let E^τ be the homogeneous vector bundle over G/K defined by τ. It is well known (see, e.g., [26] sect. 5.3) that a cross section of E^τ may be identified with a vector-valued function $f : G \to V_\tau$ which is right-K-covariant of type τ, i.e.,

$$f(gk) = \tau(k^{-1})f(g), \quad \forall g \in G, \ \forall k \in K. \tag{1}$$

We denote by $C_0^\infty(G,\tau)$ the space of compactly supported smooth functions that are right-K-covariant of type τ, and by $L^2(G,\tau)$ the Hilbert space of square integrable such functions, with scalar product

$$\langle f_1, f_2 \rangle = \int_G \langle f_1(x), f_2(x) \rangle dx. \tag{2}$$

By a *radial system of sections* we mean a map $F : G \to \mathrm{End}(V_\tau)$ such that

$$F(k_1 g k_2) = \tau(k_2^{-1})F(g)\tau(k_1^{-1}), \quad \forall g \in G, \ \forall k_1, k_2 \in K. \tag{3}$$

For any $\mathbf{v} \in V_\tau$, the vector valued function $f(g) = F(g)\mathbf{v}$ satisfies (1) and defines a *radial section* of E^τ. [We follow here Badertscher and Reimann [1], who studied vector fields over the real hyperbolic spaces.]

The radial systems of sections generalize the notion of K-biinvariant functions on G. They are called radial because, due to the Cartan decomposition $G = KAK$, they are determined by their restriction to the vector subgroup A. We denote by $C_0^\infty(G,\tau,\tau)$ and by $L^2(G,\tau,\tau)$ the obviously defined spaces of radial systems of sections, with scalar product

$$\langle F_1, F_2 \rangle = \int_G \mathrm{Tr}\,[F_1(x)F_2(x)^*]dx,$$

where $*$ denotes adjoint. For $F_1, F_2 \in C_0^\infty(G,\tau,\tau)$, define the convolution by

$$(F_1 * F_2)(x) = \int_G F_1(y^{-1}x)F_2(y)dy.$$

This definition is arranged so that $F_1 * F_2 \in C_0^\infty(G,\tau,\tau)$:

$$\begin{aligned}(F_1 * F_2)(k_1xk_2) &= \int_G F_1(y^{-1}k_1xk_2)F_2(y)dy \\ &= \tau(k_2^{-1})\int_G F_1(y^{-1}k_1x)F_2(y)dy \\ &= \tau(k_2^{-1})\int_G F_1(z^{-1}x)F_2(k_1z)dz \\ &= \tau(k_2^{-1})\int_G F_1(z^{-1}x)F_2(z)dz\ \tau(k_1^{-1}) \\ &= \tau(k_2^{-1})(F_1 * F_2)(x)\tau(k_1^{-1}).\end{aligned}$$

In general on $C_0^\infty(G)$ (the space of compactly supported smooth functions on G), the convolution is defined by the usual rule

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1})f_2(y)dy \ = \ \int_G f_1(z)f_2(z^{-1}x)dz.$$

The space $C_0^\infty(G,\tau,\tau)$ may be identified with a certain subalgebra of $C_0^\infty(G)$, which we now define. Let $I_{0,\tau}(G)$ denote the set of those $f \in C_0^\infty(G)$ which satisfy

$$f(kxk^{-1}) = f(x), \quad \forall x \in G, \ \ \forall k \in K$$

(i.e., f is K-central), and

$$d_\tau\bar{\chi}_\tau * f = f(= f * d_\tau\bar{\chi}_\tau), \tag{4}$$

where d_τ and χ_τ are the dimension and the character of τ (a bar denotes complex conjugation, and the convolutions are over K).

Then $I_{0,\tau}(G)$ is a subalgebra of $C_0^\infty(G)$ and it is (anti)-isomorphic to $C_0^\infty(G,\tau,\tau)$. Indeed given $F \in C_0^\infty(G,\tau,\tau)$ define

$$f_F(x) \equiv d_\tau \mathrm{Tr}\, F(x). \tag{5}$$

It follows from (3) that f_F is K-central and moreover it satisfies (4). Indeed

$$\begin{aligned}(f_F * d_\tau\bar{\chi}_\tau)(x) &= d_\tau\int_K f_F(xk)\chi_\tau(k)dk \\ &= d_\tau^2\int_K \mathrm{Tr}\, F(xk)\chi_\tau(k)dk \\ &= d_\tau^2\mathrm{Tr}\left[\int_K \tau(k^{-1})\chi_\tau(k)dk\, F(x)\right] \\ &= d_\tau\, \mathrm{Tr}\, F(x) = f_F(x),\end{aligned}$$

where we have used the Schur orthogonality relations for K with the normalization $\int_K dk = 1$. Thus $f_F \in I_{0,\tau}(G)$. Viceversa, given $f \in I_{0,\tau}(G)$ put

$$F_f(x) \equiv \int_K \tau(k)f(kx)dk. \tag{6}$$

Then

$$\begin{aligned} F_f(k_1xk_2) &= \int_K \tau(k)f(kk_1xk_2)dk = \int_K \tau(k)f(kxk_2)dk\,\tau(k_1^{-1}) \\ &= \int_K \tau(k)f(k_2kx)dk\,\tau(k_1^{-1}) = \tau(k_2^{-1})\int_K \tau(k)f(kx)dk\,\tau(k_1^{-1}) \\ &= \tau(k_2^{-1})F_f(x)\tau(k_1^{-1}), \end{aligned}$$

i.e., F_f is in $C_0^\infty(G,\tau,\tau)$. We have the following result (see [28] vol.II, p.3, Example 1):

Proposition 2.1. *The map $f \to F_f$ is a linear bijection of $I_{0,\tau}(G)$ onto $C_0^\infty(G,\tau,\tau)$. Its inverse is the map $F \to f_F$. These maps satisfy*

$$F_{f_1 * f_2} = F_{f_2} * F_{f_1}, \tag{7}$$

$$f_{F_1 * F_2} = f_{F_2} * f_{F_1}. \tag{8}$$

As a corollary of this proposition we see that the convolution algebra $C_0^\infty(G,\tau,\tau)$ is commutative if and only if the convolution algebra $I_{0,\tau}(G)$ is commutative.

Now let $\hat{G}$ ($\hat{K}$) denote the set of equivalence classes of irreducible unitary representations of G (K). In what follows, we shall identify a class $[U] \in \hat{G}$ ($[\tau] \in \hat{K}$) with a representative U (τ) in that class, and we shall write (somewhat incorrectly) $U \in \hat{G}$ ($\tau \in \hat{K}$). Let $m(\tau, U)$ denote the multiplicity of τ in $U|_K$. The following result, which characterizes the commutative case, is well known (for the proof see [11] p.522 the Corollary to Th.8, or [28] vol.II p.9 Prop. 6.1.1.6):

Proposition 2.2. *The following conditions are equivalent:*

1) $I_{0,\tau}(G)$ *is commutative;*
2) $m(\tau, U) \leq 1 \quad \forall U \in \hat{G}$.

For a symmetric pair (G, K), it is known that $m(\tau, U) \leq d_\tau$, $\forall U$, $\forall \tau$ [11]. Therefore the two conditions above are satisfied when τ is the trivial representation of K (i.e., in the scalar case, see [12]). It is natural to begin the investigation of vector bundles from the commutative case. Of course for a given G, $I_{0,\tau}(G)$ may or may not be commutative depending on $\tau \in \hat{K}$. An example of commutative algebras $I_{0,\tau}(G)$ may be given as follows. We say that the compact subgroup K is *multiplicity free* in G if each irreducible unitary representation of K is contained in each irreducible unitary representation of G at most once, i.e., $m(\tau, U) \leq 1$, $\forall \tau \in \hat{K}$, $\forall U \in \hat{G}$. In this case, the conditions of Prop. 2.2 are satisfied for each $\tau \in \hat{K}$. The following result is classical.

Theorem 2.3. *Let the symmetric pair (G, K) be either $(SO_0(n,1), SO(n))$ or $(SU(n,1), S(U(n) \times U(1)))$. Then K is multiplicity free in G.*

A simple proof of this result may be found in [16], where it is proved that $(G \times K, \mathrm{diag}(K))$ is a Gelfand pair. The result then follows by noting that the multiplicity of τ in U is the same as the multiplicity of the trivial representation of $\mathrm{diag}(K)$ in $U \otimes \check{\tau}$ ($\check{\tau}$ the contragredient representation of τ).

3. The spherical transform in the commutative case

We keep the notations of Section 2. Thus we fix a Riemannian symmetric pair (G,K), and an irreducible unitary representation (τ, V_τ) of K. In this section we assume that the conditions of Proposition 2.2 are satisfied for the given τ, so that the convolution algebra $C_0^\infty(G,\tau,\tau)$ is commutative.

We denote by $\hat{G}(\tau)$ the set of those U in $\hat{G}$ which contain τ upon restriction to K. For $U \in \hat{G}(\tau)$ let H_U be the Hilbert space where U acts, and let P_τ be the projector of H_U onto H_τ, the subspace of vectors which transform under K according to τ (see, e.g., [11]):

$$P_\tau = d_\tau \int_K U(k)\chi_\tau(k^{-1})dk. \tag{9}$$

Since $m(\tau,U)=1$, we can identify H_τ with V_τ. Define the (operator valued) spherical function Φ_τ^U on G by

$$\Phi_\tau^U(g) \equiv P_\tau U(g) P_\tau, \qquad g \in G. \tag{10}$$

For each $g \in G$, $\Phi_\tau^U(g)$ may be regarded as an element of $\mathrm{End}(V_\tau)$ and satisfies $\Phi_\tau^U(g)^* = \Phi_\tau^U(g^{-1})$, and

$$\Phi_\tau^U(k_1 g k_2) = \tau(k_1)\Phi_\tau^U(g)\tau(k_2), \qquad \forall g \in G, \ \ \forall k_1, k_2 \in K. \tag{11}$$

We choose an orthonormal basis $\{\mathbf{v}_A\}_{A=1\cdots\infty}$ of H_U adapted to the decomposition of $U|_K$ into different K-types, $U|_K = \sum_{\delta\in\hat{K}} \delta$. We can always assume that for $a=1,\ldots,d_\tau$ the vectors $\{\mathbf{v}_a\}$ span $H_\tau \simeq V_\tau$.

The functions Φ_τ^U generalize to vector bundles the notion of positive definite spherical functions. It is easy to see that the traces

$$\phi_\tau^U(x) = \mathrm{Tr}\ [P_\tau U(x) P_\tau] = \mathrm{Tr}\, \Phi_\tau^U(x)$$

(known as *spherical trace functions of type* τ, see [28] vol.II), are positive definite functions on G in the usual sense. The functions Φ_τ^U and ϕ_τ^U are related as follows.

Lemma 3.1. *Let the Haar measure on K be normalized by $\int_K dk = 1$. Then $\forall x \in G$*

$$\begin{aligned} \int_K \Phi_\tau^U(kxk^{-1})dk &= \frac{1}{d_\tau}\phi_\tau^U(x)\mathbf{1}, && (12)\\ d_\tau \int_K \phi_\tau^U(xk^{-1})\tau(k)dk &= \Phi_\tau^U(x), && (13) \end{aligned}$$

where $\mathbf{1}$ *in* (12) *denotes the identity operator in* V_τ.

Proof. Let $\Phi_{\tau,K}^U(x)$ denote the left-hand side of (12). Then $\Phi_{\tau,K}^U(x)$ is in $\mathrm{End}_K(V_\tau)$, $\forall x \in G$ (this follows from (11)). Since τ is irreducible, $\Phi_{\tau,K}^U(x) = \varphi(x)\mathbf{1}$, where φ is a function on G. Taking the trace gives $d_\tau\varphi(x) = \mathrm{Tr}\,\Phi_{\tau,K}^U(x) = \mathrm{Tr}\,\Phi_\tau^U(x) = \phi_\tau^U(x)$. This proves (12). Eq. (13), rewritten as

$$d_\tau \int_K \mathrm{Tr}\ \left[\Phi_\tau^U(x)\tau(k^{-1})\right] \tau(k)dk = \Phi_\tau^U(x),$$

follows from the Schur orthogonality relations. ∎

Taking the trace of Eq. (13) gives the well known relation $\phi_\tau^U * d_\tau\chi_\tau = \phi_\tau^U$, see [11].

For $f \in C_0^\infty(G)$ and $U \in \hat{G}(\tau)$, let $U(f)$ denote the operator

$$U(f) = \int_G f(x)U(x)dx.$$

Then, as is well known, $U(f)P_\tau = U(f * d_\tau\bar{\chi}_\tau)$, $P_\tau U(f) = U(d_\tau\bar{\chi}_\tau * f)$, and

$$P_\tau U(f)P_\tau = U(d_\tau\bar{\chi}_\tau * f * d_\tau\bar{\chi}_\tau).$$

In particular for $f \in I_{0,\tau}(G)$ we have the following result.

