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Metric Rigidity of Crystallographic Groups

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Abstract. Consider a finite set of Euclidean motions and ask what kind of conditions are necessary for this set to generate a crystallographic group. We investigate a set of Euclidean motions together with a special concept motivated by real crystalline structures existing in nature, called an essential crystallographic set of isometries. An essential crystallographic set of isometries can be endowed with a crystallographic pseudogroup structure. Under certain well chosen conditions on the essential crystallographic set of isometries Γ we show that the elements in Γ define a crystallographic group G, and an embedding $\Phi\colon \Gamma \to G$ exists which is an almost isomorphism close to the identity map. The subset of Euclidean motions in Γ with small rotational parts defines the lattice in the group G. An essential crystallographic set of isometries therefore contains a very slightly deformed part of a crystallographic group. This can be interpreted as a sort of metric rigidity of crystallographic groups: if there is an essential crystallographic set of isometries which is metrically close to an inner part of a crystallographic group, then there exists a local homomorphism-preserving embedding in this crystallographic group.

1. Crystallographic Groups and (Almost) Flat Manifolds

Many substances in their solid phase are crystallised. They are either mono-crystals (rock crystal, sugar crystal), or have a micro-crystalline structure, i.e., they are made up of thousands of tiny mono-crystals (steel, lump of sugar). Crystalline structures are very regular. Most of the conceptual tools for the classification of crystalline structures, the theory of lattices and space groups, had been developed by the nineteenth century. In 1830 J. F. C. Hessel determined the 32 geometric classes of point groups in three-dimensional Euclidean space. In 1850 A. Bravais derived 14 types of three-dimensional lattices. C. Jordan in 1867 listed 174 types of groups of motions, including both crystallographic and non-discrete groups. The symmetry groups of crystalline structures in three-space were found independently by E. S. Fedorov in 1885 and A. Schoenflies in 1891. The determination of all crystalline structures in three-space enabled the modern definition of a crystallographic group to be formulated. *Every discrete group of motions of n-dimensional Euclidean space for which the closure of the fundamental*

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domain is compact is an n-dimensional crystallographic group.[1]

In 1900 at the International Congress of Mathematics in Paris, D. Hilbert made an attempt to identify the important areas in contemporary mathematical research, which are known today as the twenty-three Hilbert problems. Hilbert's eighteenth problem is on crystallographic groups and on fundamental domains: *"Is there in n-dimensional Euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region?"* Already in the nineteenth century it was known with ad hoc methods that there are only finitely many different crystallographic groups in the plane and in three-space. Eleven years later, in 1911, L. Bieberbach gave a complete answer to Hilbert's question in form of his structure theory for crystallographic groups, today known as the three Bieberbach Theorems.

Theorem 1.1. (Bieberbach, [2])

(1) *An n-dimensional crystallographic group contains n linearly independent translations and the rotational group is finite.*

(2) *Any isomorphism between two n-dimensional crystallographic groups can be realised by an affine change of coordinates.*

(3) *For a fixed dimension n there are only finitely many isomorphism classes of n-dimensional crystallographic groups.*

For modern proofs see L. S. Charlap, [6]. In the seventies M. Gromov studied the original proofs of Bieberbach, to make an attempt to understand what is really going on in the proof of Bieberbach's First Theorem. This idea was very fruitful and led to the following Almost Flat Manifold Theorem of M. Gromov and E. Ruh, which is one of the most striking results in Riemannian geometry.

Theorem 1.2. (Almost Flat Manifolds, [8], [12]) *Let M be a compact n-dimensional Riemannian manifold, K the sectional curvature and* $\mathrm{diam}(M)$ *the diameter. There exists a constant $\varepsilon(n) = \exp(-\exp(\exp(n^2)))$ depending only on the dimension such that*

$$|K| \cdot \mathrm{diam}(M)^2 \leq \varepsilon(n)$$

implies that M is diffeomorphic to an infra-nilmanifold, i.e. to N/Γ, where N is a simply connected nilpotent Lie group, and Γ a discrete subgroup of $N \rtimes \mathrm{Aut}(N)$ with finite $[\Gamma : N \cap \Gamma]$. (Such a manifold M is called an $\varepsilon(n)$-almost flat manifold).

The converse is also true. A complete proof of Gromov's Almost Flat Manifold Theorem can be found in [4]. It is the proof of this theorem and the lack of perfectness of crystalline structures in nature which motivate this paper.

2. Motivation for an Essential Crystallographic Set of Isometries and the Metric Rigidity of Crystallographic Groups

In this paper we give a new characterisation of crystallographic groups, cf. Def. 2.3 and Thm. 2.5. The main aim is to find conditions under which a finite set of isometries of the Euclidean space generates a crystallographic group. In the classical

[1] Some authors call torsion-free crystallographic groups Bieberbach groups. We do not follow this convention.

mathematical model used in crystallography, only ideal unlimited crystalline structures are treated. Let us instead consider a real macro-crystal appearing in nature which is finite and not perfectly regular. To do this, throw overboard the group structure, but there is still tremendous regularity in a crystal structure appearing in nature. So let us read off the crystal all possible isometries: the identity and all isometries which leave parts of the crystalline lattice almost invariant. Represent every isometry read off the crystal by an isometry of Euclidean space and gather them together in a set Γ. This set (which is not unique) has certain immediate properties and represents the almost lattice structure of the crystal. First observe that given any crystal-point, it is possible to read off an isometry, such that the translational part of the isometry is in some neighbourhood of the point. Secondly, the inverse of an isometry in Γ is not necessarily in Γ, but since the crystal is very regular, an element close to the inverse can be read off. Similarly, the composition of two isometries in Γ needs not to be an element of Γ, but a nearby element of Γ will be identified with the composition. Thirdly, the mono-crystals in the chosen macro-crystal are of a certain minimal size. Since they may not fit together perfectly be careful not to read off several times almost the same isometry. In fact these three observations characterise the entire crystalline structure. The second observation supplies us with generators and relations of a finitely generated group, but we cannot a priori expect to obtain a group of Euclidean motions. Under certain conditions the set Γ defines a crystallographic group.

In the following we transform the above ideas into a mathematical definition. But first let us recall some preliminaries about Euclidean motions and its norms. An Euclidean motion is an ordered pair $\alpha = (A, a)$ with $A \in O(n)$ an orthogonal matrix and $a \in \mathbb{R}^n$ acting on $\mathbb{R}^n$ by $\alpha(x) = A\,x + a$. We multiply Euclidean motions by composing them $\alpha\,\beta = (A\,B, A\,b + a)$. The inverse of α is given by $\alpha^{-1} = (A^{-1}, -A^{-1}\,a)$. The identity is denoted by $id = (I, 0)$, where I is the identity matrix in $O(n)$. The group of all Euclidean motions together with the above composition is the semi-direct product $E(n) = O(n) \ltimes \mathbb{R}^n$. We call $A = \operatorname{rot}(\alpha) \in O(n)$ the rotational part and $a = \operatorname{trans}(\alpha) \in \mathbb{R}^n$ the translational part of α. For $\alpha = (A, a)$ and $\beta = (B, b)$ in $E(n)$ define the commutator $[\alpha, \beta] = \alpha^{-1}\,\beta^{-1}\,\alpha\,\beta$, more explicitly:

$$\operatorname{rot}([\alpha,\beta]) = [A, B] = A^{-1}\,B^{-1}\,A\,B$$
$$\operatorname{trans}([\alpha,\beta]) = A^{-1}\,B^{-1}\,((I-B)\,a - (I-A)\,b)$$

Moreover let us define inductively the k-times nested commutator: set $\beta_0 = \beta$ and inductively $\beta_{k+1} = [\alpha, \beta_k]$ for $k \in \mathbb{N}$.

Definition 2.1. (Operator norm on $O(n)$) For $A \in O(n)$ define the norm of A as follows $\|A\| = \max\{|(A-I)\,x| \mid x \in \mathbb{R}^n \text{ with } |x| = 1\}$.

Let $A \in O(n)$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and $T \in U(n)$ a unitary matrix. Then we obtain $\|A\| = \|T^* A\,T\| = \max\{|\lambda_i - 1| \mid \text{ all eigenvalues } \lambda_i \text{ of } A\}$.

Definition 2.2. (Distance function on $E(n)$) Let $\alpha = (A, a) \in E(n)$. Define a norm of α by $\|\alpha\| = \max\{\|A\|, \nu|a|\}$, where ν is a positive adjustable length-parameter. A distance function on $E(n)$ is then derived by

$$d_{E(n)}(\alpha,\beta) = \|\alpha^{-1}\beta\| = \max\{\|A^{-1}B\|, \nu|a-b|\}.$$

A straight-forward calculation shows that $\|id\| = 0$ and $\|\alpha\| = \|\alpha^{-1}\|$ and in addition $|\,\|\alpha\| - \|\beta\|\,| \leq \|\alpha\,\beta\| \leq \|\alpha\| + \|\beta\|$. The distance function $d_{E(n)}$ is left-invariant but not right-invariant. There is an estimate of the deviation from right-invariance $d_{E(n)}(\alpha\gamma, \beta\gamma) \leq (1 + \nu \cdot |c|) \cdot d_{E(n)}(\alpha, \beta)$ in function of $|\mathrm{trans}(\gamma)| = |c|$.

Now we are ready to give a precise definition of an essential crystallographic set of isometries:

Definition 2.3. (Essential crystallographic set of isometries) Let ζ, r, R and ε, μ, δ be non-negative numbers with $\zeta \leq r \leq \frac{R}{2}$ and ν the adjustable length-parameter in the distance $d_{E(n)}$. An essential crystallographic set of isometries Γ_R is a finite set of Euclidean motions $\alpha = (A, a) \in E(n)$ with $|a| \leq R$ together with the following properties:

(I) For all $x \in \mathbb{R}^n$ with $|x| \leq R - r$ there exists $\alpha \in \Gamma_R$ with $|x - a| \leq r$. In other words, the translational parts of elements in Γ_R are r-dense in the disc $K_{R-r}(0)$ with radius $R - r$ around 0.

(II) (a) $id \in \Gamma_R$

(b) If $\alpha \in \Gamma_R$ satisfies $|a| \leq R - \zeta$ then there exists a so called bar-element $\overline{\alpha} \in \Gamma_R$ such that $d_{E(n)}(\overline{\alpha}, \alpha^{-1}) \leq \frac{r}{R}\varepsilon$.

(c) If $\alpha, \beta \in \Gamma_R$ satisfy $|a| + |b| \leq R - \zeta$ then there exists an element $\gamma = \gamma(\alpha, \beta) \in \Gamma_R$ such that $d_{E(n)}(\gamma, \alpha\,\beta) \leq \frac{r}{R}\varepsilon$.

(III) If $\alpha, \beta \in \Gamma_R$ with $\alpha \neq \beta$ satisfy $\|A^{-1}B\| \leq \mu$ then $|a - b| > \delta$.

The size of the macro-crystal is said to be R, the minimal side length of the mono-crystals is δ, and its lattice points are r-dense. The deviation of the real crystal to its ideal is measured by ε. Isometries close to the border of the crystal are not relevant – therefore the constant ζ is used. If $\varepsilon = 0$ then (II)(b) coincides with the inverse and (II)(c) with the composition of two elements.

All this is done in the hope that an essential crystallographic set of isometries somehow contains the information of a normal free Abelian subgroup of maximal rank with finite index. Let us see. We observe that if the constants involved in the above Def. 2.3 are well chosen then an essential crystallographic set of isometries can be equipped with a local product structure, the $*$-product, cf. Cor. 6.4.
Then in a **first part** (Sec. 3. – 11.) we only consider elements with small rotational parts, i.e. elements in

$$\Delta_\rho^{1/9} = \{\alpha \in \Gamma_R \mid \|\mathrm{rot}(\alpha)\| \leq \tfrac{1}{9} \text{ and } |\mathrm{trans}(\alpha)| \leq \rho \leq R\}.$$

We gain more information about the set $\Delta_\rho^{1/9}$ using similar techniques as in the proof of the Almost Flat Manifold Theorem, [4]. We show that the set $\Delta_\rho^{1/9}$ does not only contain the identity. This is done with a pigeon hole argument. If ρ is big enough, then all elements in $\Delta_\rho^{1/9}$ have a rotational part, which is smaller than $1/27$. Therefore the set $\Delta_\rho^{1/9}$ is closed under the $*$-product. Then a short basis for $\Delta_\rho^{1/9}$ is chosen, and this short basis has at most $d_n = 2^{n(n+1)}$ elements. The norm of nested commutators in $\Delta_\rho^{1/9}$ tends to zero. Thus since $\Delta_\rho^{1/9}$ is finite a norm-controlled induction implies that all d_n-times nested commutators in $\Delta_\rho^{1/9}$

are trivial.
Now with condition (III) and the proper relations between the constants ε and δ and respectively, ε and μ, it is possible to show that elements in $\Delta_\rho^{1/9}$ $*$-commute. The following generalised Frobenius' Theorem about nested commutators is essential to show that the set $\Delta_\rho^{1/9}$ consists of almost translations.

Theorem 2.4. ([14], Thm. 1.2) *Let $A, B \in O(n)$ with $\|A\|, \|B\| \leq \frac{1}{9}$ and $\varepsilon \in [0, 1/f_n^2]$. If $\|[A,[A,B]]\| \leq \varepsilon$ then $\|[A,B]\| \leq f_n \sqrt{\varepsilon}$. The constant $f_n = (3n)^3$ depends only on n.*

Then we construct a λ-normal basis for the almost translational set trans$(\Delta_\rho^{1/9})$, cf. Sec. 11. This procedure gives us n linearly independent vectors which generate an Abelian lattice of maximal rank n.
The **second part** (Sec. 12. – 18.) contains the construction of a nearby crystallographic group $G \subset E(n)$ and an embedding of $\Gamma_{\rho/2^{n+1}}$ into G. To do this we find a partition of $\Gamma_{\rho/3}$ into finitely many equivalence classes: two elements α and β in $\Gamma_{\rho/3}$ are said to be equivalent if $\alpha * \overline{\beta} \in \Delta_\rho^{1/9}$. The set $\mathcal{H} = \Gamma_{\rho/3} / \sim$ has the structure of a finite group, which can be considered as the rotational group of the crystallographic group G. Then we slightly deform the Abelian lattice to the lattice-group of the crystallographic group G.

If the relations between the constants given in the definition of an essential crystallographic set of isometries are well chosen then we obtain the following metric rigidity theorem, providing us with a new characterisation of crystallographic groups, i.e.,

(A_1) The radius is $R = 2\,c_{d_n}\,\rho$ with $\rho \geq \rho_n = r \cdot \exp(\exp(\exp(4n^2)))$, where $c_k = 5^k$ and $d_n = 2^{n(n+1)}$ depends only on the dimension n.

(A_2) The adjustable length-parameter in Def. 2.2 is set to be $\nu = \frac{1}{9\rho}$.

(A_3) For ζ suppose $r \geq \zeta \geq 9\,\varepsilon R$.

(A_4) For δ suppose $\frac{r}{2^n} \geq \delta \geq a_n\,\varepsilon^{\frac{1}{8}}\,R$, where $a_n = 7 f_n$ and $f_n = (3n)^3$ depend only on the dimension n.

(A_5) For μ suppose $\frac{1}{9} \geq \mu \geq b_n\,\varepsilon^{\frac{1}{2}}$, where $b_n = 2 f_n c_{d_n}$ and c_{d_n} is defined in (A_1) and f_n in (A_4).

(A_6) The constant ε controlling the error satisfies

$$0 \leq \varepsilon \leq \left(\frac{1}{2^{n+1} a_n c_{d_n}} \cdot \frac{r}{\rho}\right)^8 \leq \varepsilon_n = \exp(-\exp(\exp(7n^2))).$$

Notice that the constants a_n, b_n and d_n (respectively c_k) depend only on the dimension n, (respectively the natural number k). These different constants are used in the following proofs to make the various ideas work. They depend on the particular constructions used and therefore are usually not optimal.

Theorem 2.5. (Metric rigidity of crystallographic groups) *Let $\Gamma_R \in E(n)$ be an essential crystallographic set of isometries which satisfies $(A_1) - (A_6)$. Then there exists a crystallographic group $G \subset E(n)$ and an embedding $\Phi : \Gamma_{\rho/2^{n+1}} \to G$ which satisfies the following properties:*

(1) $\Phi(id) = id$ *and* $\Phi(\alpha * \beta) = \Phi(\alpha) \cdot \Phi(\beta)$ *for all* $\alpha, \beta \in \Gamma_{\rho/2^{n+1}}$ *such that* $|a| + |b| \leq \frac{\rho}{2^{n+1}}$.

(2) $d_{E(n)}(\gamma, \Phi(\gamma)) \leq \varepsilon^{\frac{1}{4}}$ *for all* $\gamma \in \Gamma_{\rho/2^{n+1}}$.

(3) $\Phi(\Gamma_{\rho/2^{n+1}}) \supseteq G \cap \{(A, a) \in E(n) \mid |a| \leq \frac{\rho}{2^{n+1}} - 9\,\rho\,\varepsilon^{\frac{1}{4}}\}$.

In other words the embedding $\Phi|_\Gamma$ is almost the identity, and the set $\Gamma_{\rho/2^{n+1}}$ is just a slightly deformed part of a crystallographic group. This can be interpreted as a metric rigidity of crystallographic groups: if there is an essential crystallographic set of isometries which is metrically close to an inner part of a crystallographic group, then there exists a local homomorphism-preserving embedding in this crystallographic group. This metric rigidity theorem should not be confused with other rigidity results about crystallographic groups such as the Second Bieberbach Theorem (cf. Thm. 1.1(2)), which could be called an algebraic or topological rigidity result. There is no obvious reason to believe that the two kinds of rigidity are related to one another.

Note that the case where ε is set to be zero is much easier: the $*$-product turns out to be the usual product in the group of Euclidean motions and property (II) gets easier and (III) can be skipped. Then the essential crystallographic set of isometries Γ_R with $\varepsilon = 0$ generates a crystallographic group $G \subset E(n)$ which contains Γ_R as a subset. The case $n = 3$ is handled in [15]. If in addition, R is set to infinity and "finite" is replaced by "discrete", then this very special case can also be found in [3].
Given a finite set Γ of isometries of an affine Euclidean space. When is the group G generated by Γ discrete? This question was also recently treated by H. Abels, [1]. The result is phrased as a series of tests: G is discrete if and only if Γ passes all the tests. His testing procedure is algorithmic.

3. Commutator Estimates

We state some useful facts concerning the commutator, which can be found by direct calculation:

Lemma 3.1. (Commutator estimates I, [10], p. 216) *Let $\alpha = (A, a)$ and $\beta = (B, b)$ be elements of $E(n)$. Then*

(1) $\|\mathrm{rot}([\alpha, \beta])\| \leq 2\,\|A\| \cdot \|B\|$

(2) $|\mathrm{trans}([\alpha, \beta])| \leq \|B\| \cdot |a| + \|A\| \cdot |b|$

(1') $\|\mathrm{rot}([\alpha, \ldots, [\alpha, \beta] \cdots]_k)\| \leq 2^k \|A\|^k \cdot \|B\|$

(2') $|\mathrm{trans}([\alpha, \ldots, [\alpha, \beta] \cdots]_k)| \leq (2^k - 1)\|A\|^{k-1} \cdot \|B\| \cdot |a| + \|A\|^k \cdot |b|$

Now the following commutator estimates can be derived from Lem. 3.1.

Lemma 3.2. (Commutator estimates II, [10], p. 216) *Let* $\alpha = (A, a)$ *and* $\beta = (B, b)$ *be elements of* $E(n)$. *Then* $\|A\| \le \|\alpha\|$ *and* $\nu |a| \le \|\alpha\|$ *by the definition of the norm on* $E(n)$. *Therefore*

(1) $\|[\alpha, \beta]\| \le 2\, \|\alpha\| \cdot \|\beta\|$ *and*

(2) $\|[\alpha, \ldots, [\alpha, \beta] \cdots]_k\| \le 2^k\, \|\alpha\|^k \cdot \|\beta\|$.

4. Pairwise Distance in $O(n)$ and $E(n)$

The following two lemmas are important tools in the proof of the Metric Rigidity Theorem.

Lemma 4.1. (Pairwise distance in $O(n)$, [4], Prop. 7.6.1) *For given* $\theta \in \,]0, \pi[$ *there are at most*

$$N(\theta) = 2 \left(\tfrac{2\pi}{\theta}\right)^{\frac{1}{2}n(n-1)}$$

elements A_i *in* $O(n)$ *with pairwise distance* $\|A_i^{-1} A_j\| \ge 2 \sin(\frac{\theta}{2})$.

Lemma 4.2. (Pairwise distance in $E(n)$, [4], Prop. 7.6.2) *For given* $\mu \in [0, 1[$ *there are at most*

$$N(\mu) = \left(\tfrac{3-\mu}{1-\mu}\right)^{\frac{1}{2}n(n+1)}$$

non-trivial Euclidean motions α_i *in* $E(n)$ *with rotational part in* $SO(n)$, *which pairwise satisfy* $d_{E(n)}(\alpha_i, \alpha_j) \ge \max\{\|\alpha_i\| - \mu\, \|\alpha_j\|\,,\, \|\alpha_j\| - \mu\, \|\alpha_i\|\}$.

The following result from differential geometry tells us that it is possible to construct an iteration procedure which leads from an almost homomorphism $\omega_0 : H \to M$ of compact Lie groups to a homomorphism $\omega : H \to M$ near ω_0.