Proposition 3.2. *Let $f \in I_{0,\tau}(G)$ and $U \in \hat{G}(\tau)$. Then*

$$\begin{aligned} P_\tau U(f)P_\tau &= U(f), \\ U(f)U(k) &= U(k)U(f), \qquad k \in K. \end{aligned}$$

Let $U_\tau(f)$ denote the restriction of $U(f)$ to H_τ. Then the set of operators $U_\tau(f)$, $f \in I_{0,\tau}(G)$, is the centralizer of the representation $k \to \tau(k)$ of K on H_τ.

Proof. See [28] vol.I p.307, and Prop. 4.5.1.7 p.310. ∎

This proposition and (12) imply that for $f \in I_{0,\tau}(G)$

$$\begin{aligned} U_\tau(f) &= \int_G f(x)\Phi_\tau^U(x)dx = \int_G\int_K f(kxk^{-1})\Phi_\tau^U(x)dkdx \\ &= \int_G\int_K f(y)\Phi_\tau^U(k^{-1}yk)dkdy = \frac{1}{d_\tau}\mathbf{1}\int_G f(x)\phi_\tau^U(x)dx, \end{aligned}$$

and

$$\Theta_U(f) \equiv \mathrm{Tr}\, U(f) = \mathrm{Tr}\, U_\tau(f) = \int_G f(x)\phi_\tau^U(x)dx. \tag{14}$$

Since

$$U(f_1 * f_2) = U(f_1)U(f_2),$$

we see that the map $f \to \hat{f}(U) \in \mathbb{C}$, where

$$\hat{f}(U) \equiv \frac{1}{d_\tau}\int_G f(x)\phi_\tau^U(x)dx, \tag{15}$$

is a continuous homomorphism of the convolution algebra $I_{0,\tau}(G)$ into $\mathbb{C}$,

$$(\widehat{f_1 * f_2})(U) = \hat{f}_1(U)\hat{f}_2(U), \qquad \forall f_1, f_2 \in I_{0,\tau}(G). \tag{16}$$

The map $\hat{f} : \hat{G}(\tau) \to \mathbb{C}$, given in (15), is called the *spherical Fourier transform* of $f \in I_{0,\tau}(G)$.

For $F \in C_0^\infty(G,\tau,\tau)$, consider the operator $T = \int_G \Phi_\tau^U(x)F(x)dx$. It is easy to see that $T \in \mathrm{End}_K(V_\tau)$. Since τ is irreducible, $T = c\mathbf{1}$, where $c = \frac{1}{d_\tau}\int_G \mathrm{Tr}\,[\Phi_\tau^U(x)F(x)]dx$.

Definition 3.3. The spherical Fourier transform of $F \in C_0^\infty(G,\tau,\tau)$ is the function $\hat{F} : \hat{G}(\tau) \to \mathbb{C}$ defined by

$$\hat{F}(U) \equiv \frac{1}{d_\tau}\int_G \mathrm{Tr}\,[\Phi_\tau^U(x)F(x)]dx, \qquad U \in \hat{G}(\tau). \tag{17}$$

When τ is the trivial representation of K, the function $\hat{F}$, defined on the set of equivalence classes of irreducible unitary spherical representations of G, or equivalently, on the set of positive definite zonal spherical functions on G, reduces to the usual spherical Fourier transform of F, see [28] vol.II p.337.

Lemma 3.4. *In the notations of Proposition 2.1 we have for all* $U \in \hat{G}(\tau)$

$$\hat{F}(U) = \hat{f}_F(U), \qquad \hat{f}(U) = \hat{F}_f(U). \tag{18}$$

Proof. Using (13) we have

$$\begin{aligned}
\hat{F}(U) &= \frac{1}{d_\tau}\int_G \mathrm{Tr}\,[\Phi_\tau^U(x)F(x)]dx \\
&= \int_G\int_K \phi_\tau^U(xk^{-1})\mathrm{Tr}\,[\tau(k)F(x)]dkdx \\
&= \int_K\int_G \phi_\tau^U(xk^{-1})\mathrm{Tr}\,[F(xk^{-1})]dxdk \\
&= \int_G \phi_\tau^U(x)\mathrm{Tr}\,[F(x)]dx \\
&= \frac{1}{d_\tau}\int_G \phi_\tau^U(x)f_F(x)dx = \hat{f}_F(U).
\end{aligned}$$

Reading the argument backwards proves the second equality in (18). ∎

Putting together eqs. (8), (16) and (18), we see that the map $F \to \hat{F}(U)$ is a continuous homomorphism of $C_0^\infty(G,\tau,\tau)$ into $\mathbb{C}$,

$$(\widehat{F_1 * F_2})(U) = \hat{F}_1(U)\hat{F}_2(U), \quad \forall F_1, F_2 \in C_0^\infty(G,\tau,\tau).$$

This leads to the following definition of spherical function of type τ and spherical transform in the commutative case.

Definition 3.5. Assume $C_0^\infty(G,\tau,\tau)$ commutative. A function $\Phi : G \to \mathrm{End}(V_\tau)$ is called a spherical function of type τ on G if it satisfies (11) and if the map

$$F \to \hat{F}(\Phi) = \frac{1}{d_\tau}\int_G \mathrm{Tr}\,[\Phi(x)F(x)]dx \tag{19}$$

is a homomorphism of $C_0^\infty(G,\tau,\tau)$ into $\mathbb{C}$. The map $\hat{F}$, defined on the set of spherical functions of type τ by (19), is called the spherical Gelfand transform of F (spherical transform, for short). For $U \in \hat{G}(\tau)$ we write $\hat{F}(\Phi_\tau^U) = \hat{F}(U)$ (consistently with (17)).

Let Φ be a spherical function of type τ on G, and let $\phi(x) = \mathrm{Tr}\,\Phi(x)$. It is clear that Φ and ϕ are related as Φ^U_τ and ϕ^U_τ in (12)-(13) (same proof as in Lemma 3.1). Notice that ϕ is K-central and satisfies $\phi * d_\tau\chi_\tau = \phi$. Moreover, defining the *spherical transform* of $f \in I_{0,\tau}(G)$ by

$$\hat{f}(\phi) = \frac{1}{d_\tau}\int_G f(x)\phi(x)dx, \tag{20}$$

we have

$$\hat{F}(\Phi) = \hat{f}_F(\phi), \qquad \hat{f}(\phi) = \hat{F}_f(\Phi) \tag{21}$$

(same proof as in Lemma 3.4). Therefore the map $f \to \hat{f}(\phi)$ is a homomorphism of $I_{0,\tau}(G)$ into $\mathbb{C}$. We could define a spherical function of type τ to be a scalar valued function ϕ on G satisfying the three conditions (i) ϕ is K-central; (ii) $\phi * d_\tau\chi_\tau = \phi$; (iii) the map $f \to \hat{f}(\phi)$ is a homomorphism of $I_{0,\tau}(G)$ into $\mathbb{C}$ (see [28] vol.II p.14). This definition would then be equivalent to Definition 3.5. Indeed given ϕ satisfying (i)-(iii) above, the function $\Phi(x)$ given by the left hand side of (13) (with $\phi^U_\tau \to \phi$) satisfies the conditions of Definition 3.5.

The connection between spherical functions of type τ on G and representations of G is as follows. Let U be a topologically completely irreducible (TCI) Banach representation of G (see [28] vol.I p.228 for the definition of TCI). Suppose that τ occurs in $U|_K$ with multiplicity one, and define $\Phi^U_\tau(g)$ by means of eq. (10). Then Φ^U_τ is a spherical function of type τ on G. Conversely, if Φ is a spherical function of type τ on G, there exists a TCI Banach representation U of G such that $\Phi = \Phi^U_\tau$. [See [28] vol.II p.15.] We say that a spherical function Φ of type τ is of positive type if the scalar valued function $\phi(x) = \mathrm{Tr}\,\Phi(x)$ is positive definite on G. Then the spherical functions of type τ of positive type are precisely the functions Φ^U_τ with $U \in \hat{G}(\tau)$ (see [28] vol.II p.15, the remark).

As in the scalar case, spherical functions of type τ on G are characterized by certain functional equations.

Theorem 3.6. *Let Φ be a spherical function of type τ on G, and let $\phi(x) = \mathrm{Tr}\,\Phi(x)$. Then $\forall x, y \in G$*

$$d_\tau \int_K \phi(xkyk^{-1})dk = \phi(x)\phi(y), \tag{22}$$

$$d_\tau \int_K \Phi(xkyk^{-1})dk = \Phi(x)\phi(y), \tag{23}$$

$$d_\tau \int_K \Phi(xky)\chi_\tau(k^{-1})dk = \Phi(x)\Phi(y). \tag{24}$$

Conversely, let Φ be a nonzero continuous function on G with values in $\mathrm{End}(V_\tau)$ *which satisfies* (11) *and either* (23) *or* (24). *Then Φ is a spherical function of type τ on G.*

Proof. Eq. (22) is the functional equation for the spherical trace functions of height one, see [11]. The proof given here is similar to [10] Proposition I.3 p.319.

For a function $f \in C_0^\infty(G)$ put

$$f_K(x) = \int_K f(kxk^{-1})dk,$$

and $f^\tau = d_\tau \bar{\chi}_\tau * f * d_\tau \bar{\chi}_\tau$, i.e., explicitly

$$f^\tau(x) = d_\tau^2 \int_{K\times K} f(k_1 x k_2)\chi_\tau(k_1)\chi_\tau(k_2)dk_1 dk_2.$$

It is easy to see that $(f_K)^\tau = (f^\tau)_K$. Put $f^\# = (f^\tau)_K$. Then the map $f \to f^\#$ is a projection of $C_0^\infty(G)$ onto $I_{0,\tau}(G)$.

Let Φ be a spherical function of type τ on G, and let $\phi(x) = \operatorname{Tr}\Phi(x)$. The map $f \to \hat{f}(\phi)$ (cf. (20)) is a homomorphism of $I_{0,\tau}(G)$ into $\mathbb{C}$. Put $\phi(f) \equiv \int_G f(x)\phi(x)dx$. Let f_1, f_2 be in $C_0^\infty(G)$. Then

$$\begin{aligned}
0 &= \frac{1}{d_\tau}\phi(f_1^\# * f_2^\#) - \frac{1}{d_\tau^2}\phi(f_1^\#)\phi(f_2^\#) \\
&= \frac{1}{d_\tau}\int_G (f_1^\# * f_2^\#)(z)\phi(z)dz - \frac{1}{d_\tau^2}\int_G f_1^\#(x)\phi(x)dx \int_G f_2^\#(y)\phi(y)dy \\
&= \frac{1}{d_\tau}\int_{G\times G} f_1^\#(zy^{-1})f_2^\#(y)\phi(z)dydz - \frac{1}{d_\tau^2}\int_{G\times G} f_1^\#(x)f_2^\#(y)\phi(x)\phi(y)dxdy \\
&= \frac{1}{d_\tau^2}\int_{G\times G} \left[d_\tau\phi(xy) - \phi(x)\phi(y)\right] f_1^\#(x)f_2^\#(y)dxdy.
\end{aligned}$$

Using the definition of $f^\#$ and the fact that $\phi(x)$ is K-central and satisfies $\phi * d_\tau\chi_\tau = \phi$, it is easy to show that the latter expression equals

$$\frac{1}{d_\tau^2}\int_{G\times G}\left[d_\tau \int_K \phi(xkyk^{-1})dk - \phi(x)\phi(y)\right] f_1(x)f_2(y)dxdy.$$

This proves (22), since f_1, f_2 are arbitrary in $C_0^\infty(G)$. To prove (23) we use (13), to write

$$\begin{aligned}
d_\tau \int_K \Phi(xkyk^{-1})dk &= d_\tau^2 \int_{K\times K} \phi(xkyk^{-1}k_1^{-1})\tau(k_1)dk_1 dk \\
&= d_\tau^2 \int_{K\times K} \phi(xk_1^{-1}k_2 y k_2^{-1})\tau(k_1)dk_1 dk_2 \\
&= d_\tau \int_K \phi(xk_1^{-1})\phi(y)\tau(k_1)dk_1 = \Phi(x)\phi(y).
\end{aligned}$$

Finally to prove (24) we write

$$\begin{aligned}
d_\tau \int_K \Phi(xky)\chi_\tau(k^{-1})dk &= d_\tau^2 \int_K \Phi(xky)\left[\int_K \tau(k_1 k^{-1} k_1^{-1})dk_1\right] dk \\
&= d_\tau^2 \int_{K\times K} \Phi(xkyk_1k^{-1})\tau(k_1^{-1})dk_1 dk \\
&= d_\tau^2 \int_K \left[\int_K \Phi(xkyk_1k^{-1})dk\right]\tau(k_1^{-1})dk_1 \\
&= d_\tau \int_K \Phi(x)\phi(yk_1)\tau(k_1^{-1})dk_1 = \Phi(x)\Phi(y),
\end{aligned}$$

as claimed.