Theorem 4.3. ([9], Thm. 4.3.) *Let* H *and* M *be compact Lie groups with bi-invariant metrics satisfying the following conditions: The volume of* H *is normalised to one. The bi-invariant metric on* M *is normalised such that for all* $X, Y \in T_e M$ *the commutator satisfies* $\|[X, Y]\| \le \|X\| \cdot \|Y\|$, *and the injectivity radius of the exponential map is at least* π.
Let $\omega_0 : H \to M$ *be a* q*-almost homomorphism, i.e., assume for all* $h_1, h_2 \in H$

$$d(\omega_0(h_1 \cdot h_2)\,,\, \omega_0(h_1) \cdot \omega_0(h_2)) \le q \le \tfrac{\pi}{6}.$$

Then there exists a homomorphism $\omega : H \to M$ *near* ω_0, *i.e. for all* $h \in H$

$$d(\omega_0(h), \omega(h)) \le q + \tfrac{1}{2} q^2 + q^4 \le 2q.$$

5. Generalised Frobenius' Theorem

The following fact is known as Frobenius' Theorem: *Let* $A, B \in O(n)$ *with* $\|B\| < \sqrt{2}$. *If* $[A, [A, B]] = I$ *then* $[A, B] = I$. Since A and $[A, B]$ commute we can assume, using a unitary change of basis if necessary, that A and $[A, B]$ are simultaneously diagonal. Set $C = [A, B]$ then $A\,B = B\,A\,C$ with A and C

diagonal. Compare the diagonal entries, then $a_{ii}b_{ii} = b_{ii}a_{ii}c_{ii}$ for all $i \in \{1, \dots, n\}$. We have $|a_{ii}| = 1$ since $A \in U(n)$ is diagonal, and $b_{ii} \neq 0$ since $\|B\| < \sqrt{2}$. Therefore $c_{ii} = 1$ for all $i \in \{1, \dots, n\}$. Hence $[A, B] = I$.
This fact can be extended to almost commuting matrices, cf. Thm. 2.4 in the introduction. The proof of this theorem follows the lines of the exact case. But it is not anymore possible to assume that A and $[A, B]$ are simultaneously diagonal. Therefore we construct a change of basis such that A and $[A, B]$ are simultaneously almost diagonal, cf. [14]:

Lemma 5.1. ([14], Lem. 3.3) *Let $A, C \in O(n)$. If $\|[A, C]\| \leq \varepsilon$ with $\varepsilon \in [0, \frac{1}{2n^2}]$ then there exists $V \in U(n)$ such that the matrices $V^* A V$ and $V^* C V$ are simultaneously almost diagonal, i.e. $|d|^2 \geq 1 - 9n^3\varepsilon$ for all diagonal entries $d \in \{(V^* A V)_{ii} \mid i \in \{1, \dots, n\}\} \cup \{(V^* C V)_{ii} \mid i \in \{1, \dots, n\}\}$.*
From Lem. 5.1 we can conclude a weaker real analogue. For the proof remember that the non-trivial (i.e. imaginary) eigenvalues of an orthogonal matrix appear as conjugate pairs. The set $\mathrm{Mat}(n \times m, \mathbb{R})$ denotes the $(n \times m)$-matrices with real entries.

Corollary 5.2. *Let $A, B \in O(n)$ with $\|A\|, \|B\| \leq \frac{1}{9}$. If $\|[A, B]\| \leq \varepsilon \leq \frac{1}{n^2}$ then there exists $V \in O(n)$ depending only on A such that*

$$V^t A V = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix},$$

where $A' \in O(2k)$ with $\|A'\| > \eta - n\sqrt{\varepsilon}$ and $A'' \in O(n-2k)$ with $\|A''\| \leq \eta + n\sqrt{\varepsilon}$. The positive number η is an adjustable parameter with $\eta > n\sqrt{\varepsilon}$ and $0 \leq 2k \leq n$. Also,

$$V^t B V = \begin{pmatrix} B' & F' \\ F'' & B'' \end{pmatrix},$$

where $B' \in \mathrm{Mat}(2k \times 2k, \mathbb{R})$ and $B'' \in \mathrm{Mat}((n-2k) \times (n-2k), \mathbb{R})$ and $|f'_{ij}|, |f''_{ij}| \leq n\sqrt{\varepsilon}$ for all possible i, j-combinations.

6. The Crystallographic Pseudogroup

The abstract definition of an essential crystallographic set of isometries will now become clearer. If we suppose a relatively weak condition on the constants in Def. 2.3, then every essential crystallographic set of isometries has a local group structure, i.e., is a crystallographic pseudogroup. But first let us say something about equality of two elements in Γ_R:

Lemma 6.1. *Let $\alpha = (A, a), \beta = (B, b) \in \Gamma_R$.*

(a) *If $\|A^{-1}B\| \leq \mu$ and $|a - b| \leq \delta$ then $\alpha = \beta$.*

(b) *If $d_{E(n)}(\alpha, \beta) \leq \min\{\mu, \nu\delta\}$ then $\alpha = \beta$.*

Proof. Suppose $\alpha \neq \beta$, then property (III) can be interpreted as follows: if $\|A^{-1}B\| \leq \mu$ then $|a - b| > \delta$, otherwise $\|A^{-1}B\| > \mu$, therefore $d_{E(n)}(\alpha, \beta) = \max\{\|A^{-1}B\|, \nu|a - b|\} > \min\{\mu, \nu\delta\}$. ■

Lemma 6.2. (Unique bar-element and unique element $\gamma = \gamma(\alpha, \beta)$) *If the constants in Def. 2.3 are supposed to satisfy* $\min\{\mu, \nu\delta\} > 2\frac{r}{R}\,\varepsilon$*, then the bar operation* $^{-} : \{\alpha \in \Gamma_R \mid |a| \leq R - \zeta\} \to \Gamma_R$ *and the multiplicative element* $\gamma : \{(\alpha, \beta) \in \Gamma_R \times \Gamma_R \mid |a| + |b| \leq R - \zeta\} \to \Gamma_R$ *are unique.*

Proof. *For* $\alpha \in \Gamma_R$ *satisfying* $|a| \leq R - \zeta$ *there is only one element* $\overline{\alpha} \in \Gamma_R$ *which satisfies* (II)(b)*:* Indeed, if there were two $\overline{\alpha}$ and $\overline{\alpha}'$, then

$$d_{E(n)}(\overline{\alpha}, \overline{\alpha}') \ \leq d_{E(n)}(\overline{\alpha}, \alpha^{-1}) + d_{E(n)}(\alpha^{-1}, \overline{\alpha}') \leq 2\tfrac{r}{R}\,\varepsilon,$$

so Lem. 6.1 implies $\overline{\alpha} = \overline{\alpha}'$. The proof for the uniqueness of γ is similar. ∎

With the same assumptions on the constants as in Lem. 6.2 we can summarise:

(IV) (Neutral element, inverse and product in Γ_R) By property (II)(a) $id \in \Gamma_R$. For all $\alpha \in \Gamma_R$ with $|a| \leq R - \zeta$ the $^{-}$-inversion $\alpha \mapsto \overline{\alpha}$ and for all $\alpha, \beta \in \Gamma_R$ with $|a| + |b| \leq R - \zeta$ the $*$-product $\alpha * \beta = \gamma(\alpha, \beta)$ are well defined in Γ_R.

If the assumptions on the constants in Def. 2.3 are sharpened a bit, then we can derive several properties of this product. Using the left-invariance and the deviation from the right-invariance of $d_{E(n)}$ we estimate the distance between from $\alpha * \overline{\alpha}$ to $\overline{\alpha} * \alpha$, for instance. Then Lem. 6.1 implies that both expressions must be equal:

Theorem 6.3. (Properties of the $*$-product) *Assume that* $\min\{\mu, \nu\delta\} > \frac{7}{2}\,\varepsilon$, $\nu\varepsilon\frac{r}{R} \leq \zeta$ *and the adjustable length-parameter* $\nu \in \left]0, \frac{1}{2r}\right]$*. Then the* $*$*-product satisfies.*

(V) *Well-defined inverse:* $\alpha * \overline{\alpha} = id = \overline{\alpha} * \alpha$ if $2\,|a| \leq R - 2\,\zeta \leq R - 2\,\nu\varepsilon\frac{r}{R}$.

(VI) *Idempotent inverse:* $\overline{\overline{\alpha}} = \alpha$ if $|a| \leq R - 2\,\zeta$.

(VII) *Antisymmetric inverse:* $\overline{\alpha * \beta} = \overline{\beta} * \overline{\alpha}$ if $|a| + |b| \leq R - 3\,\zeta$.

(X) *Associative multiplication:* $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ if $|a| + |b| + |c| \leq R - 2\,\zeta$.

Summarising the above we get the following $*$-structure for our essential crystallographic set of isometries.

Corollary 6.4. (Crystallographic Pseudogroup) *The finite set* Γ_R *together with the* $*$*-structure has the following properties:*

(a) *If* $\alpha \in \Gamma_R$ *satisfies* $|a| \leq R - \zeta$ *then* $\bar{\alpha} \in \Gamma_R$.

(b) *If* $\alpha, \beta \in \Gamma_R$ *satisfy* $|a| + |b| \leq R - \zeta$ *then* $\alpha * \beta \in \Gamma_R$.

Notice that the $^{-}$-inversion and $*$-product are not defined on the entire set Γ_R but only on a subset. Therefore we will speak of a local $*$-structure on Γ_R or of a crystallographic pseudogroup.[2]

[2] There are other definitions of a pseudogroup which are in general not equivalent. It is possible to assume that the inverse is defined everywhere, cf. [7], Def. 7.1. For an example see the pseudo fundamental group of an almost flat manifold in [3]. In the older literature we find the name *local group* for such a concept, cf. [11], Sec. 23. We will not follow this convention.

If a product of several factors is given without any brackets, then associativity is understood, so any bracket setting is the same. If a term with $^-$-inversions and $*$-products is given, then every such operation gives rise to an error of at most $(1+\nu R)\frac{r}{R}\,\varepsilon \le \varepsilon$ compared to usual inversion and product in $E(n)$, since $\nu \in \left]0, \frac{1}{2r}\right]$ and $2\,r \le R$ are supposed. Since we have a local group structure ready, let us abbreviate

$$\alpha^{*l} = \alpha * \cdots * \alpha \quad \text{ if } l\cdot|a| \le R - l\cdot\zeta$$

for the $*$-potency, and use the symbol

$$[\alpha\,;\beta] = \overline{\alpha} * \overline{\beta} * \alpha * \beta \quad \text{ if } 2\,|a| + 2\,|b| \le R - 5\,\zeta$$

for the $*$-commutator in Γ_R. The maximal number c_k of $^-$-inversions and $*$-products in a k-times nested $*$-commutator $[\alpha\,;\ldots;[\alpha\,;\beta]\cdots]_k$ does not exceed $c_k = 5^k$. From Lem. 3.1 and Lem. 3.2 we derive some inequalities which put the $^-$-inversion and $*$-product into relation with the usual inversion and product in the group of Euclidean motions:

Corollary 6.5. *Let $\alpha, \beta \in \Gamma_R$. Then the following inequalities are valid:*

(a) inversion: $\|\alpha^{-1}\| - \varepsilon \le \|\overline{\alpha}\| \le \|\alpha^{-1}\| + \varepsilon$

(b) product: $\|\alpha\,\beta\| - \varepsilon \le \|\alpha * \beta\| \le \|\alpha\,\beta\| + \varepsilon$

(c) commutator: $\|[\alpha\,;\beta]\| \lesseqgtr \|[\alpha\,,\beta]\| \pm 5\,\varepsilon$

(d) nested commutator: $\|[\alpha\,;\ldots;[\alpha\,;\beta]\cdots]_k\| \lesseqgtr \|[\alpha\,,\ldots,[\alpha\,,\beta]\cdots]_k\| \pm c_k\,\varepsilon$

7. Examples of Essential Crystallographic Sets of Isometries

The most obvious example of an essential crystallographic set of isometries would be a crystallographic group itself. The only problem with this example is that for technical reasons, Def. 2.3 requires essential crystallographic sets of isometries to be finite.

Below, we discuss some other examples of essential crystallographic sets of isometries. We begin with examples where $\varepsilon = 0$.

Example 7.1. To get a finite essential crystallographic set of isometries Γ_R, take the subset

$$\Gamma_R = \{\alpha = (A, a) \in \Pi \mid \operatorname{trans}(\alpha) \le R\}$$

of any crystallographic group $\Pi \subset E(n)$ and adjust the constants ζ, r, R and μ, δ, ν such that Γ_R turns into an essential crystallographic set of isometries as follows. Let D be a fundamental domain of the crystallographic group Π and let δ be the minimal distance between two vertices of D. Set $\zeta = \varepsilon = 0$ and $r = \frac{1}{2}\operatorname{diam}(D)$, $R = 10\operatorname{diam}(D)$, where $\operatorname{diam}(D)$ denotes the diameter of D. From [5] we know that if $\alpha = (A, a) \in \Pi$ with $\|A\| \le \frac{1}{2}$, then α is a pure translation. We can therefore set $\mu = \frac{1}{2}$ and leave ν arbitrary.

Example 7.2. Let $(\mathbb{Z}^n, +, 0)$ be the standard lattice. Consider the set

$$\Gamma_{10} = \{(I, a) \in E(n) \mid a \in \mathbb{Z}^n \text{ with } |a| \le 10\}$$

and set $r = \delta = 1$ and $\zeta = \varepsilon = 0$. For any non-negative values of μ and ν, if the dimension n is $1, 2, 3$ or 4 then Γ_{10} is an essential crystallographic set of isometries. For $n \ge 5$, property (I) in Def. 2.3 fails, since half of the diameter of the fundamental domain of $\mathbb{Z}^n$ is bigger than $r = 1$.

Example 7.3. To make Ex. 7.2 work in all dimensions, we can consider the set

$$\Gamma'_{10} = \left\{(I, a) \in E(n) \mid a \in \left(\tfrac{1}{\sqrt{n}}\mathbb{Z}\right)^n \text{ with } |a| \le 10\right\}.$$

Setting the constants $r = 1$, $\delta = \frac{1}{\sqrt{n}}$ and $\zeta = \varepsilon = 0$ and leaving μ and ν arbitrary, Γ'_{10} becomes an essential crystallographic set of isometries for all $n \in \mathbb{N}$.

Now we give two examples with $\varepsilon \ge 0$.

Example 7.4. Let Γ'_{10}, r, δ and ν be as in Ex. 7.3. Let $\alpha \in \Gamma'_{10}$ and $\tilde{\alpha}$ be any element of $E(n)$ such that $d_{E(n)}(\alpha, \tilde{\alpha}) < \frac{\delta}{100}$. Denote by $\tilde{\Gamma}'_{10}$ the set obtained by replacing each element $\alpha \in \Gamma'_{10} - \{id\}$ by $\tilde{\alpha}$. By slightly adjusting the constants ζ and μ chosen in Ex. 7.3, we obtain an essential crystallographic set of isometries $\tilde{\Gamma}'_{10}$ with $\varepsilon \ge 0$. It is a slightly deformed part of the crystallographic group $\left(\frac{1}{\sqrt{n}}\mathbb{Z}\right)^n$.

Example 7.5. Let $\{e_1, e_2\}$ be the standard basis for the Abelian lattice $\mathbb{Z}^2$. Further define two Euclidean motions $\alpha = (A, e_1)$ and $\beta = (B, e_2)$ of the plane with A and B non-trivial rotations. Suppose without loss of generality that $\|\alpha\| \ge \|\beta\|$. In general the Euclidean motions α and β do not commute, but if the rotations A and B are small they almost commute. Therefore define the set

$$\Gamma_R = \{\alpha^k\beta^l \mid k, l \in \mathbb{Z} \text{ with } |k| + |l| \le R\}.$$

If the density constant is $r = 1$, and if $\mu, \nu\delta$ and ζ are chosen to be much smaller than $r = 1$ and if they satisfy $\min\{\mu, \nu\delta\} \ge 2\,R^2\,\|\alpha\|^2$ and $\zeta \ge 2\,R^2\,\|\alpha\|^2$ then Γ_R is an essential crystallographic set of isometries with positive ε. Indeed, consider the case

$$d_{E(n)}(\beta^l\alpha^k, \alpha^k\beta^l) \le 2\,\|\alpha^k\| \cdot \|\beta^l\| \le 2\,|k|\,\|\alpha\| \cdot |l|\,\|\beta\| \le 2\,R^2\|\alpha\|^2 \le \min\{\mu, \nu\delta\},$$

so $\alpha^k\beta^l$ and $\beta^l\alpha^k$ are close since we assumed that $\min\{\mu, \nu\delta\}$ is much smaller than $r = 1$. Therefore they can be considered as one element. In other words, we have the following $*$-structure on Γ_R: if $|k| + |l| \le R - \zeta$ then $\overline{\alpha^k\beta^l} = \alpha^{-k}\beta^{-l}$ and if $|k + k'| + |l + l'| \le R - \zeta$ then $\alpha^k\beta^l * \alpha^{k'}\beta^{l'} = \alpha^{k+k'}\beta^{l+l'}$. Therefore Γ_R turns out to be an Abelian essential crystallographic set of isometries. The maximal deviation of the translational parts in Γ_R from the ideal lattice $\mathbb{Z}^2$ generated by $\text{trans}(\alpha) = e_1$ and $\text{trans}(\beta) = e_2$ is therefore at most

$$\text{dist}(ke_1 + le_2, \text{trans}(\alpha^k\beta^l)) \le |k| \cdot \|A\| + |l| \cdot \|B\| \le R\,\|\alpha\|,$$

where the notation $\mathrm{dist}(ke_1 + le_2, \mathrm{trans}(\alpha^k\beta^l))$ denotes the distance from the point $\mathrm{trans}(\alpha^k\beta^l)$ to the lattice point $ke_1 + le_2$. If every element $\alpha^k\beta^l$ in Γ_R is replaced by the element $(I, ke_1 + le_2)$ in the Abelian crystallographic group $G = (\{I\} \ltimes \mathbb{Z}^2, \cdot, id)$ then this embedding is almost the identity, i.e., the maximal deviation in $E(2)$ is smaller than the square-root of $\min\{\mu, \nu\delta\}$ which was chosen much smaller than $r = 1$.

8. Nilpotency of the Set $\Delta_\rho^{1/9}$ in Γ_R

First we introduce some general definitions and concepts.

Definition 8.1. (The set of motions with small rotational parts) For $\lambda \in [0, 2]$ and $0 \le \xi \le R$ define the set

$$\Delta_\xi^\lambda = \{\alpha \in \Gamma_R \mid \|\mathrm{rot}(\alpha)\| \le \lambda \text{ and } |\mathrm{trans}(\alpha)| \le \xi\}.$$

In what follows it is elegant if the largest rotational part and translational part allowed in the set Δ_ξ^λ have the same weight. Therefore, the adjustable parameter ν in the definition of the distance function on $E(n)$ is set to be $\nu = \frac{\lambda}{\xi}$. In other words $\Delta_\xi^\lambda = \{\alpha \in \Gamma_R \mid \|\alpha\| \le \lambda\}$.

In the Bieberbach case the set $\Delta_\rho^{1/9}$ would be the set of Euclidean motions with trivial rotational parts, i.e. the Abelian lattice group. In the almost flat case the set $\Delta_\rho^{1/9}$ is shown to be an almost translational set whose elements $*$-commute.

Definition 8.2. (Norm-controlled generation) For any subset $A \subseteq \Gamma_R$ and $\lambda \in [0, 2]$ and $0 \le \xi \le R$ define the set $\langle A\rangle_\xi^\lambda$ inductively by

(I) $\{id\} \cup A \subseteq \langle A\rangle_\xi^\lambda$

(II) If $\alpha \in A$ and $\|\mathrm{rot}(\overline{\alpha})\| \le \lambda$ and $|\mathrm{trans}(\overline{\alpha})| \le \xi$ then $\overline{\alpha} \in \langle A\rangle_\xi^\lambda$.

(III) If $\alpha, \beta \in \langle A\rangle_\xi^\lambda$ and $\|\mathrm{rot}(\alpha*\beta)\| \le \lambda$ and $|\mathrm{trans}(\alpha*\beta)| \le \xi$ then $\alpha*\beta \in \langle A\rangle_\xi^\lambda$.

Note that we have to be careful with associativity, e.g. $\alpha * (\beta * \gamma) \in \langle A\rangle_\xi^\lambda$ does in general not imply $\alpha * \beta \in \langle A\rangle_\xi^\lambda$. In what follows a short basis $\{\gamma_0, \dots, \gamma_m\}$ for Δ_ξ^λ is defined in such a way that it reflects nilpotent properties of Δ_ξ^λ, as we will see later, cf. Cor. 8.6.

Definition 8.3. (Short basis) Let $\lambda \in [0, 2]$. The elements of a short basis $\{\gamma_0, \dots, \gamma_m\}$ of Δ_ξ^λ are inductively selected:

(I) $\gamma_0 = id$

(II) $\gamma_1 \in \Delta_\xi^\lambda - \{id\}$ is such that $\|\gamma_1\|$ is minimal in $\Delta_\xi^\lambda - \{id\}$.

(III) If $\{\gamma_0, \dots, \gamma_i\} \subset \Delta_\xi^\lambda$ have been selected, then $\gamma_{i+1} \in \Delta_\xi^\lambda - \langle\{\gamma_0, \dots, \gamma_i\}\rangle_\xi^\lambda$ is chosen such that $\|\gamma_{i+1}\|$ is minimal in $\Delta_\xi^\lambda - \langle\{\gamma_0, \dots, \gamma_i\}\rangle_\xi^\lambda$.

We start with a couple of lemmas, which teach us first something about the set $\Delta_\rho^{1/9}$ in Γ_R. The setting $\lambda = \frac{1}{9}$ and $\xi = \rho$ is such that Lem. 8.5 can be proved. Furthermore we need associativity in Γ_R for at most c_{d_n} factors of elements in Γ_ρ. Thus set $R = 2c_{d_n}\rho$ and assume

$$6\varepsilon \leq 2c_{d_n}\varepsilon \leq \min\{\mu, \tfrac{\delta}{9\rho}\} < \min\{\|\alpha\| \mid \alpha \in \Gamma_R - \{id\}\}, \tag{1}$$

which is certainly satisfied by assumptions $(A_1) - (A_6)$. This implies that all $*$-products of at most $2c_{d_n}$ factors $\alpha_j \in \Gamma_\rho$ with $\sum_{j=1}^{2c_{d_n}} |a_j| \leq R - \zeta$ are well defined and any setting of parenthesis is the same. Therefore a very rough estimation for the constant ζ, which is tight enough for our further considerations can be found: in calculations we need $\zeta \leq 9\rho \cdot 2c_{d_n}\varepsilon = 9\varepsilon R$, thus by assumption (A_3) the constant ζ is considered to be smaller than r.
A priori it is not clear how many elements apart from the identity are contained in the finite set $\Delta_\rho^{1/9}$. Is it possible that the identity is the only element in Γ_ρ with rotational part smaller than $\frac{1}{9}$? – No, the next lemma tells us more:

Lemma 8.4. ([3], p. 85) *Let η be an adjustable parameter in $]2\varepsilon, 2]$. If*

$$\rho = \rho_{N(\eta/2)} = 2r\left(\tfrac{2}{\eta}\right)^{N(\eta/2)+1} \qquad \textit{with} \qquad N(\theta) = 2\left(\tfrac{2\pi}{\theta}\right)^{\frac{1}{2}n(n-1)},$$

then for all $x \in \mathbb{R}^n$ with $|x| \leq \frac{\rho}{2}$ there is $\alpha \in \Gamma_\rho \subset \Gamma_R$ with $|a - x| \leq \eta\rho$ and $\|A\| \leq \eta$.
Now it is already clear that the set $\Delta_\rho^{1/9}$ is not trivial. Let us see what else can be discovered about its elements. There is a crucial fact in P. Buser's new proof of the First Bieberbach Theorem, [5]: elements with small rotational parts are indeed pure translations. This cannot be true for all elements in $\Delta_\rho^{1/9}$, but it is still true that there is a certain gap: as shown in the following lemma, there are in fact no elements with rotational part between $\frac{1}{27}$ and $\frac{1}{9}$.