Conversely, suppose that $\Phi : G \to \mathrm{End}(V_\tau)$ satisfies (11) and (23). Then $\phi(x) \equiv \mathrm{Tr}\,\Phi(x)$ satisfies (22). Clearly Φ and ϕ are related as Φ^U_τ and ϕ^U_τ in (12)-(13) (same proof as in Lemma 3.1). For $F \in C^\infty_0(G,\tau,\tau)$ and $f \in I_{0,\tau}(G)$, define $\hat{F}(\Phi)$ and $\hat{f}(\phi)$ by (19) and (20). Then (21) holds (same proof as in Lemma 3.4). Now ϕ is K-central, and $\phi * d_\tau\chi_\tau = \phi$. Let $f_1, f_2 \in I_{0,\tau}(G)$. Then

$$\begin{aligned}
(\widehat{f_1 * f_2})(\phi) &= \frac{1}{d_\tau}\int_G\int_G f_1(xy^{-1})f_2(y)\phi(x)dydx \\
&= \frac{1}{d_\tau}\int_G\int_G f_1(x)f_2(y)\phi(xy)dxdy \\
&= \frac{1}{d_\tau}\int_G\int_G\int_K f_1(x)f_2(kyk^{-1})\phi(xy)dkdxdy \\
&= \frac{1}{d_\tau}\int_G\int_G f_1(x)f_2(y)\left[\int_K \phi(xkyk^{-1})dk\right]dxdy \\
&= \frac{1}{d_\tau^2}\int_G\int_G f_1(x)f_2(y)\phi(x)\phi(y)dxdy = \hat{f}_1(\phi)\hat{f}_2(\phi).
\end{aligned}$$

The relation $(\widehat{F_1 * F_2})(\Phi) = \hat{F}_1(\Phi)\hat{F}_2(\Phi)\ \forall F_1, F_2 \in C^\infty_0(G,\tau,\tau)$, follows immediately from (21) and (8). Thus Φ is a spherical function of type τ on G.

Finally let $\Phi : G \to \mathrm{End}(V_\tau)$ satisfy (11) and (24). Then Φ satisfies (23). Indeed

$$\begin{aligned}
d_\tau\int_K \Phi(xkyk^{-1})dk &= d_\tau^2\int_K \Phi(xkyk^{-1})\left[\int_K \tau(k_1)\chi_\tau(k_1^{-1})dk_1\right]dk \\
&= d_\tau^2\int_{K\times K} \Phi(xkyk^{-1}k_1)\chi_\tau(k_1^{-1})dk_1dk \\
&= d_\tau^2\int_K\left[\int_K \Phi(xk_1k_2yk_2^{-1})\chi_\tau(k_1^{-1})dk_1\right]dk_2 \\
&= d_\tau\int_K \Phi(x)\Phi(k_2yk_2^{-1})dk_2 = \Phi(x)\phi(y).
\end{aligned}$$

This completes the proof of the theorem. ∎

Theorem 3.7. *Let Φ be a nonzero spherical function of type τ on G, and let $\phi(x) = \mathrm{Tr}\,\Phi(x)$. Then $\Phi(e) = \mathbf{1}$, and $\forall x \in G$*

$$\int_G \Phi(xy)F(y)dy = \Phi(x)\hat{F}(\Phi), \qquad \forall F \in C^\infty_0(G,\tau,\tau), \tag{25}$$

$$\int_G \phi(xy)f(y)dy = \phi(x)\hat{f}(\phi), \qquad \forall f \in I_{0,\tau}(G). \tag{26}$$

Conversely, let Φ be a nonzero continuous function on G with values in $\mathrm{End}(V_\tau)$ satisfying (11) and $\Phi(e) = \mathbf{1}$. Suppose that for any $F \in C^\infty_0(G,\tau,\tau)$ there exists a complex number $\hat{F}(\Phi)$ such that (25) holds. Then $\hat{F}(\Phi) = \frac{1}{d_\tau}\int_G \mathrm{Tr}\,[\Phi(x)F(x)]dx$, and Φ is a spherical function of type τ on G.

Proof. Let Φ be a spherical function of type τ on G, not identically zero. Letting $g = e$ and $k_1 = k = k_2^{-1}$ in $\Phi(k_1 g k_2) = \tau(k_1)\Phi(g)\tau(k_2)$, gives $\Phi(e)\tau(k) = \tau(k)\Phi(e)$ $\forall k \in K$. Then $\Phi(e) = c\mathbf{1}$. Moreover $\Phi(k) = \Phi(ke) = \tau(k)\Phi(e) = c\,\tau(k)$. Letting $x = y = e$ in (24) gives $c^2 = c$. The function $\phi(x) = \operatorname{Tr}\Phi(x)$ can not vanish identically, because $d_\tau \int_K \phi(xk^{-1})\tau(k)dk = \Phi(x)$. Let $x_0 \in G$ be such that $\phi(x_0) \neq 0$. Letting $x = x_0$ and $y = e$ in (22) gives $\phi(e) = d_\tau$, whence $c = 1$.

Now using Proposition 2.1, (23) and (21) we have

$$\begin{aligned}
\int_G \Phi(xy)F(y)dy &= \int_G \int_K \Phi(xy)\tau(k)f_F(ky)dkdy \\
&= \int_K \int_G \Phi(xk^{-1}y)\tau(k)f_F(y)dydk \\
&= \int_G \left[\int_K \Phi(xk^{-1}yk)dk\right] f_F(y)dy \\
&= \frac{1}{d_\tau}\Phi(x)\int_G \phi(y)f_F(y)dy \\
&= \Phi(x)\hat{f}_F(\phi) = \Phi(x)\hat{F}(\Phi),
\end{aligned}$$

which is (25). Eq. (26) can be proved in a similar way, using the K-centrality of f and (22).

Conversely, let $\Phi : G \to \operatorname{End}(V_\tau)$ satisfy (11), $\Phi(e) = \mathbf{1}$, and (25), for some $\hat{F}(\Phi) \in \mathbb{C}$. Setting $x = e$ in (25) gives $\hat{F}(\Phi)\mathbf{1} = \int_G \Phi(y)F(y)dy$. Taking the trace of this equation gives $\hat{F}(\Phi) = \frac{1}{d_\tau}\int_G \operatorname{Tr}[\Phi(x)F(x)]dx$.

Given $F_1, F_2 \in C_0^\infty(G,\tau,\tau)$, it is easy to see that

$$(\widehat{F_1 * F_2})(\Phi) = \frac{1}{d_\tau}\operatorname{Tr}\int_G\int_G \Phi(xy)F_1(y)F_2(x)dxdy.$$

Then (25) implies $(\widehat{F_1 * F_2})(\Phi) = \hat{F}_1(\Phi)\hat{F}_2(\Phi)$, i.e., Φ is a spherical function of type τ on G. ■

From the differential point of view, the spherical functions of type τ are the radial joint eigenfunctions of $\mathbf{D}(G,\tau)$, the algebra of G-invariant differential operators mapping sections of E^τ to sections of E^τ. [See [20] for a Lie-algebraic description of $\mathbf{D}(G,\tau)$.] By methods similar to those of [25], one can show that $\mathbf{D}(G,\tau)$ is commutative if and only if $C_0^\infty(G,\tau,\tau)$ is commutative (see [8]). Thus $\mathbf{D}(G,\tau)$ is commutative in our case.

Let $C^\infty(G,\tau)$ (resp. $C^\infty(G,\tau,\tau)$) be the space of C^∞ maps from G to V_τ (resp. $\operatorname{End}(V_\tau)$) satisfying (1) (resp. (3)). $\mathbf{D}(G,\tau)$ acts naturally on $C^\infty(G,\tau)$, in view of the identification of $C^\infty(G,\tau)$ with $\Gamma(E^\tau)$, the space of C^∞ cross sections of E^τ. For $F \in C^\infty(G,\tau,\tau)$ and $v \in V_\tau$, the function $f(g) = F(g)v$ is in $C^\infty(G,\tau)$. Therefore we can let $\mathbf{D}(G,\tau)$ act on $C^\infty(G,\tau,\tau)$ by

$$(DF)(\cdot)v \equiv D(F(\cdot)v), \qquad \forall D \in \mathbf{D}(G,\tau),\ \ \forall F \in C^\infty(G,\tau,\tau),\ \ \forall v \in V_\tau.$$

Clearly $DF \in C^\infty(G,\tau,\tau)$. Let η be the map on G defined by sending g to g^{-1}. Let Φ be a spherical function of type τ on G. Then $\Phi \circ \eta \in C^\infty(G,\tau,\tau)$. We shall now prove that $\Phi \circ \eta$ is an eigenfunction of each $D \in \mathbf{D}(G,\tau)$, using the functional equation (24).

Theorem 3.8. *Let $\Phi : G \to \mathrm{End}(V_\tau)$ be a spherical function of type τ on G. Let $\eta : g \to g^{-1}$, and let $\tilde{\Phi} = \Phi \circ \eta \in C^\infty(G,\tau,\tau)$. Then for each $D \in \mathbf{D}(G,\tau)$ there exists a complex number $\mu_\Phi(D)$ such that*

$$D\tilde{\Phi} = \mu_\Phi(D)\tilde{\Phi}. \tag{27}$$

Moreover $(D\tilde{\Phi})(e) = \mu_\Phi(D)\mathbf{1}$, and the map $D \to \mu_\Phi(D)$ is a homomorphism of $\mathbf{D}(G,\tau)$ into $\mathbb{C}$. Conversely, let $T \in C^\infty(G,\tau,\tau)$, and suppose there is a homomorphism, $D \to \mu_T(D)$, of $\mathbf{D}(G,\tau)$ into $\mathbb{C}$ such that

$$DT = \mu_T(D)T, \qquad \forall D \in \mathbf{D}(G,\tau),$$

with $DT(e) = \mu_T(D)\mathbf{1}$. Then there is a spherical function Φ of type τ on G such that $T = \Phi \circ \eta$.

Proof. Let Φ be a spherical function of type τ on G. Using (24), it is easy to derive the following functional equation for $\tilde{\Phi}$:

$$d_\tau \int_K \tilde{\Phi}(xky)\,\chi_\tau(k)dk = \tilde{\Phi}(y)\tilde{\Phi}(x), \qquad x, y \in G. \tag{28}$$

Acting with $D \in \mathbf{D}(G,\tau)$ on the y variable in (28), using the fact that D is left-invariant, letting $y = e$, and observing that $D\tilde{\Phi} \in C^\infty(G,\tau,\tau)$, we obtain

$$\begin{aligned}[D\tilde{\Phi}](e)\tilde{\Phi}(x) &= d_\tau \int_K \chi_\tau(k)[D\tilde{\Phi}](xk)dk \\ &= d_\tau \int_K \chi_\tau(k)\tau(k^{-1})dk\,[D\tilde{\Phi}](x) = [D\tilde{\Phi}](x).\end{aligned} \tag{29}$$

Now notice that if $T \in C^\infty(G,\tau,\tau)$, then $T(e) \in \mathrm{End}_K(V_\tau)$, so that $T(e) = c\mathbf{1}$, with $c = \frac{1}{d_\tau}\mathrm{Tr}\,T(e) \in \mathbb{C}$. Applying this to $T = D\tilde{\Phi} \in C^\infty(G,\tau,\tau)$, gives

$$D\tilde{\Phi}(e) = \mu_\Phi(D)\mathbf{1},$$

with $\mu_\Phi(D) = \frac{1}{d_\tau}\mathrm{Tr}\,[D\tilde{\Phi}(e)] \in \mathbb{C}$. Using this in (29) proves (27).

Finally, we have

$$\begin{aligned}\mu_\Phi(D_1D_2) &= \frac{1}{d_\tau}\mathrm{Tr}\,[(D_1D_2\tilde{\Phi})(e)] \\ &= \frac{1}{d_\tau}\mathrm{Tr}\,[\mu_\Phi(D_2)(D_1\tilde{\Phi})(e)] \\ &= \frac{1}{d_\tau}\mathrm{Tr}\,[\mu_\Phi(D_2)\mu_\Phi(D_1)\mathbf{1}] \\ &= \mu_\Phi(D_1)\mu_\Phi(D_2).\end{aligned}$$

The proof of the converse is analogous to [28] Theorem 6.1.2.3. One proves first that T is analytic, using the fact that $\mathbf{D}(G,\tau)$ has an elliptic element. Then using Taylor's formula, one proves that T satisfies the functional equation (28). The result follows then from Theorem 3.6. ∎

We shall now use the isomorphism between $C_0^\infty(G,\tau,\tau)$ and $I_{0,\tau}(G)$ to derive an inversion formula on $C_0^\infty(G,\tau,\tau)$ from the Plancherel (inversion) formula on the group G.

Theorem 3.9. *The spherical transform* (19) *is inverted by*

$$F(g) = \frac{1}{d_\tau} \int_{\hat{G}(\tau)} \Phi_\tau^U(g^{-1}) \hat{F}(U) d\mu(U), \qquad F \in C_0^\infty(G, \tau, \tau), \tag{30}$$

where $d\mu(U)$ is the Plancherel measure on $\hat{G}$ (suitably normalized).