Lemma 8.5. *If $\alpha \in \Delta_\rho^{1/9}$ then $\|\mathrm{rot}(\alpha)\| \leq \frac{1}{27}$.*

Proof. The proof proceeds in three steps. Since Γ_R is finite, the set $\Delta_\rho^{1/9}$ is finite too. Set $\nu = \frac{1}{9\rho}$ then write $\Delta_\rho^{1/9} = \{\alpha \in \Gamma_R \mid \|\alpha\| \leq \frac{1}{9}\}$. Now chose a short basis $\{\gamma_0, \ldots, \gamma_d\}$ for $\Delta_\rho^{1/9}$ and define $G_i = \langle\{\gamma_0, \ldots, \gamma_i\}\rangle_\rho^{1/9}$ for all $i \in \{0, \ldots, d\}$. Thus we obtain a finite ascending chain $\{id\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_d = \Delta_\rho^{1/9}$.

(a) *If $\alpha \in \Delta_\rho^{1/9}$ and $\beta \in G_i$ then $[\alpha\,;\beta] \in G_{i-1}$ for all $i \in \{1, \ldots, d\}$.*

Indeed, fix $i \in \{1, \ldots, d\}$ and use induction:

For γ_i in the short basis we obtain, using Lem. 3.2 and Cor. 6.5 (c),

$$\|[\alpha\,;\gamma_i]\| \leq 2\,\|\alpha\| \cdot \|\gamma_i\| + 5\varepsilon \leq \tfrac{2}{9}\,\|\gamma_i\| + 5\varepsilon < \|\gamma_i\|.$$

Since γ_i is minimal in $\Delta_\rho^{1/9} - G_{i-1}$, we have $[\alpha\,;\gamma_i] \in G_{i-1}$. For γ_i in the short basis and if $\overline{\gamma}_i \in G_i$ then the same argument as above is valid for $\overline{\gamma}_i$.

If (a) holds for $\beta, \beta' \in G_i$ and arbitrary $\alpha \in \Delta_\rho^{1/9}$ then the identity

$$[\alpha\,;\beta * \beta'] = [\alpha\,;\beta'] * [\alpha\,;\beta] * [[\alpha\,;\beta]\,;\beta'] \tag{2}$$

is still valid since associativity holds for much more than 18 factors in Γ_ρ. By assumption each factor of (2) belongs to G_{i-1} and has norm smaller than $\frac{2}{9\cdot 9}+5\varepsilon$, thus $\|[\alpha\,;\beta*\beta']\| \le \frac{1}{9}$ by (2). By Def. 8.2 we obtain $[\alpha\,;\beta*\beta'] \in G_{i-1}$. Since $i \in \{1,\dots,d\}$ was arbitrary the claim follows for all $i \in \{1,\dots,d\}$.

(b) *If $k \le 2^{n(n+1)}$ then all $\alpha \in G_k$ satisfy $\|\mathrm{rot}(\alpha)\| \le \frac{1}{27}$.*

Suppose not, i.e. $\|\mathrm{rot}(\alpha)\| = \|A\| > \frac{1}{27}$ for some $\alpha \in G_k$ with $k \le 2^{n(n+1)}$. Let $E_A \subset \mathbb{R}^n$ be the plane of maximal rotation of A, i.e. $|(A-I)\,x| = \|A\|\cdot|x|$ for all $x \in E_A$. Take $x_0 \in E_A$ with $|x_0| = \frac{\rho}{2}$ and $\beta_0 = (B_0, b_0) \in \Delta_\rho^\eta \subseteq \Delta_\rho^{1/9}$ with $|x_0 - b_0| \le \eta\rho$ and $\|B_0\| \le \eta$, this is indeed possible by Lem. 8.4. Define inductively for $i \in \{1,\dots,k\}$ the points $x_i = (I - A^{-1})\,x_{i-1} \in E_A$, then

$$\begin{aligned} |x_k| &= |(I - A^{-1})^k x_0| = \|A\|^k \cdot |x_0| \quad \text{since } x_i \in E_A \text{ for all } i \in \{0,\dots,k\} \\ &> \tfrac{1}{2}\left(\tfrac{1}{27}\right)^k \rho > 0 \end{aligned}$$

by our assumption. Moreover

$$\|\mathrm{rot}([\alpha,\dots,[\alpha,\beta_0]\cdots]_k)\| = \|B_k\| \le 2^k \|A\|^k\, \|B_0\| \le \eta$$

and Lem. 3.1 (2') implies $|b_k| \le \rho$, thus

$$\begin{aligned} |x_k - b_k| &\le \|A\| \cdot |x_{k-1} - b_{k-1}| + (1 + \|A\|)\,\rho\,\eta \\ &\le \|A\|^k \cdot |x_0 - b_0| + (1 + \|A\|)\,\rho\,\eta \sum_{r=0}^{k-1} \|A\|^r \le 2\rho\,\eta. \end{aligned}$$

So on one hand $|b_k| \ge |x_k| - 2\rho\,\eta \ge \frac{1}{2}\left(\frac{1}{27}\right)^k \rho - 2\rho\,\eta$ and on the other hand, (a) implies $[\alpha\,;\dots;[\alpha\,;\beta_0]\cdots]_i \in G_{k-i}$ for all $i \in \{0,\dots,k\}$, thus $[\alpha\,;\dots;[\alpha\,;\beta_0]\cdots]_k = id$. So Cor. 6.5 implies that $|b_k| \le 9\,\rho\, c_k\varepsilon$. Together $9c_k\varepsilon \ge \frac{1}{2}\left(\frac{1}{27}\right)^k - 2\eta$. Since η in Lem. 8.4 can be chosen to be arbitrarily small without contradicting assumption (A_1), we obtain a contradiction if $\eta < \frac{1}{4}\left(\frac{1}{27}\right)^k - \frac{9}{2}c_k\varepsilon$, so $\|\mathrm{rot}(\alpha)\| \le \frac{1}{27}$ for all $\alpha \in G_k$.

(c) *There exists a number $d \le 2^{n(n+1)}$ such that $G_d = \Delta_\rho^{1/9}$.*

The crucial point in the proof is that d, the number of short basis elements, has an upper bound, which depends only on the dimension n. Observe that by construction of a short basis, $\|\gamma_i\| \le \|\gamma_j\|$ for $i \le j$. Also if $i < j$, then

$$\|\gamma_j * \overline{\gamma}_i\| \ge \|\gamma_j\|, \tag{3}$$

since otherwise $\|\gamma_j * \overline{\gamma}_i\| < \|\gamma_j\|$ implies $\gamma_j * \overline{\gamma}_i \in \langle\{\gamma_0,\dots,\gamma_{j-1}\}\rangle_\rho^{1/9}$, hence

$$(\gamma_j * \overline{\gamma}_i) * \gamma_i = \gamma_j * (\overline{\gamma}_i * \gamma_i) = \gamma_j \in \langle\{\gamma_0,\dots,\gamma_{j-1}\}\rangle_\rho^{1/9},$$

which contradicts the short basis construction.

Cor. 6.5 and assumption (1), i.e. $\|\gamma_i\| \ge 6\,\varepsilon$ for all $i \in \{1,\dots,d\}$, and (3) with $i \ne j$ imply

$$\|\gamma_j \cdot \gamma_i^{-1}\| \ge \max\left\{\|\gamma_i\| - \tfrac{1}{3}\,\|\gamma_j\|, \|\gamma_j\| - \tfrac{1}{3}\,\|\gamma_i\|\right\}.$$

Lem. 4.2 with $\mu = \frac{1}{3}$ gives an upper bound $d \le 2^{n(n+1)}$. Therefore (c) is true.

Taking (c) and (b) together Lem. 8.5 is proven. ∎

From Lem. 8.5 we derive two facts which are important in what follows. The set $\Delta_\rho^{1/9}$ is closed under multiplication if the length of the translational part is controlled, and it has nilpotent properties:

Corollary 8.6. (Closedness and nilpotency of $\Delta_\rho^{1/9}$)

(a) *The set $\Delta_\rho^{1/9}$ is closed under the $*$-product, i.e. for all $\alpha, \beta \in \Delta_\rho^{1/9}$ with $|\mathrm{trans}(\alpha * \beta)| \leq \rho$ also $\alpha * \beta \in \Delta_\rho^{1/9}$.*

(b) *The set $\Delta_\rho^{1/9}$ is d_n-step nilpotent with $d_n = 2^{n(n+1)}$, i.e., all d_n-times nested commutators of elements in $\Delta_\rho^{1/9}$ are trivial.*

The above proof of Lem. 8.5 needs the lower bound $R = 2c_{d_n}\rho$ for the radius of Γ_R. By Lem. 8.4 we obtain $\rho_{N(\eta/2)} = 2r\,(2/\eta)^{N(\eta/2)+1}$. The proof of Lem. 8.5 needs in (b) that the constant η satisfies

$$\eta \leq \tfrac{1}{8}\left(\tfrac{1}{27}\right)^{d_n} \leq \tfrac{1}{4}\left(\tfrac{1}{27}\right)^{d_n} - \tfrac{9}{2}c_{d_n}\varepsilon.$$

In assumption (A_6) we supposed $0 \leq \varepsilon \leq \varepsilon_n$ small enough such that $9\,c_{d_n}\varepsilon \leq \frac{1}{4}\left(\frac{1}{27}\right)^{d_n}$ is valid. By Lem. 4.1 the maximal number $N(\theta)$ of elements $A_j \in O(n)$ with pairwise distance bigger than $\theta = \eta/2$ is $N(\eta/2) \leq \cdots \leq \exp(\exp(3n^2))$. Therefore a lower bound ρ_n for the radius $R = 2c_{d_n}\rho$ is immediately derived:

$$\rho_{N(\eta/2)} \leq 2r\left(8 \cdot (27)^{d_n}\right)^{N(\eta/2)+1} \leq \cdots \leq r \cdot \exp(\exp(\exp(4n^2))) = \rho_n.$$

These estimations with exponential functions are very clumsy and do not represent exact values for N and ρ_n but give a vague idea of the enormous size of the constants.

Now we have enough preliminaries together to prove a first important fact, which has its analogue in the Bieberbach case. For a better understanding of the following proof the reader is recommended to set $\varepsilon = 0$ in a first reading and then to go through the proof once more with $\varepsilon \geq 0$ in mind. We will also see where Def. 2.3 (III) and assumptions (A_4) and (A_5) enter the proof.

Lemma 8.7. *Let $\alpha, \beta \in \Delta_\rho^{1/9}$. Then $\alpha * \beta = \beta * \alpha$.*

Proof. The set $\Delta_\rho^{1/9}$ is d_n-step nilpotent with $d_n = 2^{n(n+2)}$, cf. Cor. 8.6 (b): $[\alpha\,; \ldots; [\alpha\,; \beta] \cdots]_{d_n} = id$ for all $\alpha, \beta \in \Delta_\rho^{1/9}$. Thus Cor. 6.5 tells us that $\|\beta_{d_n}\| = \|[\alpha, \ldots, [\alpha, \beta] \cdots]_{d_n}\| \leq c_{d_n}\,\varepsilon$, and this signifies for the rotational and translational part of $\beta_{d_n} = (B_{d_n}, b_{d_n})$:

$$\|B_{d_n}\| = \|[A, \ldots, [A, B] \cdots]_{d_n}\| \leq c_{d_n}\,\varepsilon \tag{4}$$

$$|b_{d_n}| = |\mathrm{trans}([\alpha, \ldots, [\alpha, \beta] \cdots]_{d_n})| \leq 9\,\rho \cdot c_{d_n}\,\varepsilon \tag{5}$$

The proof uses two inductive arguments: one for the rotational part and the other for the translational part of a k-times nested commutator. This induction results in $[\alpha\,; [\alpha\,; \beta]] = id$, then another argument is used to obtain $[\alpha\,; \beta] = id$. (The constants ν_i with $i \in \{1, \ldots, 6\}$ serve to abbreviate complicated expressions containing ε. As ε tends to zero so do the ν_i.)

(a) Fix $k \in \{3, \ldots, d_n\}$ and suppose that $[\alpha\,; \ldots ; [\alpha\,; \beta] \cdots]_k = id$. Consider the rotational part of $[\alpha, \ldots, [\alpha, \beta] \cdots]_k$, then apply Thm. 2.4 twice to get

$$\|B_{k-1}\| \le f_n c_{d_n}^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} = \nu_1 \quad \text{and} \quad \|B_{k-2}\| \le f_n^{\frac{3}{2}} c_{d_n}^{\frac{1}{4}} \varepsilon^{\frac{1}{4}} = \nu_2,$$

where $f_n = (3n)^3$. At this step we do not proceed further inductively; like this it is possible to get a better dependence in assumptions (A_4) and (A_5).

For the translational part, see Lem. 3.1 (2'), we have $|b_k| \le \rho$, which is valid for all $k \in \mathbb{N}$. We need the above estimate in the next inequality, which is valid for $k = 2$ too,

$$|b_k| \le |(I - A)\, b_{k-1}| + \|B_{k-1}\| \cdot |a| \le |(I - A)\, b_{k-1}| + \nu_1\, \rho. \tag{6}$$

On the other hand

$$\begin{aligned} |b_k| &\ge |(I - A)\, B_{k-2}^{-1}(I - A)\, b_{k-2}| - \|B_{k-1}\| \cdot |a| - \|A\| \cdot \|B_{k-2}\| \cdot |a| \\ &\ge |(I - A)\, B_{k-2}^{-1}(I - A)\, b_{k-2}| - (\nu_1 + \tfrac{1}{9}\nu_2)\rho. \end{aligned} \tag{7}$$

Now we need

$$\begin{aligned} |((I - A)^2 - (I - A)\, B_{k-2}^{-1}(I - A))b_{k-2}| &\le \|A\|^2 \cdot \|B_{k-2}\| \cdot |b_{k-2}| \\ &\le \tfrac{1}{9^2}\nu_2\rho, \end{aligned} \tag{8}$$

so inequalities (7) and (8) imply

$$|b_k| \ge |(I - A)^2 b_{k-2}| - (\nu_1 + \tfrac{1}{9}\nu_2 + \tfrac{1}{81}\nu_2)\rho = |(I - A)^2 b_{k-2}| - \nu_3\rho. \tag{9}$$

Now all the preliminaries are ready: by Cor. 6.5 and assumption (A_6) it follows that $\|\mathrm{rot}([\alpha\,; \ldots ; [\alpha\,; \beta] \cdots]_{k-1})\| \le \|B_{k-1}\| + c_{k-1}\varepsilon \le 2\, f_n c_{d_n} \varepsilon^{\frac{1}{2}} \le \mu$. Let us suppose by contradiction that

$$|b_{k-1}| > \delta - 9\, \rho \cdot c_{d_n} \varepsilon. \tag{10}$$

This and (6) used with $k-1$ instead of k implies

$$|(I - A)\, b_{k-2}| > (\delta - 9\, \rho \cdot c_{d_n} \varepsilon) - \nu_1\rho. \tag{11}$$

First suppose that $b_{k-2} = 0$. Then $|b_{k-1}| = |(I - B_{k-2})a| \le \|B_{k-2}\| \cdot |a| \le \nu_2\, \rho$ and therefore $\delta \le (9\, c_{d_n} + \nu_2)\, \rho$. This contradicts assumption (A_4), and therefore (10) fails.
Secondly consider the case $b_{k-2} \neq 0$: Using $|(I - A)^2 x| \ge \frac{1}{|x|} |(I - A)\, x|^2$ for all $x \in \mathbb{R}^n - \{0\}$ and inequality (11), then (9) becomes

$$|b_k| \ge \frac{1}{|b_{k-2}|} |(I - A) b_{k-2}|^2 - \nu_3\rho > \tfrac{1}{\rho}(\delta - 9\, \rho \cdot c_{d_n}\varepsilon - \nu_1\rho)^2 - \nu_3\rho.$$

Together with estimation (5) this contradicts $\delta \ge 14\, f_n c_{d_n} \varepsilon^{\frac{1}{8}} \rho$ in assumption (A_4), and therefore (10) fails.

So the rotational part of $[\alpha\,; \ldots ; [\alpha\,; \beta] \cdots]_{k-1}$ is smaller than μ and the translational part smaller than δ. Therefore Lem. 6.1 implies that

$$[\alpha\,; \ldots ; [\alpha\,; \beta] \cdots]_{k-1} = id.$$

Iterating the above procedure we finally arrive at $k = 3$. Therefore the last inductive step gives $[\alpha\,; [\alpha\,; \beta]] = id$. So it is time to find another argument to show that $[\alpha\,; \beta] = id$.

The reader might ask why we do not proceed as above, using induction until $k=2$. The answer is simple: we cannot hope any more that $\|B_{k-2}\| = \|B_0\| = \|B\| \le \mu$, which would be used in (a) to get inequality (7).

(b) Now $[\alpha\,;[\alpha\,;\beta]] = id$ implies $\|B_2\| \le c_2\,\varepsilon$ and $|b_2| \le 9\,\rho\,c_2\,\varepsilon$. Therefore Thm. 2.4 implies $\|B_1\| = \|[A,B]\| \le 5 f_n \varepsilon^{\frac{1}{2}} = \nu_4$. So Lem. 5.1 guarantees the existence of a unitary change of basis $V \in U(n)$, if necessary, such that we can assume that $A=(a_{ij})$ and $B=(b_{ij})$ are almost diagonal with $|a_{ii}|^2, |b_{ii}|^2 \ge 1 - 3n^3\sqrt{\nu_4}$ for all $i \in \{1,\dots,n\}$ and translational parts $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$.

Now investigate the translational part $b_2 = A^{-1}B_1^{-1}((I-B_1)\,a-(I-A)\,b_1)$ and its length $|b_2|$. We have $|(I-A)\,b_1| \le |(I-B_1)\,a| + |b_2| \le 230\, f_n\varepsilon^{\frac{1}{2}}\rho = \nu_5$. Now changing the roles of $\alpha=(A,a)$ and $\beta=(B,b)$ in $\Delta_\rho^{1/9}$ we obtain again $|(I-B)\,a_1| \le \nu_5$, where $a_1 = -B_1^{-1}b_1$. Set

$$u = (I-B)\,a-(I-A)\,b,$$

then $|(I-A)\,b_1| = |(I-A)B^{-1}\,u| \le \nu_5$ and $|(I-B)\,a_1| = |(I-B)A^{-1}\,u| \le \nu_5$. Moreover conclude $|(I-A)B^{-1}\,u - B^{-1}(I-A)\,u| = \|B_1\|\cdot|u| \le \nu_4\rho$ and $|(I-B)A^{-1}\,u - A^{-1}(I-B)\,u| = \|B_1\|\cdot|u| \le \nu_4\rho$. We summarise the above as:

$$|(I-A)\,u| \le \nu_5+\nu_4\rho \le 235\, f_n\varepsilon^{\frac{1}{2}}\rho \quad \text{and} \quad |(I-B)\,u| \le 235\, f_n\varepsilon^{\frac{1}{2}}\rho \tag{12}$$

From the above and Cor. 6.5 (c) we obtain $\|\mathrm{rot}([\alpha\,;\beta])\| \le 10\, f_n\varepsilon^{\frac{1}{2}} \le \mu$, hence let us suppose by contradiction that

$$|b_1| = |u| = (|u_1|^2+\dots+|u_n|^2)^{\frac{1}{2}} > \delta - 45\rho\varepsilon. \tag{13}$$

So there exists at least one $j \in \{1,\dots,n\}$ with $|u_j| > \frac{\delta-45\rho\varepsilon}{\sqrt{n}}$.

Our aim is to contradict (13) by looking at the j-component of (12):

$$\begin{aligned}\nu_5+\nu_4\rho \ge \left|\sum_{l=1}^{n}(\delta_{jl}-a_{jl})u_l\right| &\ge |1-a_{jj}|\cdot|u_j| - (1-|a_{jj}|^2)^{\frac{1}{2}}|u| \\ &\ge |1-a_{jj}|\tfrac{\delta-45\rho\varepsilon}{\sqrt{n}} - f_n^2\,\varepsilon^{\frac{1}{4}}\rho,\end{aligned}$$

using $AA^*=I$, A is $f_n^2\varepsilon^{\frac{1}{4}}$-almost diagonal and $(\sum_{l=1,l\neq j}^{n}|u_l|^2)^{\frac{1}{2}} \le |u| \le \rho$. Therefore

$$|1-a_{jj}| \le 236\,\sqrt{n} f_n^2\varepsilon^{\frac{1}{4}}\tfrac{\rho}{\delta-45\rho\varepsilon} = \nu_6. \tag{14}$$

If we exchange the roles of α and β then $u=(I-B)\,a-(I-A)\,b$ changes its sign and the above estimation remains valid. From (13) follows now

$$\begin{aligned}\tfrac{\delta-45\rho\varepsilon}{\sqrt{n}} &< |u_j| \\ &\le |1-b_{jj}|\,\rho + |1-a_{jj}|\,\rho + (1-|a_{jj}|^2)^{\frac{1}{2}}|b| + (1-|b_{jj}|^2)^{\frac{1}{2}}|a| \\ &\le |1-b_{jj}|\,\rho + \nu_6\rho + 2f_n^2\,\varepsilon^{\frac{1}{4}}\rho.\end{aligned}$$

Estimation (12) gives again $|1-b_{jj}| < \nu_6$, (cf. inequality (14) which is also valid for the exchanged roles of α and β). This contradicts assumption (A_4), hence (13) fails. Thus $\|\mathrm{rot}([\alpha\,;\beta])\| \le \mu$ and $|\mathrm{trans}([\alpha\,;\beta])| \le \delta$. Hence $[\alpha\,;\beta] = id$ by Lem. 6.1.