Proof. Let $U \in \hat{G}$, and let Θ_U denote the global character of U. Consider the *Fourier component* $\Theta_{U,\tau}$ of Θ_U, defined by the rule

$$\Theta_{U,\tau}(f) \equiv \Theta_U(f * d_\tau \bar{\chi}_\tau), \quad f \in C_0^\infty(G),$$

the convolution being over K. As proved in [28] vol.II p.18 (the remark), the distribution $\Theta_{U,\tau}$ coincides with the spherical trace function ϕ_τ^U, i.e., for $f \in C_0^\infty(G)$,

$$\Theta_{U,\tau}(f) = \int_G \phi_\tau^U(x) f(x) dx. \tag{31}$$

Clearly if U is not in $\hat{G}(\tau)$, then

$$\Theta_{U,\tau}(f) = 0.$$

Let $F \in C_0^\infty(G, \tau, \tau)$, and consider the function $f_F \circ L_g$, where $f_F \in I_{0,\tau}(G)$ is defined in (5), and L_g denotes left-translation on G, $L_g(x) = gx$. This function satisfies

$$(f_F \circ L_g) * d_\tau \bar{\chi}_\tau = f_F \circ L_g,$$

as follows immediately from the definition of $I_{0,\tau}(G)$. Therefore

$$\Theta_U(f_F \circ L_g) = \Theta_U((f_F \circ L_g) * d_\tau \bar{\chi}_\tau) = \Theta_{U,\tau}(f_F \circ L_g),$$

and if $U \notin \hat{G}(\tau)$

$$\Theta_U(f_F \circ L_g) = \Theta_{U,\tau}(f_F \circ L_g) = 0.$$

We now use these relations and (31) in the Plancherel (inversion) formula for $f_F \in I_{0,\tau}(G)$:

$$\begin{aligned}
f_F(g) &= \int_{\hat{G}} \Theta_U(f_F \circ L_g) d\mu(U) \\
&= \int_{\hat{G}(\tau)} \Theta_{U,\tau}(f_F \circ L_g) d\mu(U) \\
&= \int_{\hat{G}(\tau)} \int_G \phi_\tau^U(x) f_F(gx) dx d\mu(U) \\
&= \int_{\hat{G}(\tau)} \int_G \phi_\tau^U(g^{-1}y) f_F(y) dy d\mu(U) \\
&= \int_{\hat{G}(\tau)} \phi_\tau^U(g^{-1}) \hat{f}_F(U) d\mu(U) = \int_{\hat{G}(\tau)} \phi_\tau^U(g^{-1}) \hat{F}(U) d\mu(U),
\end{aligned}$$

where we have used (26) and Lemma 3.4. To pass from f_F to F we apply the map $f \to F_f$ (cf. (6)), and observe that $F_{f_F} = F$. Then

$$\begin{aligned} F(g) &= \int_K \tau(k) f_F(kg)dk \\ &= \int_K \int_{\hat{G}(\tau)} \tau(k)\phi_\tau^U(g^{-1}k^{-1})\hat{F}(U)d\mu(U)dk \\ &= \int_{\hat{G}(\tau)} \left(\int_K \phi_\tau^U(g^{-1}k^{-1})\tau(k)dk\right)\hat{F}(U)d\mu(U) \\ &= \frac{1}{d_\tau}\int_{\hat{G}(\tau)} \Phi_\tau^U(g^{-1})\hat{F}(U)d\mu(U), \end{aligned}$$

which is the inversion formula (30). ■

The inversion formula for the spherical transform $\hat{f}$ of $f \in I_{0,\tau}(G)$ (cf. (20)) is

$$f(g) = \int_{\hat{G}(\tau)} \phi_\tau^U(g^{-1})\hat{f}(U)d\mu(U), \qquad f \in I_{0,\tau}(G).$$

The Plancherel theorem for the spherical transform follows now from Theorem 3.9 by well known standard arguments.

Corollary 3.10. *The spherical Fourier transform $F \to \hat{F}$, defined in (17), extends to an isometry of the Hilbert space $L^2(G,\tau,\tau)$ onto the Hilbert space $L^2(\hat{G}(\tau), d\mu(U))$.*

Remark 3.11. The map $F \to \hat{F}$ given in (19) is the Gelfand transform on the commutative convolution algebra $C_0^\infty(G,\tau,\tau)$. We see from (30) that the Gelfand measure in the inversion formula on $C_0^\infty(G,\tau,\tau)$ may be identified with the Plancherel measure on $\hat{G}$, restricted to those representations of G which contain τ.

We now derive an "inversion formula" for $f \in C_0^\infty(G,\tau)$, i.e., for arbitrary (smooth compactly supported) sections of E^τ (not necessarily radial). We can proceed in two ways. We can either imitate the argument in the proof of Theorem 3.9, by observing that if f satisfies (1) then $f * d_\tau\bar{\chi}_\tau = f$, and similarly $(f \circ L_g) * d_\tau\bar{\chi}_\tau = f \circ L_g$. Then we can apply the Plancherel formula to each component f_a of $f(g) = \sum_{a=1}^{d_\tau} f_a(g)\mathbf{v}_a$ and obtain (in index-free notation)

$$\begin{aligned} f(g) &= \int_{\hat{G}(\tau)} \int_G \phi_\tau^U(g^{-1}x)f(x)dxd\mu(U) \\ &= \int_{\hat{G}(\tau)} \int_G \int_K \phi_\tau^U(g^{-1}x)\tau(k)f(xk)dkdxd\mu(U) \\ &= \int_{\hat{G}(\tau)} \int_G \int_K \phi_\tau^U(g^{-1}yk^{-1})\tau(k)f(y)dkdyd\mu(U) \\ &= \frac{1}{d_\tau}\int_{\hat{G}(\tau)} \int_G \Phi_\tau^U(g^{-1}y)f(y)dyd\mu(U). \end{aligned} \tag{32}$$

Alternatively, we can use (30). Observe that any $f \in C_0^\infty(G,\tau)$ can be written as

$$f(g) = F(g)\mathbf{v}, \qquad g \in G,$$

where $\mathbf{v}$ is an arbitrary (fixed) unit vector in V_τ, and $F(g) \in \mathrm{End}(V_\tau)$ is defined by

$$F(g)\mathbf{w} \equiv f(g)\langle \mathbf{w}, \mathbf{v}\rangle, \qquad \mathbf{w} \in V_\tau.$$

Notice that F satisfies

$$F(gk) = \tau(k^{-1})F(g), \qquad \forall g \in G,\ \forall k \in K,$$

and $\forall \mathcal{O} \in \mathrm{End}(V_\tau)$,

$$\mathrm{Tr}\ [\mathcal{O}F(g)] = \langle \mathcal{O}f(g), \mathbf{v}\rangle. \tag{33}$$

Put

$$F_g(x) \equiv \int_K F(gkx)\tau(k)dk. \tag{34}$$

It is easy to see that $F_g \in C_0^\infty(G,\tau,\tau)$. Hence, we can expand it according to (30),

$$F_g(x) = \frac{1}{d_\tau}\int_{\hat{G}(\tau)} \Phi_\tau^U(x^{-1})\hat{F}_g(U)d\mu(U). \tag{35}$$

The spherical Fourier transform of F_g is given by

$$\begin{aligned}
d_\tau \hat{F}_g(U) &= \int_G \mathrm{Tr}\ \left[\Phi_\tau^U(y)F_g(y)\right] dy \\
&= \int_G \mathrm{Tr}\ \left[\Phi_\tau^U(y)\int_K F(gky)\tau(k)dk\right] dy \\
&= \mathrm{Tr}\int_K\int_G \Phi_\tau^U(k^{-1}g^{-1}x)F(x)\tau(k)dxdk \\
&= \int_G \mathrm{Tr}\ \left[\Phi_\tau^U(g^{-1}x)F(x)\right] dx = \int_G \langle \Phi_\tau^U(g^{-1}x)f(x), \mathbf{v}\rangle dx,
\end{aligned} \tag{36}$$

where we have used (33). Now set $x = e$ in (34) and take the trace to get

$$\begin{aligned}
\mathrm{Tr}\, F_g(e) &= \mathrm{Tr}\int_K F(gk)\tau(k)dk \\
&= \mathrm{Tr}\int_K \tau(k^{-1})F(g)\tau(k)dk \\
&= \mathrm{Tr}\, F(g) = \langle f(g), \mathbf{v}\rangle,
\end{aligned} \tag{37}$$

where we have used (33) with $\mathcal{O} = \mathbf{1}$. Setting $x = e$ in (35) and taking the trace gives

$$\mathrm{Tr}\, F_g(e) = \int_{\hat{G}(\tau)} \hat{F}_g(U)d\mu(U).$$

Using here (36) for $\hat{F}_g(U)$ and equating to (37), we obtain, since $\mathbf{v}$ is arbitrary,

$$f(g) = \frac{1}{d_\tau}\int_{\hat{G}(\tau)}\int_G \Phi_\tau^U(g^{-1}x)f(x)dxd\mu(U), \tag{38}$$

in agreement with (32).

Notice that for a radial section f [i.e., $f(g) = F(g)\mathbf{v}$, with $\mathbf{v} \in V_\tau$, and $F \in C_0^\infty(G,\tau,\tau)$], eq. (38) reduces to (30) upon using (25).

We define the convolution of Φ_τ^U and f by

$$(\Phi_\tau^U * f)(g) = \int_G \Phi_\tau^U(g^{-1}x)f(x)dx, \qquad f \in C_0^\infty(G,\tau),\ g \in G,$$

so that eq. (38) becomes

$$f(g) = \frac{1}{d_\tau}\int_{\hat{G}(\tau)} (\Phi_\tau^U * f)(g)d\mu(U).$$

The space $\Phi_\tau^U * C_0^\infty(G,\tau)$ can be given the positive definite inner product

$$\begin{aligned}\langle \Phi_\tau^U * f_1, \Phi_\tau^U * f_2\rangle &= \frac{1}{d_\tau}\int_{G\times G} \langle \Phi_\tau^U(g^{-1}x)f_1(x), f_2(g)\rangle dx\, dg \\ &= \frac{1}{d_\tau}\int_{G\times G} \langle f_1(x), \Phi_\tau^U(x^{-1}g)f_2(g)\rangle dx\, dg.\end{aligned}$$

Denoting its completion by $L_U^2(G,\tau)$, we have the direct integral decompositions

$$L^2(G,\tau) = \int_{\hat{G}(\tau)}^{\oplus} L_U^2(G,\tau)d\mu(U),$$

$$\|f\|^2 = \int_{\hat{G}(\tau)} \|\Phi_\tau^U * f\|^2 d\mu(U), \qquad \forall f \in C_0^\infty(G,\tau), \tag{39}$$

where $L^2(G,\tau)$ is the completion of $C_0^\infty(G,\tau)$ with respect to the scalar product (2). From the representation theoretic point of view, this result gives the direct integral decomposition over $\hat{G}$ of the induced representation $\mathrm{ind}_K^G(\tau)$, according to

$$\mathrm{ind}_K^G(\tau) = \int_{\hat{G}(\tau)}^{\oplus} m(\tau,U)U d\mu(U) \tag{40}$$

(see [21] Lemma 1, or [18] p.58). This result may be regarded as a generalization to the noncompact case of the classical Frobenius Reciprocity Theorem. That is, the multiplicity with which U occurs in $\mathrm{ind}_K^G(\tau)$ coincides with the multiplicity of τ in $U|_K$. Of course the notion of multiplicity in a direct integral requires a little care. For U in the discrete series, $d\mu(U)$ is discrete and (40) takes the same form as in the classical (compact) case, see [18] p.58.

There are two important differences between the compact and the noncompact case: 1) in the noncompact case the Plancherel measure may have both a discrete and a continuous part, rather than only discrete; 2) in the noncompact case a representation in $\hat{G}$ may contain τ and still not appear in the decomposition of $\mathrm{ind}_K^G(\tau)$ (if it is not tempered). In fact the support of $d\mu(U)$ is, in general, a proper subset of $\hat{G}$. For example the so called complementary series of $SL(2,\mathbb{R})$ have zero Plancherel measure.

Remark 3.12. In this section we have fixed $\tau \in \hat{K}$ so that $m(\tau,U) = 1$, $\forall U \in \hat{G}(\tau)$. However for τ generic in $\hat{K}$, the multiplicity $\xi_U \equiv m(\tau,U)$ will exceed 1. (It is known that $m(\tau,U) \le d_\tau$, giving an estimate independent of U, see [11].) In this more general case, the results of this section will be modified as follows: 1) the algebras $I_{0,\tau}(G)$ and $C_0^\infty(G,\tau,\tau)$ will no longer be commutative; 2) the spherical functions $\Psi_\tau^U(g) = P_\tau U(g) P_\tau$ will take values in $\mathrm{End}(V_\tau \otimes \mathbb{C}^{\xi_U})$; 3) the spherical transform $\hat{F}(U)$ will no longer be scalar valued, but will take values in $\mathrm{End}(\mathbb{C}^{\xi_U})$. However the inversion formula on $C_0^\infty(G,\tau,\tau)$ can still be derived from the Plancherel formula on $C_0^\infty(G)$, and is similar to (30). The noncommutative case will be discussed in another paper.

Remark 3.13. The direct integral decomposition (39) does not involve any Fourier transform concept for general $f \in C_0^\infty(G,\tau)$. [The spherical transform applies only to radial sections.] However in the scalar case, Helgason has defined a Fourier transform for general functions $f \in C_0^\infty(G/K)$, see [13]. The Helgason Fourier transform on Riemannian symmetric spaces G/K has recently been generalized to homogeneous vector bundles, see [7].