Therefore any two elements $\alpha, \beta \in \Delta_\rho^{1/9}$ $*$-commute. In other words, the set $\Delta_\rho^{1/9}$ is not only d_n-step nilpotent, and 2-step nilpotent after (a) but even Abelian. This proof explains the origin of assumptions (A_4) and (A_6): for it to work, we are obliged to take ε such that $0 \le \varepsilon \le \varepsilon(\rho) \le \varepsilon_n$, where $\rho \ge \rho_n$. ■

9. Equivalence Classes of $\Gamma_{\rho/3}$

In the next step we want to define an equivalence relation on $\Gamma_{\rho/3}$. Later we consider the quotient of $\Gamma_{\rho/3}$ modulo this equivalence relation, which has the structure of a finite group. The product is given by the product of short representatives. The main application is the development of still more precise information about the $*$-structure.

Definition 9.1. (Equivalence relation) Call $\alpha, \beta \in \Gamma_{\rho/3}$ equivalent modulo $\Delta_\rho^{1/9}$ if and only if $\alpha * \overline{\beta} \in \Delta_\rho^{1/9}$. Notation: $\alpha \sim \beta \bmod \Delta_\rho^{1/9}$.

It is shown in [4], Chap. 3.6 that $\sim \bmod \Delta_\rho^{1/9}$ is an equivalence relation. We denote $\{\beta \in \Gamma_{\rho/3} \mid \alpha \sim \beta \bmod \Delta_\rho^{1/9}\}$ the equivalence class of $\alpha \in \Gamma_{\rho/3}$ by $[\alpha]$, and the set of all equivalence classes of $\Gamma_{\rho/3}$ by $\mathcal{H}$. Since representatives α and β of distinct equivalence classes $[\alpha]$ and $[\beta]$ in $\mathcal{H}$ have their rotational parts at pairwise distance $\|A^{-1}B\| \ge 1/10$, there are by Lem. 4.1 at most $w_n = 2(20\pi)^{\frac{1}{2}n(n-1)}$ such equivalence classes.

Lemma 9.2. (Short representatives) *In each equivalence class* $[\alpha]$ *of* $\mathcal{H}$ *there is a representative* α *with* $|\mathrm{trans}(\alpha)| \le 4\,r \cdot w_n$.

We call such a representative α a short representative, since it has a translational part which is much shorter than $\rho/3$.

Proof. We first want to show that *any element* $\alpha \in \Gamma_{\rho/3}$ *can be presented as* $\alpha = \alpha_1 * \cdots * \alpha_k$ *with* $\alpha_i \in \Gamma_R$ *and* $|a_i| \le 3\,r$ *for all* $i \in \{1, \ldots, k\}$. *Moreover, each partial product* $\alpha_1 * \cdots * \alpha_{k'}$ *with* $k' \le k$ *is an element of* $\Gamma_{\rho/3+r}$. We call $\alpha = \alpha_1 * \cdots * \alpha_k$ a normal word. Indeed, connect $0 \in \mathbb{R}^n$ and $\mathrm{trans}(\alpha) = a \in \mathbb{R}^n$ with a straight line, which must have length equal to $|a| \le \rho/3$. Now subdivide this line into pieces of length smaller than η, where $0 < \frac{1}{2}\eta \le r$. By Def. 2.3 (I) we find near each division point p_i on the line an element $\beta_i \in \Gamma_{\rho/3+r}$ such that $|b_i - p_i| \le r$. Define $\alpha_i = \overline{\beta}_{i-1} * \beta_i$ for all division points. With $|p_i - p_{i-1}| \le \eta$ and Cor. 6.5 (a) and (b) we obtain $|a_i| \le 3\,r$. Furthermore $\alpha_1 * \cdots * \alpha_{k'} = \beta_{k'} \in \Gamma_{\rho/3+r}$ for all $k' \in \{1, \ldots, k\}$. Any partial product of consecutive factors $\alpha_j * \cdots * \alpha_{j+l} = \overline{\beta}_{j-1} * \beta_{j+l}$ in the decomposition of α has a translational part shorter than $|a| + 2\,r + 18\rho\,\varepsilon$.

Therefore take $\Gamma_{3r} = \{\alpha = (A, a) \in \Gamma_R \mid |a| \le 3\,r\}$ as a set of generators for $\Gamma_{\rho/3}$ and call word length of α the number $l(\alpha)$ of generators needed to present the normal word $\alpha = \alpha_1 * \cdots * \alpha_l$. The set of equivalence classes which can be presented by normal words of length smaller than m is denoted by $\mathcal{H}_m$. We have $\mathcal{H}_m \subseteq \mathcal{H}_{m+1}$ for all $m \in \mathbb{N}$. By Lem. 4.1 the set $\mathcal{H}$ and all sets $\mathcal{H}_m$ contain at most w_n equivalence classes. Consider the equivalence classes in $\mathcal{H}_m$, then we have exactly two possibilities for those in $\mathcal{H}_{m+1}$:

(i) All normal words of length $m+1$ are equivalent to normal words of maximal length m. Therefore we show that there exists a number m_0 such that $\mathcal{H}_m = \mathcal{H}_{m+1}$ for all $m \geq m_0$: Indeed assume by induction that each element $\alpha \in \Gamma_{\rho/3}$, which can be written as a normal word of length $l(\alpha) \geq m_0+1$ has a representative α', that is $\alpha' \sim \alpha \bmod \Delta_\rho^{1/9}$, of length $l(\alpha') \leq m_0$. Then each $\alpha * \alpha_i \in \Gamma_{\rho/3}$ with $\alpha_i \in \Gamma_{3r}$ of length $l(\alpha * \alpha_i) = l(\alpha)+1$ is equivalent modulo $\Delta_\rho^{1/9}$ to $\alpha' * \alpha_i$, which is a normal word of length $l(\alpha' * \alpha_i) = l(\alpha')+1 \leq m_0+1$ and is therefore by assumption equivalent to a normal word with $l \leq m_0$.

(ii) At least one normal word of length $m+1$ is not equivalent to a normal word of length m, i.e. $\mathcal{H}_m \neq \mathcal{H}_{m+1}$. Each time we add a letter $\alpha \in \Gamma_{3r}$, we obtain at least one more equivalence class. This procedure terminates after at most w_n steps, since there are not more than w_n equivalence classes in the set $\mathcal{H}$, i.e. $\mathcal{H}_{w_n} = \mathcal{H}$.

Now all normal words generated by elements of Γ_{3r} with length $l \leq w_n$ have a translational part which is smaller than $l\,(3r + 9\rho\,\varepsilon) \leq 4r \cdot w_n$. So any normal word in $\Gamma_{\rho/3}$ is equivalent to a normal word in $\Gamma_{4r \cdot w_n}$, hence there is a short representative in each equivalence class. ∎

Corollary 9.3. (Relative denseness of $\Delta_\rho^{1/9}$) *Let $x \in \mathbb{R}^n$ with $|x| \leq \rho/3 - r$. Then there is an element $\gamma = (C, c) \in \Delta_\rho^{1/9}$ such that $|x - c| \leq 5r \cdot w_n$.*

Lemma 9.4. (Multiplication of equivalence classes in $\mathcal{H}$, [4], Cor. 3.6.4)

(a) *The finite set $\Delta_\rho^{1/9}$ has the following property of a normal subgroup: if $\alpha \in \Gamma_{\rho/3}$ and $\gamma = (C, c) \in \Delta_\rho^{1/9}$ with $|c| \leq \rho/3$ then $\alpha * \gamma * \overline{\alpha} \in \Delta_\rho^{1/9}$.*

(b) *If $\alpha_i \in [\alpha_i]$ for $i \in \{1, 2\}$ are short representatives of equivalence classes modulo $\Delta_\rho^{1/9}$ then the product $[\alpha_1] * [\alpha_2] = [\alpha_1 * \alpha_2]$ is well defined.*

(c) *The equivalence classes modulo $\Delta_\rho^{1/9}$ together with the product defined in* (b) *form a group $\mathcal{H}$ of order $|\mathcal{H}| \leq w_n = 2(20\pi)^{\frac{1}{2}n(n-1)}$.*

10. The Almost Translational Set $\Delta_\rho^{1/9}$

Lemma 10.1. (Small rotational parts) *Let $\alpha \in \Delta_\rho^{1/9}$. Then*

$$\|\mathrm{rot}(\alpha)\| \leq 10\sqrt{5}n^3\sqrt{\varepsilon}.$$

Lem. 8.5 tells us that all elements in $\Delta_\rho^{1/9}$ have norm of its rotational parts of at most $\frac{1}{27}$. Looking at the proof of Lem. 8.5 and Lem. 8.4 we find out that the proof works for any positive number $\xi \leq \frac{1}{27}$, but with the problem that if ξ gets smaller then ε gets smaller and ρ grows fast. In the following proof we do not touch the size of the constants ε_n and ρ_n as they are defined in assumption (A_1) and (A_6).

Proof. The set $\Delta_\rho^{1/9}$ equipped with the $*$-product is Abelian by Lem. 8.7, thus $\|[A,B]\| \le 5\varepsilon$ and $|A^{-1}B^{-1}((I-B)a-(I-A)b)| \le 45\rho\varepsilon$ by Cor. 6.5 (c). Using Cor. 5.2 we may assume, by applying an orthogonal change of basis of $\mathbb{R}^n$ if necessary, that the element $\alpha=(A,a)\in\Delta_\rho^{1/9}$ has the form

$$A=\begin{pmatrix} A' & 0\\ 0 & A''\end{pmatrix}\in O(2k)\times O(n-2k) \qquad\text{and}\qquad a=(a',a'')\in\mathbb{R}^{2k}\times\mathbb{R}^{n-2k},$$

where all eigenvalues $a_1,\dots,a_{2k}$ of A' satisfy $|1-a_i|>\eta-n\sqrt{5\varepsilon}$ and all eigenvalues $a_{2k+1},\dots,a_n$ of A'' satisfy $|1-a_i|\le\eta+n\sqrt{5\varepsilon}$; in addition all other elements $\beta=(B,b)\in\Delta_\rho^{1/9}$ have the form (after the orthogonal change of basis)

$$B=\begin{pmatrix} B' & F'\\ F'' & B''\end{pmatrix} \qquad\text{and}\qquad b=(b',b'')\in\mathbb{R}^{2k}\times\mathbb{R}^{n-2k},$$

where $B'\in\mathrm{Mat}(2k\times 2k,\mathbb{R})$, $B''\in\mathrm{Mat}((n-2k)\times(n-2k),\mathbb{R})$ and $|f'_{ij}|,|f''_{ij}|\le n\sqrt{5\varepsilon}$ for all possible i,j-combinations. Set $\eta=9n^3\sqrt{5\varepsilon}$.
Suppose by contradiction that $k\ge 1$. Then define a vector $t=(t',0)\in\mathbb{R}^{2k}\times\mathbb{R}^{n-2k}$ by $(A'-I)t'=a'$. The matrix $A'-I$ is not singular, since $\eta\ge n\sqrt{5\varepsilon}$ is big enough. For a better understanding of the idea of the following proof compare with the Bieberbach case: for all β we have

$$(B'-I)t'=(B'-I)(A'-I)^{-1}a'=(A'-I)^{-1}(B'-I)a'=b',$$

which is impossible by a denseness argument. Our aim is to show that the vector $(B'-I)t'$ is close to b' for all $\beta\in\Delta_\rho^{1/9}$. Using $[\alpha\,;\beta]=id$ and $\eta=9n^3\sqrt{5\varepsilon}$ estimate:

$$\begin{aligned} |(A'-I)^{-1}(B'-I)a'-b'| &\le \tfrac{2n}{9n^2-1}\,\rho\\ |(A'-I)^{-1}(B'-I)a'-(B'-I)(A'-I)^{-1}a'| &\le \tfrac{1}{(9n^3-n)^2}\,\rho\end{aligned}$$