We have not used, so far, the structure theory of semisimple Lie groups (Cartan or Iwasawa decomposition). In fact, the theory of spherical functions of type τ can be formulated for any pair (G,K) of a locally compact unimodular Lie group G and a compact subgroup $K \subset G$, provided that every $U \in \hat{G}$ is K-finite (see [11] section 3). As mentioned in the introduction, if K is uniformly large in G (see [28] vol.I p. 305), then the results obtained up to this point apply (for $I_{0,\tau}(G)$ commutative).

We now consider the semisimple case in more detail. It is well known (see, e.g., [15, 27, 28]) that for a noncompact semisimple Lie group with finite center, the irreducible unitary representations that appear in the Plancherel formula (the so-called tempered spectrum of G) are the (generalized) principal series (constructed from a complete set of cuspidal parabolic proper subgroups by the method of induced representation), and the discrete series, which exist if and only if G has a compact Cartan subgroup, i.e., if and only if $\operatorname{rank} G = \operatorname{rank} K$. The Plancherel measure has been obtained by Harish-Chandra. Using Harish Chandra's Plancherel Theorem and Subquotient Theorem, we shall write down in a more precise way the spherical transform and the inversion formula on $C_0^\infty(G,\tau,\tau)$ in the commutative case.

Let $G = KAN$ be an Iwasawa decomposition of G, and write

$$x = \mathbf{k}(x)\exp[H(x)]\,n(x), \qquad \forall x \in G,$$

where $H(x) \in \mathfrak{a}$ (the Lie algebra of A), $\mathbf{k}(x) \in K$ and $n(x) \in N$. Let M be the centralizer of A in K, and let $P = MAN$ be the corresponding minimal parabolic subgroup of G. Given $\sigma \in \hat{M}$ and a linear function $\lambda : \mathfrak{a} \to \mathbb{C}$, let $U^{\sigma\lambda}$ denote the representation $\mathrm{ind}_P^G(\sigma \otimes e^{i\lambda} \otimes \mathbf{1})$ in the minimal principal series of G with parameters σ and λ. $U^{\sigma\lambda}$ is unitary if and only if λ is real valued.

The Subquotient Theorem of Harish Chandra (see, e.g., [28] Theorem 5.5.1.5) implies that each $U \in \hat{G}$ (more generally every TCI Banach representation of G) is infinitesimally equivalent to a subquotient representation of a nonunitary principal series $U^{\sigma\lambda}$, for suitable $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (the complexification of $\mathfrak{a}^*$).

In the commutative case we have, for the given τ, $m(\tau, U^{\sigma\lambda}) \le 1$ $\forall \sigma \in \hat{M}$, $\forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*$. By Frobenius Reciprocity, the multiplicity $m(\sigma,\tau)$ of σ in $\tau|_M$ is also ≤ 1 $\forall \sigma \in \hat{M}$. Then the Subquotient Theorem implies that every (nonzero) spherical function of type τ on G can be written as $\Phi_\tau^{\sigma\lambda} \equiv P_\tau U^{\sigma\lambda} P_\tau$, for suitable $\sigma \subset \tau|_M$, and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Let $\phi_\tau^{\sigma\lambda}(x) = \operatorname{Tr} \Phi_\tau^{\sigma\lambda}(x)$ be the corresponding spherical trace function of type τ. By [28] Corollary 6.2.2.3 one has

$$\phi_\tau^{\sigma\lambda}(x) = d_\tau \int_K (\chi_\tau * \chi_\sigma)(\mathbf{k}(k^{-1}xk))\; e^{(i\lambda-\rho)(H(xk))} dk, \tag{41}$$

where $\rho \in \mathfrak{a}^*$ is half the sum of the positive restricted roots, χ_σ is the character of σ, and the convolution is over M. One can easily calculate $\Phi_\tau^{\sigma\lambda}(x)$ using (41), (13), and the Schur orthogonality relations. The result is the following integral representation for $\Phi_\tau^{\sigma\lambda}$:

$$\Phi_\tau^{\sigma\lambda}(x) = \frac{d_\tau}{d_\sigma} \int_K \tau\left(\mathbf{k}(xk)\right) P_\sigma \tau(k^{-1})\; e^{(i\lambda-\rho)(H(xk))} dk, \tag{42}$$

where

$$P_\sigma = d_\sigma \int_M \tau(m^{-1}) \chi_\sigma(m) dm$$

is the projector of V_τ onto $V_\sigma \subset V_\tau$ (the subspace of vectors of V_τ which transform under M according to σ), and d_σ is the dimension of σ.

The spherical transform of $F \in C_0^\infty(G, \tau, \tau)$ can then be defined as the set of functions $\hat{F}_\sigma : \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$ (where $\sigma \subset \tau|_M$), given by

$$\hat{F}_\sigma(\lambda) = \frac{1}{d_\tau} \int_G \operatorname{Tr}\left[\Phi_\tau^{\sigma\lambda}(x) F(x)\right] dx, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{43}$$

The inversion formula (30) can be written more explicitly as follows. Let P' be a cuspidal parabolic subgroup of G such that $P' \supseteq P$ and $A' \subseteq A$, where $P' = M'A'N'$ is a Langlands decomposition of P'. Given σ' in the discrete series of M' and $\nu' \in \mathfrak{a}'^*$ (the real dual of the Lie algebra of A'), let $U^{\sigma'\nu'}$ denote the representation $\operatorname{ind}_{P'}^G(\sigma' \otimes e^{i\nu'} \otimes \mathbf{1})$ in the unitary principal P'-series with parameters σ' and ν'. If $\tau \subset U^{\sigma'\nu'}|_K$, put as usual $\Phi_\tau^{U^{\sigma'\nu'}}(x) = P_\tau U^{\sigma'\nu'}(x) P_\tau$, and

$$\hat{F}(U^{\sigma'\nu'}) = \frac{1}{d_\tau} \int_G \operatorname{Tr}\left[\Phi_\tau^{U^{\sigma'\nu'}}(x) F(x)\right] dx, \qquad F \in C_0^\infty(G, \tau, \tau).$$

(For $U^{\sigma'\nu'}$ irreducible, this is the spherical Fourier transform (17) with $U (= U^{\sigma'\nu'})$ in the tempered spectrum of G). By the Subquotient Theorem we can identify $U^{\sigma'\nu'}$ with a subquotient of a nonunitary (minimal) principal series $U^{\tilde{\sigma}'\lambda}$, for suitable $\tilde{\sigma}' \in \hat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

More precisely, let $K' = K \bigcap M'$ be maximal compact in M', and let $M' = K'A_1N_1$ be an Iwasawa decomposition of M' such that $A = A'A_1$ and $N = N'N_1$. We choose parameters $\tilde{\sigma}' \in \hat{M}$ and $\mu_1 \in (\mathfrak{a}_1)_{\mathbb{C}}^*$ ($\mathfrak{a}_1$ the Lie algebra of A_1) by the Subquotient Theorem, so that σ' is infinitesimally equivalent to a subquotient of the nonunitary (minimal) principal series of M' given by

$$\omega_{\tilde{\sigma}'\mu_1} = \operatorname{ind}_{MA_1N_1}^{M'}(\tilde{\sigma}' \otimes e^{\mu_1} \otimes \mathbf{1}). \tag{44}$$

Then the (generalized) principal series $U^{\sigma'\nu'}$ is infinitesimally equivalent to a subquotient of the nonunitary (minimal) principal series of G

$$U^{\tilde{\sigma}',\nu'-i\mu_1} = \operatorname{ind}_{MAN}^G(\tilde{\sigma}' \otimes e^{i\nu'+\mu_1} \otimes \mathbf{1}). \tag{45}$$

This follows by a double induction formula, as in [15] pp. 171, 240.

Then the parameter λ above equals $\nu'-i\mu_1$, and $\Phi_\tau^{U^{\sigma'\nu'}}(x)$, for $\tau \subset U^{\sigma'\nu'}|_K$, is given by

$$\Phi_\tau^{U^{\sigma'\nu'}}(x) = \Phi_\tau^{\tilde{\sigma}',\nu'-i\mu_1}(x) = \frac{d_\tau}{d_{\tilde{\sigma}'}} \int_K \tau(\mathbf{k}(xk)) P_{\tilde{\sigma}'} \tau(k^{-1}) e^{(i\nu'+\mu_1-\rho)(H(xk))} dk$$

(cf. (42)). Thus the spherical Fourier transform $\hat{F}(U^{\sigma'\nu'})$ of $F \in C_0^\infty(G,\tau,\tau)$ relative to $U^{\sigma'\nu'}$ equals $\hat{F}_{\tilde{\sigma}'}(\nu'-i\mu_1)$, given by (43).

Finally, let $d\mu(U^{\sigma'\nu'}) = p_{\sigma'}(\nu')d\nu'$ be the Plancherel measure associated with $U^{\sigma'\nu'}$, where $d\nu'$ is a properly normalized Euclidean measure on $\mathfrak{a}'^*$. Then, for a suitable normalization of the relevant Haar measures, and for suitable constants $c_{P'} > 0$, we have the following inversion formula for the spherical transform (43) of $F \in C_0^\infty(G,\tau,\tau)$:

$$F(x) = \frac{1}{d_\tau} \sum_{P'} c_{P'} \sum_{\sigma'} \int_{\mathfrak{a}'^*} \Phi_\tau^{\tilde{\sigma}',\nu'-i\mu_1}(x^{-1}) \hat{F}_{\tilde{\sigma}'}(\nu'-i\mu_1) p_{\sigma'}(\nu') d\nu'. \tag{46}$$

Here the sum $\sum_{P'}$ is over all cuspidal parabolic subgroups P' of G such that $P' \supseteq P$ and $A' \subseteq A$, and the sum $\sum_{\sigma'}$ is over all discrete series σ' of M' such that $U^{\sigma'\nu'}|_K \supset \tau$. [These are in finite number only.] This result follows from (30) and from Harish Chandra's Plancherel Theorem for semisimple Lie groups (see, e.g., [27] Theorem 13.4.1, or [15] Theorem 13.11). If $\operatorname{rank} G = \operatorname{rank} K$, then G itself is cuspidal parabolic, and the contribution of $P' = G$ in (46) is the discrete series contribution. In this case the parameter ν' is trivial, and the integral over $\mathfrak{a}'^*$ drops out. For $P' = P$ (the minimal parabolic subgroup), the parameter μ_1 is trivial. The corresponding contribution in (46) is

$$F_{\text{minimal}}(x) = \frac{1}{d_\tau} c_P \sum_\sigma \int_{\mathfrak{a}^*} \Phi_\tau^{\sigma\lambda}(x^{-1}) \hat{F}_\sigma(\lambda) p_\sigma(\lambda) d\lambda,$$

where the sum is over all M-types contained in $\tau|_M$.

Now due to the Cartan decomposition $G = KAK$, the spherical functions $\Phi_\tau^{\sigma\lambda}$ are determined by their restriction to A. Notice that $\Phi_\tau^{\sigma\lambda}(a) \in \operatorname{End}_M(V_\tau)$, $\forall a \in A$. Write $\tau|_M = \sum_{j=1}^n \sigma_j$, with $\sigma_j \in \hat{M}$. Then $\sigma_i \not\sim \sigma_j$ for $i \neq j$, and σ is just one of the σ_j, say $\sigma = \sigma_k$. We have a direct sum decomposition $V_\tau = \sum_j V_j$, where each V_j may be identified with the representation space of σ_j. Let P_j be the orthogonal projection of V_τ onto V_j. Schur's Lemma implies that

$$\Phi_\tau^{\sigma_k\lambda}(a) = \sum_j \phi_{\tau j}^{k\lambda}(a) \mathbf{1}_j, \qquad \forall a \in A$$

(direct sum of linear operators), where $\mathbf{1}_j$ is the identity in V_j. The scalar functions $\phi_{\tau j}^{k\lambda}(a) = \frac{1}{d_j} \operatorname{Tr}[P_j \Phi_\tau^{\sigma_k\lambda}(a) P_j]$ ($d_j \equiv d_{\sigma_j}$) admit the integral representation (by (42))

$$\phi_{\tau j}^{\kappa\lambda}(a) = \frac{d_\tau}{d_j d_\kappa} \int_K \operatorname{Tr}[P_j \tau(\mathbf{k}(ak)) P_\kappa \tau(k^{-1}) P_j]\, e^{(i\lambda-\rho)(H(ak))} dk.$$

In a similar way, each $F \in C_0^\infty(G,\tau,\tau)$ is determined by restriction to A. Since $F(a) \in \operatorname{End}_M(V_\tau)$, $\forall a \in A$, then $F(a) = \sum_j F_j(a)\mathbf{1}_j$, where $F_j(a)$ are scalar functions on A. Let us write the integral formula for the Cartan decomposition as (see, e.g., [15] Prop.5.28 p.141):

$$\int_G f(g)dg = \int_{K\times A^+\times K} f(k_1 a k_2)\delta(a)dk_1 da dk_2, \tag{47}$$

where $A^+ = \exp \mathfrak{a}^+$ ($\mathfrak{a}^+$ the positive Weyl chamber in $\mathbf{a}$), and

$$\delta(a) \equiv \prod_+ (\sinh \alpha(\log a))^{m_\alpha}, \tag{48}$$

where the product is over the positive restricted roots of the symmetric space (m_α is the multiplicity of the root α). The Haar measure on K is normalized by $\int_K dk = 1$.

The spherical transform (43) can then be written as

$$\begin{aligned}\hat{F}_k(\lambda) \equiv \hat{F}_{\sigma_k}(\lambda) &= \frac{1}{d_\tau}\int_{A^+} \mathrm{Tr}\,[\Phi_\tau^{\sigma_k\lambda}(a)F(a)]\delta(a)da \\ &= \frac{1}{d_\tau}\sum_j d_j \int_{A^+} \phi_{\tau j}^{k\lambda}(a)F_j(a)\delta(a)da.\end{aligned} \tag{49}$$

Thus the spherical transform can be described as follows in the commutative case. Let n be the number of M-types contained in $\tau|_M$. Then each $F \in C_0^\infty(G,\tau,\tau)$ is determined by n scalar functions $F_j(a)$ on A ($j = 1,\ldots,n$). The spherical transform associates to this set of functions the n scalar functions $\hat{F}_k(\lambda)$ on $\mathfrak{a}_{\mathbb{C}}^*$ ($k = 1,\ldots,n$) given by (49).

If G/K has rank one the functions $\phi_{\tau j}^{k\lambda}(a)$ can be calculated explicitly in terms of Jacobi functions (using the radial part of the Casimir operator of G). Examples of this will be seen in the next section.

4. The case of real hyperbolic spaces

In this section we apply the above considerations to $G = \mathrm{Spin}(N,1)$ and $K = \mathrm{Spin}(N)$, the double covers of the Lorentz and orthogonal groups $SO_0(N,1)$ and $SO(N)$, respectively. (For $N > 2$ these are also the universal covering groups.) In this way, we can include in our discussion the spinor bundles over the real hyperbolic spaces $H^N(\mathbb{R}) \simeq G/K$. Using Theorem 2.3, one can prove that $\mathrm{Spin}(N)$ is multiplicity free in $\mathrm{Spin}(N,1)$, so that the algebras $C_0^\infty(G,\tau,\tau)$ are commutative $\forall \tau \in \hat{K}$.

For N odd all Cartan subgroups of G are conjugate, and there are no discrete series. For N even there are two conjugacy classes of Cartan subgroups, one for a compact Cartan subgroup contained in K, the other for a noncompact Cartan subgroup with one generator in G/K. In this case there are discrete series. If the irreducible representation τ of $\mathrm{Spin}(N)$ is contained in some discrete series, then the vector bundle E^τ has square-integrable eigensections of the Laplacian, corresponding to the discrete spectrum. (An example of this will be seen later.) Otherwise the L^2-spectrum of the Laplacian on E^τ is purely continuous (e.g., for τ the trivial representation).

The Plancherel measure for the Lorentz group has been given by Hirai [14]. We shall now summarize his results, and write down the Plancherel (inversion) formula for the double cover $\mathrm{Spin}(N,1)$. As specific examples we shall consider the homogeneous vector bundles of Dirac spinors, differential forms, and symmetric traceless tensor fields over $H^N(\mathbb{R})$.

Case 1. $N = 2k+2$ $(k = 1, 2, \ldots)$: The unitary principal series representations are denoted $U^{\sigma\lambda}$, where λ is a real number, and $\sigma = (n_1, n_2, \ldots, n_k)$ is a row of numbers that are either all integers or all half-odd-integers and satisfy

$$0 \le n_1 \le n_2 \le \cdots \le n_k.$$

In the standard Cartan-Weyl labeling scheme, σ is the highest weight of an irreducible representation of $M = \mathrm{Spin}(N-1)$ (see [2] Th.2 p.221). In fact, $U^{\sigma\lambda}$ is nothing but the representation $\mathrm{ind}_P^G(\sigma \otimes e^{i\lambda} \otimes \mathbf{1})$ unitarily induced from a minimal parabolic subgroup $P = MAN$, where $G = KAN$ is an Iwasawa decomposition of G. Thus $\sigma \in \hat{M}$, and it is well known that M (the centralizer of A in K) can be identified with $\mathrm{Spin}(N-1)$ in this case. We define $l_j = n_j + j - \frac{1}{2}$ $(j = 1, 2, \ldots, k)$, and denote the global character of $U^{\sigma\lambda}$ by $\Theta_{\sigma\lambda}$.

There are two sets of representations in the discrete series denoted $U_+^{\sigma n_0}$ and $U_-^{\sigma n_0}$, where $\sigma = (n_j)$ is as above, $n_0 \in \mathbb{Z}$ (resp. $\mathbb{Z}+\frac{1}{2}$) $\Longleftrightarrow n_j \in \mathbb{Z}$ (resp. $\mathbb{Z}+\frac{1}{2}$), and

$$\frac{1}{2} < n_0 \le n_1 \le n_2 \le \cdots \le n_k.$$

(For $n_0 = \frac{1}{2}$ we have the two "limits of discrete series", which are irreducible unitary representations, but are not square-integrable.) Define l_j $(j = 0, \ldots, k)$ as above, and denote the sum of the characters of $U_+^{\sigma n_0}$ and $U_-^{\sigma n_0}$ by $\Theta_{\sigma n_0}$. Then the inversion formula on $C_0^\infty(\mathrm{Spin}(N,1))$ takes the following form (we have corrected the continuous part by a factor of 2; this makes the formula consistent with ref. [22]):

$$\begin{aligned} c\, f(e) = &\sum_{0<l_1<\cdots<l_k} \int_0^\infty iP(-i\lambda, l_1, \ldots, l_k)\, g(\lambda)\, \Theta_{\sigma\lambda}(f)\, d\lambda \\ &+ \sum_{0<l_0<l_1<\cdots<l_k} P(l_0, l_1, \ldots, l_k)\, \Theta_{\sigma n_0}(f), \end{aligned} \tag{50}$$

where $c > 0$ is a normalization constant (to be determined later),

$$g(\lambda) = \begin{cases} \tanh(\pi\lambda), & l_j \ \text{half-odd-integers}, \\ \coth(\pi\lambda), & l_j \ \text{integers}, \end{cases}$$

and P is the following polynomial, corresponding to the product over the positive roots of the $\mathrm{Spin}(N,1)$ Lie algebra:

$$P(x_1, x_2, \ldots, x_{k+1}) = x_1 x_2 \cdots x_{k+1} \prod_{1 \le s < r \le k+1} (x_r^2 - x_s^2).$$

The case of $N = 2$ $(k = 0)$ corresponds to $G = SL(2,\mathbb{R}) \simeq \mathrm{Spin}(2,1)$. In this case, the continuous part is the sum of two terms, one with $g(\lambda) = \tanh(\pi\lambda)$, and the other with $g(\lambda) = \coth(\pi\lambda)$, see [15] p. 42.

For σ fixed, the continuous part of the Plancherel measure $d\mu(U^{\sigma\lambda}) = p_\sigma(\lambda)d\lambda$ has the following λ-dependence:

$$p_\sigma(\lambda) \propto \lambda(\lambda^2 + l_1^2)(\lambda^2 + l_2^2)\cdots(\lambda^2 + l_k^2)\, g(\lambda),$$

and the proportionality constant depends on l_j.

Case 2. $N = 2k + 1$ $(k = 1, 2, \ldots)$: The principal series representations are denoted $U^{\sigma\lambda}$, with $\lambda \in \mathbb{R}$ and $\sigma = (n_1, n_2, \ldots, n_k)$, where the numbers n_j are either all integers or all half-odd-integers and satisfy

$$|n_1| \leq n_2 \leq \cdots \leq n_k.$$

The number n_1 can be negative, and again σ defines a representation of $\mathrm{Spin}(N-1)$ (see [2] Th.2 p.221). Let $l_j = n_j + j - 1$ $(j = 1, 2, \ldots, k)$. Then the inversion formula on $C_0^\infty(\mathrm{Spin}(N, 1))$ reads

$$c\, f(e) = \sum_{|l_1|<l_2<\cdots<l_k} \int_0^\infty P(i\lambda, l_1, \ldots, l_k)\, \Theta_{\sigma\lambda}(f)\, d\lambda, \tag{51}$$

where $c > 0$ is a normalization constant, and P is the polynomial corresponding to the product over the positive roots of the $\mathrm{Spin}(N, 1)$ Lie algebra:

$$P(x_1, x_2, \ldots, x_{k+1}) = \prod_{1\leq s<r\leq k+1} (x_r^2 - x_s^2).$$

For σ fixed, the Plancherel density is just a polynomial in λ^2:

$$p_\sigma(\lambda) \propto (\lambda^2 + l_1^2)(\lambda^2 + l_2^2)\cdots(\lambda^2 + l_k^2).$$

Now let τ be an irreducible representation (irrep) of $\mathrm{Spin}(N)$, and let E^τ be the corresponding homogeneous vector bundle over $H^N(\mathbb{R})$. In order to find the Plancherel measure for the cross sections of E^τ, we simply have to identify the irreducible unitary representations in the Plancherel formula on $\mathrm{Spin}(N, 1)$ which contain τ upon restriction to $\mathrm{Spin}(N)$. Thus we need the branching rule for $\mathrm{Spin}(N, 1) \supset \mathrm{Spin}(N)$ for principal and discrete series (see, e.g., [24]). Again we distinguish the cases with N odd and N even.

Let $N = 2k + 2$, and let τ be the irrep of $\mathrm{Spin}(N)$ labelled by $(f_1, f_2, \ldots, f_{k+1})$, where f_j are either all integers or all half-odd-integers satisfying (see [2] p.221)

$$|f_1| \leq f_2 \leq \cdots \leq f_{k+1},$$

and f_1 can be negative. Then the principal series representation $U^{\sigma\lambda}$ $(\sigma = \{n_j\})$ contains τ if and only if 1) $n_j \in \mathbb{Z}$ (resp. $\mathbb{Z}+\frac{1}{2}$) $\Longleftrightarrow f_j \in \mathbb{Z}$ (resp. $\mathbb{Z}+\frac{1}{2}$), and 2)

$$|f_1| \leq n_1 \leq f_2 \leq n_2 \leq \cdots \leq f_k \leq n_k \leq f_{k+1}. \tag{52}$$

Using the branching rule for $\mathrm{Spin}(N) \supset \mathrm{Spin}(N-1)$, it is easy to see that (52) is equivalent to the condition that τ contain the irrep σ of $\mathrm{Spin}(N-1)$ (see

[2] Th.2 p.228). Therefore $U^{\sigma\lambda}|_K$ contains τ if and only if $\tau|_M$ contains σ, i.e., in symbols

$$\tau|_{\mathrm{Spin}(N-1)} = \oplus_{j=1}^{n}\sigma_j \iff U^{\sigma_j\lambda}|_{\mathrm{Spin}(N)} \supset \tau, \quad \forall j = 1,\ldots,n. \tag{53}$$

(Of course this follows by Frobenius Reciprocity, since $U^{\sigma\lambda}|_K = \mathrm{ind}_M^K(\sigma)$.)

A geometric interpretation of (53) may be given as follows. According to the so-called polar coordinates decomposition [12], a (noncompact) Riemannian symmetric space G/K is diffeomorphic to $A^+ \times K/M$ (up to a zero-measure set), where A^+ is a fundamental domain of the Weyl group in A. The coset space K/M is diffeomorphic to the orbits of K in G/K. When a field on G/K (i.e., a cross section of E^τ, $\tau \in \hat{K}$) is restricted to the K-orbits, we obtain a set of fields on K/M. These fields are cross sections of homogeneous vector bundles E^{σ_j} over K/M, defined by irreps σ_j of M, with $\sigma_j \subset \tau|_M$. In our case the above decomposition reads $H^N(\mathbb{R}) \simeq \mathbb{R}^+ \times S^{N-1}$, and the representations σ_j are all different. For example if τ is the defining vector representation of $SO(N)$ in $\mathbb{R}^N$, then E^τ is the tangent bundle over $H^N(\mathbb{R})$. When we restrict a vector in $H^N(\mathbb{R})$ to S^{N-1}, we get a vector and a scalar, i.e., $\tau|_M = \sigma_1 \oplus \sigma_2$, where σ_1 is the trivial representation of $SO(N)$, and σ_2 is the vector one.

Now according to (53), the principal series that contain τ and enter in the decomposition of the induced representation $\mathrm{ind}_K^G(\tau)$ (i.e., in the right hand side of (40)), are precisely the $U^{\sigma_j\lambda}$. In the previous example we have that the principal series contributing to the harmonic analysis of vector fields over $H^N(\mathbb{R})$ are $U^{\sigma_1\lambda}$ and $U^{\sigma_2\lambda}$. It is possible to show that the vector valued functions on G given by $f_{\sigma_1\lambda}(g) = P_\tau U^{\sigma_1\lambda}(g^{-1})v$ (v any vector of H_U) correspond to pure gradients, whereas the functions $f_{\sigma_2\lambda}(g) = P_\tau U^{\sigma_2\lambda}(g^{-1})v$ correspond to divergence-free vectors (see [1] Prop. 4.1 and 4.2).

Concerning the discrete series, we have the following branching rule. A representation in the discrete series $U_\pm^{\sigma n_0}$ contains $\tau = (f_1,\ldots,f_{k+1})$ if and only if, in addition to (52), the following condition is satisfied (as before $n_j \in \mathbb{Z}$ (or $\mathbb{Z}+\frac{1}{2}$) $\iff f_j \in \mathbb{Z}$ (or $\mathbb{Z}+\frac{1}{2}$)):

$$\frac{1}{2} < n_0 \le f_1 \le n_1, \qquad \text{for } U_+^{\sigma n_0}, \tag{54}$$

$$\frac{1}{2} < n_0 \le -f_1 \le n_1, \qquad \text{for } U_-^{\sigma n_0}. \tag{55}$$

Thus f_1 must be nonzero (positive for $U_+^{\sigma n_0}$, and negative for $U_-^{\sigma n_0}$). [See, e.g., ref. [24] Th.5 p.36, and Th.2 p.28, Case 2; in our notation $|\nu| + \frac{1}{2} = n_0$, and $\Lambda(\omega)_m = f_1$.]

Let $N = 2k+1$. The branching rule for $U^{\sigma\lambda}|_K \supset \tau = (f_1,\ldots,f_k)$ is

$$|n_1| \le f_1 \le n_2 \le f_2 \le \ldots \le n_k \le f_k. \tag{56}$$

This is equivalent to $\tau|_M \supset \sigma$, and (53) is again true (see [2] Th.2 p.228).

Example 4.1. Let τ be the trivial representation of $\mathrm{Spin}(N)$. Then E^τ is the trivial bundle, whose sections are the scalar functions on $H^N(\mathbb{R})$. In this case the spherical transform reduces to the Jacobi transform [17].

Since G has real rank one, $\mathfrak{a}^+ \simeq \mathbb{R}^+$, and each $a \in A$ can be written as $a_t = \exp(tH)$, where H is the element of $\mathfrak{a}$ such that $\beta(H) = 1$ (β the positive restricted root). In this normalization $\delta(a_t) = [\sinh(t)]^{N-1}$ in (48). Moreover $C_0^\infty(G, \tau, \tau)$ and $I_{0,\tau}(G)$ both coincide with $C_0^\#(G)$, the set of compactly supported smooth functions on G which are biinvariant under K. The spherical transform (49) reduces to

$$\hat{f}(\lambda) = \int_0^\infty f(a_t)\phi_\lambda(a_t)(\sinh t)^{N-1} dt, \qquad f \in C_0^\#(G), \tag{57}$$

and we have the inversion formula [17]

$$f(a_t) = \frac{2^{N-2}}{\pi} \int_0^\infty \hat{f}(\lambda)\phi_\lambda(a_t)|C(\lambda)|^{-2} d\lambda. \tag{58}$$

Here $\phi_\lambda(a_t) = F(i\lambda + \rho, -i\lambda + \rho, \frac{N}{2}, -\sinh^2 \frac{t}{2})$ are the spherical functions ($\rho \equiv (N-1)/2$, F is the hypergeometric function), and $C(\lambda)$ is the Harish-Chandra function

$$C(\lambda) = \frac{2^{N-2}\Gamma(N/2)}{\sqrt{\pi}} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda + \rho)}. \tag{59}$$

It is easy to see that (50) and (51) agree with (58) at $t = 0$. Indeed for any N, the highest weight of τ is $(f_j) = (0, 0, \ldots, 0)$. Thus there is no discrete series containing τ (a well known result, see [15] p.455). Since $\tau|_M$ is just the trivial representation of $\mathrm{Spin}(N-1)$, we have $\sigma = (n_j) = (0, 0, \ldots, 0)$ $\forall N$. Thus the only representations in $\hat{G}$ which contain τ and have nonzero Plancherel measure are the *spherical* principal series $U^{\sigma\lambda}$. The only term that survives in (50) and (51) (for f K-biinvariant) is the term with $(l_j) = (\frac{1}{2}, \frac{3}{2}, \ldots, k - \frac{1}{2})$ for $N = 2k+2$, and $(l_j) = (0, 1, \ldots, k-1)$ for $N = 2k+1$. Using the given values of l_j in (50) and (51) we verify that the Plancherel measure as a function of λ is indeed proportional to $|C(\lambda)|^{-2}$. Moreover $\Theta_{\sigma\lambda}(f) = \hat{f}(\lambda)$ for f K-biinvariant [use (47) in (14), and compare with (57)]. By writing $d\mu(U^{\sigma\lambda}) = p_0(\lambda)d\lambda$, with scalar Plancherel density $p_0(\lambda) = \frac{2^{N-2}}{\pi}|C(\lambda)|^{-2}$, we can match eqs. (50) and (51) with eq. (58), to obtain the value of the normalization constant c of Hirai (which is independent of τ!). The result is

$$c = k! \prod_{s=1}^{k} (2s)!, \qquad N = 2k+2,$$

$$c = 2^k (\Gamma(k + \frac{1}{2}))^2 \prod_{s=1}^{k-1} (2s)!, \qquad N = 2k+1.$$

Example 4.2. Consider Dirac spinors on $H^N(\mathbb{R})$. The spinor bundle E^τ is defined by

$$\tau = \tau_+ \oplus \tau_- = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus (-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}), \quad N \text{ even},$$

$$\tau = (\frac{1}{2}, \frac{1}{2}, \dots \frac{1}{2}), \qquad N \text{ odd},$$

i.e., τ is the fundamental spinor representation of $\mathrm{Spin}(N)$, for N odd, and it is the direct sum of the two fundamental spinor representations of $\mathrm{Spin}(N)$, for N even (see [2] p.222-224). Notice that for N even the bundle is reducible.

Using the above values of (f_j) in (54)-(55), we see that no discrete series contain τ. Using these values in (52) and (56), we find that τ is contained in the principal series $U^{\sigma\lambda}$ with

$$\sigma = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}), \qquad N \text{ even},$$
$$\sigma = \sigma_+, \sigma_-, \quad \sigma_\pm = (\pm\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}), \qquad N \text{ odd}.$$

By Frobenius Reciprocity, this is simply the statement that for N even one has $\tau_\pm|_{\mathrm{Spin}(N-1)} = \sigma$, whereas for N odd $\tau|_{\mathrm{Spin}(N-1)} = \sigma_+ \oplus \sigma_-$.

Thus for $f \in I_{0,\tau}(G)$ only the terms corresponding to the above principal series survive in (50) and (51). The Plancherel measure so obtained agrees with the analytic calculation performed in [3]. Let σ denote either σ_+ or σ_- in the case of N odd. Then we find, for any N, $d\mu(U^{\sigma\lambda}) = p_\sigma(\lambda)d\lambda$, with the spinor Plancherel density

$$p_\sigma(\lambda) = \frac{2^{N-2}}{\pi} d_\sigma |C_\sigma(\lambda)|^{-2},$$

where $d_\sigma = 2^{[N/2]-1}$ is the dimension of σ, and

$$C_\sigma(\lambda) = \frac{2^{N-2}\Gamma(N/2)}{\sqrt{\pi}} \frac{\Gamma(i\lambda + \frac{1}{2})}{\Gamma(i\lambda + \frac{N}{2})}.$$

The spinor spherical functions for the principal series [i.e., $\Phi_+^\lambda = P_{\tau_+} U^{\sigma\lambda} P_{\tau_+}$ and $\Phi_-^\lambda = P_{\tau_-} U^{\sigma\lambda} P_{\tau_-}$ (for N even), and $\Phi^{\lambda\pm} = P_\tau U^{\sigma_\pm\lambda} P_\tau$ (for N odd)] have been calculated in [6]. As in the scalar case, the spinor Harish-Chandra function $C_\sigma(\lambda)$ is determined by the asymptotic form at infinity of the (normalized) spherical trace functions ϕ_λ, given by (see [3, 6])

$$\phi_\lambda(a_t) = \cosh\frac{t}{2}\, F(i\lambda + \frac{N}{2}, -i\lambda + \frac{N}{2}, \frac{N}{2}, -\sinh^2\frac{t}{2}).$$

Example 4.3. Let τ be the representation of $\mathrm{Spin}(N)$ defined by the highest weight $(f_j) = (0, \dots, 0, s)$, where s is a nonnegative integer. Then E^τ is the bundle of totally symmetric traceless tensor fields of rank s over $H^N(\mathbb{R})$ (see [2] Th.6 p.301). Again no discrete series contain τ. From (52) and (56) we find that τ is contained in $s+1$ different principal series representations, namely in $U^{\sigma_r\lambda}$, where $\sigma_r = (0, \dots, 0, r)$, and $r = 0, \dots, s$. The representations $U^{\sigma_s\lambda}$ correspond to the symmetric traceless and transverse (i.e., divergence-free) (STT) tensor fields of rank s (see [4]).

The Plancherel measure for STT tensor fields has been calculated analytically in [4], and the two methods give the same result. From (50) and (51) we find

$d\mu(U^{\sigma_s\lambda}) = p_s(\lambda)d\lambda$, with density $p_s(\lambda) = \frac{2^{N-2}}{\pi}d_s|C_s(\lambda)|^{-2}$, where

$$d_s = \frac{(2s+N-3)(s+N-4)!}{s!(N-3)!}$$

is the dimension of σ_s, and

$$C_s(\lambda) = \frac{2^{N-2}\Gamma(N/2)}{\sqrt{\pi}} \frac{\Gamma(i\lambda)}{(i\lambda+s+\rho-1)\Gamma(i\lambda+\rho-1)}. \tag{60}$$

For $s = 0$ this reduces to the scalar result (59). The spherical functions $\Phi_s^{\sigma_r\lambda} = P_\tau U^{\sigma_r\lambda} P_\tau$ $(r = 0, \ldots, s)$ can be obtained from the results of ref. [4].

Example 4.4. We now consider the example of p-forms (i.e., totally antisymmetric tensor fields of rank p) over $H^N(\mathbb{R})$ (see also [5]).

Let $N = 2k+2$. The bundle of p-forms on $H^N(\mathbb{R})$ is defined by the following irreps τ of $\mathrm{Spin}(N)$ (see [2] Th.5 p.299):

$$\begin{array}{llll}
p & = 0, 2k+2: & \tau = (0,\ldots,0); \\
p & = 1, 2k+1: & \tau = (0,\ldots,0,1); \\
p & = 2, 2k: & \tau = (0,\ldots,0,1,1); \\
\vdots & & \\
p & = k, k+2: & \tau = (0,1,\ldots,1) \\
p & = k+1: & \tau = (1,\ldots,1) \oplus (-1,1,\ldots,1) \equiv \tau_+ \oplus \tau_-.
\end{array}$$

The bundles of p-forms and $(N-p)$-forms correspond to the same τ as a consequence of duality. Notice that for $p = k+1 = N/2$ the bundle is reducible.

From (54)-(55) we see that the only p-forms contained in the discrete series are for $p = k+1$, namely $\tau_+ \subset U_+^{\sigma n_0}|_K$, and $\tau_- \subset U_-^{\sigma n_0}|_K$, where $(\sigma, n_0) = (1,\ldots,1)$. This identifies in group theoretic terms the square-integrable harmonic k-forms on H^{2k} which may be found by a delicate spectral analysis of the Hodge-de Rham operator Δ (see [9, 5]). Thus for N even and $p = N/2$ (and only in that case), Δ has discrete spectrum. The discrete part of the Plancherel measure (i.e., $\frac{1}{c}P(l_0,\ldots,l_k)$ in eq. (50)) is essentially the formal degree of the discrete series (see ref. [28] vol.II p. 407). In our case $l_j = j + \frac{1}{2}$, and a simple calculation gives

$$P(\frac{1}{2}, \frac{3}{2}, \ldots, k+\frac{1}{2}) = \frac{(2k+2)!}{(k+1)!2^{2k+2}} \prod_{s=1}^{k} (2s)!$$

Concerning the principal series we obtain [using (52)] the following list of irreps $\sigma = (n_j)$ of $M = \mathrm{Spin}(N-1)$ such that $U^{\sigma\lambda}|_K \supset \tau$:

$$\begin{array}{lll}
p & = 0, 2k+2: & \sigma = (0,\ldots,0); \\
p & = 1, 2k+1: & \sigma = \left\{ \begin{array}{l} (0,\ldots,0), \\ (0,\ldots,0,1); \end{array} \right.
\end{array}$$

$$p \quad = 2, 2k: \qquad \sigma = \begin{cases} (0, \ldots, 0, 1), \\ (0, \ldots, 0, 1, 1); \end{cases}$$

$$\vdots$$

$$p \quad = k, k+2: \qquad \sigma = \begin{cases} (0, 1, \ldots, 1), \\ (1, 1, \ldots, 1); \end{cases}$$

$$p \quad = k+1: \qquad \sigma = (1, 1, \ldots, 1) \quad \text{for both } \tau_{\pm}.$$

For $p = 0, 2k+2$ and $p = k+1$ we have a "singlet", for the other values of p we get a "doublet". It is possible to show that for $p = 1, \ldots, k$ and for σ equal to the second member of each doublet, the vector valued functions on G given by $f_{\sigma\lambda}(g) = P_\tau U^{\sigma\lambda}(g^{-1})v$ $(v \in H_U)$ correspond to coexact p-forms. When σ equals the first member of the doublet we get exact p-forms instead. For $p = k+2, \ldots, 2k+1$, the role of the two members of a doublet is reversed, i.e., the first corresponds to coexact forms, and the second to exact ones.

Let $N = 2k+1$. The discussion proceeds as before. For $p = 0, 1, \ldots, k$, the irreps of Spin(N) defining p-forms are given by $f_j = 0$ $(j = 1, \ldots, |k-p|)$, and $f_j = 1$ $(j = |k-p|+1, \ldots, k)$. For $p = k+1, \ldots, 2k+1$, τ is the same as for $(N-p)$-forms. All bundles are now irreducible. Applying (56) for $p = 0, \ldots, k-1$ (and the corresponding $(N-p)$-forms), we get the same list of σ's we had before. For $p = k, k+1$ we obtain a "triplet", namely $\sigma = (\epsilon, 1, \ldots, 1)$, where $\epsilon = 0, \pm 1$.

Again for $p = 1, \ldots, k-1$ the second member of each "doublet" corresponds to coexact p-forms, while the first member of each doublet corresponds to exact p-forms. For $p = k+2, \ldots, 2k$ the role of the members of each doublet gets reversed. For $p = k$ the terms of the triplet with $\epsilon = \pm 1$ $(\epsilon = 0)$ correspond to coexact (exact) k-forms, while for $p = k+1$ it is the opposite.

The continuous part of the Plancherel measure may be written in unified form for any N, as follows. For $p = 0, 1, \ldots, [\frac{N-1}{2}]$, let σ_p be the irrep of Spin$(N-1)$ with highest weight

$$\sigma_p = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{p-\text{times}}). \tag{61}$$

Then the continuous part of the Plancherel measure for coexact p-forms is given by $d\mu(U^{\sigma_p\lambda}) = \mu_p^{CE}(\lambda)d\lambda$, with density

$$\mu_p^{CE}(\lambda) = \frac{2^{N-2}}{\pi} d_p |C_p^{CE}(\lambda)|^{-2},$$

where d_p is the dimension of σ_p [i.e., $d_p = \binom{N-1}{p}$ for $p \neq \frac{N-1}{2}$, and $d_p = \frac{1}{2}\binom{N-1}{p}$ for $p = \frac{N-1}{2}$], and

$$C_p^{CE}(\lambda) = \frac{2^{N-2}\Gamma(N/2)}{\sqrt{\pi}} \frac{(i\lambda+\rho-p)\,\Gamma(i\lambda)}{\Gamma(i\lambda+\rho+1)}. \tag{62}$$

For $p = 0$ we reobtain (59). For $p = 1$ (i.e., for divergence-free vector fields), (62) gives the same result as (60) for $s = 1$.

In all cases we verify the equality of the Plancherel measures for exact p-forms and for coexact $(p-1)$-forms, and of those for coexact p-forms and coexact $(N-p-1)$-forms:

$$\mu_p^E(\lambda) = \mu_{p-1}^{CE}(\lambda), \qquad \mu_p^{CE}(\lambda) = \mu_{N-p-1}^{CE}(\lambda).$$

[This follows also by duality.] In particular, for N even and $p = N/2$, exact p-forms and coexact p-forms have the same Plancherel measure.

The spherical functions for p-forms can be deduced from the results of ref. [5], where the eigenfunctions (spherical or not) of the Hodge-de Rham Laplacian have been calculated by working in geodesic polar coordinates. The spherical functions have also been calculated by Pedon [23]. We give here their expression for $p \neq (N-1)/2$.

First let $p \neq \frac{N}{2}, \frac{N-1}{2}$. Let τ_p be the irrep of $K = \mathrm{Spin}(N)$ with highest weight vector given by the right hand side of (61). Then $\tau_p|_M = \sigma_{p-1} \oplus \sigma_p$. This induces a direct sum decomposition $V_{\tau_p} = V_{\sigma_{p-1}} \oplus V_{\sigma_p}$. Define the τ_p-spherical functions

$$\Phi_p^{\sigma_p\lambda}(g) = P_{\tau_p} U^{\sigma_p\lambda}(g) P_{\tau_p},$$
$$\Phi_p^{\sigma_{p-1}\lambda}(g) = P_{\tau_p} U^{\sigma_{p-1}\lambda}(g) P_{\tau_p}.$$

As M centralizes A in K, we get from Schur's lemma

$$\Phi_p^{\sigma_p\lambda}(a_t) = \gamma_{\lambda p}(t)\mathbf{1}_{p-1} \oplus \delta_{\lambda p}(t)\mathbf{1}_p,$$
$$\Phi_p^{\sigma_{p-1}\lambda}(a_t) = \alpha_{\lambda p}(t)\mathbf{1}_{p-1} \oplus \beta_{\lambda p}(t)\mathbf{1}_p,$$

where $\mathbf{1}_{p-1}$ and $\mathbf{1}_p$ denote the identity operators in $V_{\sigma_{p-1}}$ and V_{σ_p}, respectively. Using the radial part of the Casimir operator one obtains a system of differential equations for the scalar functions $\gamma_{\lambda p}, \delta_{\lambda p}$ (or $\alpha_{\lambda p}, \beta_{\lambda p}$). The solution is as follows:

$$\gamma_{\lambda p}(t) = \beta_{\lambda p}(t) = F(i\lambda + \frac{N+1}{2}, -i\lambda + \frac{N+1}{2}, 1 + \frac{N}{2}, -\sinh^2 \frac{t}{2}),$$

$$\delta_{\lambda p}(t) = \alpha_{\lambda, N-p}(t) = \frac{1}{N-p} \sinh t \, \frac{d\gamma_{\lambda p}}{dt} + \cosh t \, \gamma_{\lambda p}(t),$$

where F is the hypergeometric function. Notice that $\gamma_{\lambda p}$ and $\beta_{\lambda p}$ are independent of p.

Now let $p = N/2$ (thus N is even). In this case the bundle of p-forms over $H^N(\mathbb{R})$ is reducible, namely we have $\tau = \tau_+ \oplus \tau_-$ and $E^\tau = E^{\tau_+} \oplus E^{\tau_-}$, where τ_+ and τ_- are the representations of $\mathrm{Spin}(N)$ with highest weights

$$\tau_\pm = (\pm 1, 1, 1, \dots, 1),$$

and dimensions

$$d_{\tau_+} = d_{\tau_-} = \frac{1}{2} \begin{pmatrix} N \\ N/2 \end{pmatrix}.$$

We have the branching rule

$$\tau_+|_M = \tau_-|_M = \sigma = (1, \dots, 1).$$

Consider the spherical functions

$$\Phi_+^\lambda(g) = P_{\tau_+} U^{\sigma\lambda}(g) P_{\tau_+},$$
$$\Phi_-^\lambda(g) = P_{\tau_-} U^{\sigma\lambda}(g) P_{\tau_-}.$$

Schur's Lemma gives

$$\Phi_\pm^\lambda(a_t) = f_{\lambda\pm}(t)\mathbf{1}_\pm,$$

where $f_{\lambda\pm}$ are scalar functions. Using the radial part of the Casimir operator one finds

$$f_{\lambda+}(t) = f_{\lambda-}(t) = \cosh^2(\frac{t}{2})F(i\lambda + \frac{N+1}{2}, -i\lambda + \frac{N+1}{2}, \frac{N}{2}, -\sinh^2\frac{t}{2}). \tag{63}$$

The spherical functions for the discrete series are $\Phi_+(g) = P_{\tau_+} U_+^{\sigma n_0}(g) P_{\tau_+}$ and $\Phi_-(g) = P_{\tau_-} U_-^{\sigma n_0}(g) P_{\tau_-}$, where $(\sigma, n_0) = (1, \ldots, 1)$. From Schur's Lemma $\Phi_\pm(a_t) = f_\pm(t)\mathbf{1}_\pm$. The scalar functions $f_\pm(t)$ can be obtained from $f_{\lambda\pm}(t)$ in (63) by analytic continuation in λ, by letting $\lambda = i/2$ (or $-i/2$).

Indeed let us recall that the Casimir operator of G induces minus the Hodge-de Rham Laplacian, $\Omega_G = -\Delta$. For general p, the eigenvalues of Δ acting on the coclosed p-forms $f_{\sigma_p\lambda}(g) = P_{\tau_p} U^{\sigma_p\lambda}(g^{-1})v$ $(v \in H_U)$ are given by $\omega_{\lambda p} = \lambda^2 + (\rho - p)^2$, where $\rho = (N-1)/2$ [5, 9]. Now for $p = N/2$, it is well known that the square-integrable p-forms corresponding to the discrete series are harmonic. [Moreover they are both closed and coclosed, in agreement with a general theorem of Andreotti and Vesentini, see [5].]

By the Subquotient Theorem, the discrete series $U_+^{\sigma n_0}$ is infinitesimally equivalent with a subquotient representation of a nonunitary principal series $U^{\sigma\lambda}$, for suitable $\lambda \in \mathbb{C}$. [A similar statement holds for $U_-^{\sigma n_0}$.] The condition $\omega_{\lambda, N/2} = 0$ gives $\lambda = \pm i/2$. Using this value in (63), we find

$$f_+(t) = f_-(t) = \cosh^2(t/2)F(\frac{N}{2}+1, \frac{N}{2}, \frac{N}{2}, -\sinh^2\frac{t}{2}) = (\cosh\frac{t}{2})^{-N}.$$

The case of $p = (N-1)/2$ (N odd) is slightly more complicated, and will not be given here.

References

[1] Badertscher, E., and H. M. Reimann, *Harmonic analysis for vector fields on hyperbolic spaces*, Math. Z. **202** (1989), 431–456.

[2] Barut, A. O., and R. Raczka, "Theory of Group Representations and Applications," World Scientific, 1986.

[3] Camporesi, R., *The spinor heat kernel in maximally symmetric spaces*, Commun. Math. Phys. **148** (1992), 283–308.

[4] Camporesi, R., and A. Higuchi, *Spectral functions and zeta functions in hyperbolic spaces*, Journal of Math. Phys. **35** (1994), 4217–4246.

[5] Camporesi, R., and A. Higuchi, *The Plancherel measure for p-forms in real hyperbolic spaces*, Journal of Geometry and Physics **15** (1994), 57–94.

[6] Camporesi, R., and A. Higuchi, *On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces*, Journal of Geometry and Physics **20** (1996), 1–19.

[7] Camporesi, R., *The Helgason Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces*, Pac. J. of Math. (1998), to appear.

[8] Deitmar, A., *Invariant operators on higher K-types*, J. reine angew. Math. **412** (1990), 97–107.

[9] Donnelly, H., *The differential form spectrum of hyperbolic space*, Manuscripta Math. **33** (1981), 365–385.

[10] Faraut, J., *Analyse harmonique sur le paires de Gelfand et les espaces hyperboliques*, in: "Analyse Harmonique," Nice, C.I.M.P.A. 1983, 315–437.

[11] Godement, R., *A theory of spherical functions*, Trans. Amer. Math. Soc. **73** (1952), 496–556.

[12] Helgason, S., "Groups and Geometric Analysis," Academic Press New York etc., 1984.

[13] Helgason, S., "Geometric Analysis on Symmetric Spaces," Amer. Math. Soc., 1994.

[14] Hirai, T., *The Plancherel formula for the Lorentz group of n-th order*, Proc. Japan Acad. **42** (1966), 323–326.

[15] Knapp, A., "Representation Theory of Semisimple Groups, an Overview based on Examples," Princeton University Press, 1986.

[16] Koornwinder, T. H., *A note on the multiplicity free reduction of certain orthogonal and unitary groups*, Indag. Math. **44** (1982), 215–218.

[17] Koornwinder, T. H., *Jacobi functions and analysis on noncompact semisimple Lie groups*, in "Special functions, group theoretical aspects and applications," Reidel 1984, 1–85.

[18] Lipsman, R., "Group Representations," Lecture Notes in Math. **388** Springer-Verlag Berlin etc., 1974.

[19] Mautner, F. I., *Induced representations*, Amer. J. Math. **74** (1952), 737–758.

[20] Minemura, K., *Invariant differential operators and spherical sections on a homogeneous vector bundle*, Tokyo J. Math **15** (1992), 231–245.

[21] Okamoto, K., *On induced representations*, Osaka J. Math. **4** (1967), 85–94.

[22] Okamoto, K., *The Plancherel formula for the universal covering group of De Sitter group*, Proc. Japan Acad. **41** (1965), 23–28.

[23] Pedon, E., Thesis, Université de Nancy, in preparation.

[24] Thieleker, E., *On the integrable and square-integrable representations of* Spin*(1,2m)*, Trans. Amer. Math. Soc. **230** (1977), 1–40.

[25] Thomas, E. G. F., *An infinitesimal characterization of Gelfand pairs*, Proceedings of the Conference on Harmonic Analysis and Probability in honour of Shizuo Kakutani, Yale University, 1982.

[26] Wallach, N. R., "Harmonic Analysis on Homogeneous Spaces," Marcel Dekker New York, 1973.

[27] Wallach, N. R., "Real Reductive Groups I, II," Academic Press New York etc., 1988, 1992.

[28] Warner, G., "Harmonic Analysis on Semi-Simple Lie Groups I, II," Springer Verlag, 1972.

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Received April 18, 1996