Therefore $|(B'-I)t'-b'|\le\frac{\rho}{3n}$ for any $\beta=(B,b)\in\Delta_\rho^{1/9}$. Hence $a'=A't'-t'$ and all $B't'-t'$ lie in the disc $K'_{|t'|}(-t')$ in $\mathbb{R}^{2k}$. Thus a' and all b' lie in a $\frac{\rho}{3n}$-neighbourhood of the disc $K'_{|t'|}(-t')$ in $\mathbb{R}^n$. But Cor. 9.3 tells us that the translational parts of elements in $\Delta_\rho^{1/9}$ are $5\,r\cdot w_n$-dense in $K_{\rho/3-r}(0)$. This contradicts the assumption $k\ge 1$, therefore $k=0$, i.e., all eigenvalues of A satisfy $|1-a_i|\le 10\sqrt{5}n^3\sqrt{\varepsilon}$. ∎

Corollary 10.2. *If $\alpha\sim\beta$ mod $\Delta_\rho^{1/9}$ then $\|A\,B^{-1}\|\le 10\sqrt{5}n^3\sqrt{\varepsilon}+2\varepsilon$.*

Therefore the set $\Delta_\rho^{1/9}$ with the $*$-structure is an Abelian crystallographic pseudogroup which contains only almost translations.

11. Basis for the Almost Translational Set $\Delta_\rho^{1/9}$

In what follows we define a λ-normal basis for lattices in $\mathbb{R}^n$. Then we introduce almost translational subsets of $\mathbb{R}^n$ and give bounds for the deviation of products from purely translational behaviour. Guided by the translational case we study orbits, suitable representatives and their projection onto a hyperplane. Projections

of almost translational sets are again almost translational. All this is used to find a λ-normal basis $\{d_1, \ldots, d_n\}$ for the almost translational set $T_{(\delta,\rho)} = \mathrm{trans}(\Delta_\rho^{1/9})$, such that every element $d \in T_{(\delta,\rho)}$ with $|d| \leq \frac{1}{2^n}\rho$ can uniquely be written as $d = d_1^{\oplus l_1} \oplus \cdots \oplus d_n^{\oplus l_n}$ with $l_j \in \mathbb{Z}$ for all $j \in \{1, \ldots, n\}$.
The following is an adapted version to our problem of [4], Chap. 4.1 – 4.4.

Definition 11.1. (λ-normal basis for $\mathbb{R}^n$) A λ-normal basis for $\mathbb{R}^n$ is defined by induction over n as follows, with $\lambda \in [1, \infty[$.

(I) Any basis for $\mathbb{R}^1$ is a λ-normal basis for each λ.

(II) A basis $\{b_1, \ldots, b_n\}$ for $\mathbb{R}^n$ is λ-normal, if it satisfies $|b'_j| \leq |b_j| \leq \lambda |b'_j|$ for all indices $j \in \{2, \ldots, n\}$, where $b'_j = b_j - \frac{\langle b_j, b_1\rangle}{\langle b_1, b_1\rangle} b_1$ is the orthogonal projection of b_j onto $\{b_1\}^\perp$. The set $\{b'_2, \ldots, b'_n\}$ is a λ-normal basis for $\{b_1\}^\perp \cong \mathbb{R}^{n-1}$.

It is useful to study this notion first in the purely translational case: Any uniform discrete subgroup Λ of $\mathbb{R}^n$ has a $\sqrt{2}$-normal basis, cf. [4], Prop. 4.1.3. Later we modify the procedures to handle error terms. Additional problems arise if one tries to generalise this fact to the case where Λ is not any longer a lattice, but only an almost translational set. Orbits are not straight and therefore more difficult to project. Products are only defined for sufficiently short elements. The following notion helps to generalise the arguments.

Definition 11.2. (Almost translational set) The finite set $T_{(\delta,\rho)} \subset \mathbb{R}^n$ is called κ-almost translational, σ-dense of radii (δ, ρ), if it satisfies:

(I) There is $0 \in T_{(\delta,\rho)}$ and if $a \in T_{(\delta,\rho)} - \{0\}$ then $\delta \leq |a| \leq \rho$.

(II) For all $x \in \mathbb{R}^n$ with $|x| \leq \frac{\rho}{4}$, there is some $c \in T_{(\delta,\rho)}$ such that $|x - c| \leq \sigma$.

(III) For all $a, b \in T_{(\delta,\rho)}$ with $|a + b| \leq \rho - \kappa$, the sum $a \oplus b \in T_{(\delta,\rho)}$ is defined such that $|a \oplus b - (a + b)| \leq \kappa$. For each $a \in T_{(\delta,\rho)}$ with $|a| \leq \rho - \kappa$, there is a unique negative $\ominus a \in T_{(\delta,\rho)}$ such that $a \ominus a = a \oplus (\ominus a) = 0$.

(IV) Commutativity $a \oplus b = b \oplus a$ and associativity $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ hold, if all sums involved are defined as in (III).

Lemma 11.3. *The set $T_{(\delta,\rho)} = \{a = \mathrm{trans}(\alpha) \mid \alpha \in \Delta_\rho^{1/9}\}$ together with the addition*

$$a \oplus b = \mathrm{trans}(\alpha * \beta) \quad \textit{if} \quad |a + b| \leq \rho - 25n^3\sqrt{\varepsilon}\rho$$

is $25n^3\sqrt{\varepsilon}\rho$-almost translational and $5r \cdot w_n$-dense of radii (δ, ρ) which satisfy $\frac{r}{2^n} \geq \delta \geq 2\, a_n c_{d_n} \varepsilon^{\frac{1}{8}} \rho$.

Proof. Of course $id \in \Delta_\rho^{1/9}$. If $\alpha \in \Delta_\rho^{1/9}$, then $\|\mathrm{rot}(\alpha)\| \leq 10\sqrt{5}n^3\sqrt{\varepsilon} \leq \mu$ and if $\alpha \neq id$, then $\delta \leq |\mathrm{trans}(\alpha)| \leq \rho$, so (I) holds. For (II) see Cor. 9.3 with $\sigma = 5r \cdot w_n$. To get (III) use Lem. 10.1 and observe $|a \oplus b - (a + b)| \leq 25n^3\sqrt{\varepsilon}\rho$, if $|a + b| \leq \rho - 25n^3\sqrt{\varepsilon}\rho$. We know that $\alpha * \overline{\alpha} = \overline{\alpha} * \alpha = id$, thus $\ominus a = \mathrm{trans}(\overline{\alpha})$, which is unique. Since $\Delta_\rho^{1/9}$ is Abelian by Lem. 8.7 and associative, also the $\oplus$-operation in $T_{(\delta,\rho)}$ is, thus (IV) follows. ∎

The induction step starts with this example of an almost translational set. In order to control the deviation of the $\oplus$-addition from ordinary vector sums, we have the following lemma.

Lemma 11.4. *Every $\oplus$-operation causes an error of at most κ, compared to ordinary vector addition in $\mathbb{R}^n$. In other words $T_{(\delta,\rho)}$ with the $\oplus$-operation satisfies:*

(a) *Shortest elements $a \in T_{(\delta,\rho)} - \{0\}$ satisfy $\delta \le |a| \le 2\sigma$.*

(b) *Let $a \in T_{(\delta,\rho)}$ with $|a| \le \rho - \kappa$. Then $|\ominus a + a| \le \kappa$.*

(c) *Let $a, b \in T_{(\delta,\rho)}$ with $|a+b| \le \rho - 2\kappa$. Then $|a \oplus (\ominus b) - (a-b)| \le 2\kappa$.*

(d) *Let $a_1, \ldots, a_k \in T_{(\delta,\rho)}$ with $|\sum_{j=1}^k a_j| \le \rho - \frac{2c_{d_n}\rho}{\delta}\kappa$. Then $|a_1 \oplus \cdots \oplus a_k - (a_1 + \cdots + a_k)| \le k\kappa$ and $k \le \frac{2c_{d_n}\rho}{\delta}$.*

Proof. The Claim (a) is clear from Def. 11.2 (I) and (II). For (b) use $\ominus a \oplus a = 0$ in Def. 11.2 (III), thus $|\ominus a \oplus a - (\ominus a + a)| = |\ominus a + a| \le \kappa$. To see (c), use $a \oplus (\ominus b)$ in Def. 11.2 (III), then $|a \oplus (\ominus b) - (a-b)| \le |a \oplus (\ominus b) - (a + \ominus b)| + |\ominus b + b| \le 2\kappa$. For (d) use $a_1 \oplus \cdots \oplus a_k$ in Def. 11.2 (III). By Def. 11.2 (I) the smallest non-trivial element in $T_{(\delta,\rho)}$ has $|a| \ge \delta$, thus the assumption that $a_1 \oplus \cdots \oplus a_k$ is defined in Γ_R, i.e., $\sum_{j=1}^k |a_j| \le R - \zeta \le 2c_{d_n}\rho$, needs $k \le \frac{2c_{d_n}\rho}{\delta}$. ■

From Def. 11.2 (II), Lem. 11.4 (c) and with $\frac{\kappa}{\delta} \le \frac{1}{4}$ we conclude:

Corollary 11.5. *Let $a, b \in T_{(\delta,\rho)}$ with $a \ne b$. Then $|a-b| \ge \delta - 2\kappa \ge \frac{1}{2}\delta$.*

Almost straight orbits: Let $T_{(\delta,\rho)}$ be as in Def. 11.2 and $d_1 \in T_{(\delta,\rho)} - \{0\}$ a shortest non-trivial element, i.e. $\delta \le |d_1| \le 2\sigma$. For each $d \in T_{(\delta,\rho)}$ define the d_1-orbit

$$O_d = \left\{ d \oplus d_1^{\oplus l_1} \mid l_1 \in \mathbb{Z} \text{ with } |d + l_1 \cdot d_1| \le \rho - \tfrac{2c_{d_n}\rho}{\delta}\kappa \right\}$$

through d. By the above and Lem. 11.4 (d) we get $|d \oplus d_1^{\oplus l_1} - (d + l_1 d_1)| \le \frac{2c_{d_n}\rho}{\delta}\kappa$. In other words every d_1-orbit O_d is contained in a tube of radius $\frac{2c_{d_n}\rho}{\delta}\kappa$ and centre-line $d + \mathbb{R} \cdot d_1$.

Representatives of orbits: In every d_1-orbit through $d \in T_{(\delta,\rho)}$ we find a unique representative $\tilde{d}$, which is characterised by the inequalities

$$\langle d_1, \tilde{d} \rangle > 0 \quad \text{and} \quad \langle d_1, \tilde{d} \ominus d_1 \rangle \le 0. \tag{15}$$

Define the set $\tilde{T}_{(\delta,\rho)}$ of all representatives of orbits.

Projection of representatives of orbits: Let $\{d_1\}^\perp$ be the hyperplane through the origin of $\mathbb{R}^n$ which is perpendicular to d_1. Now map the set $\tilde{T}_{(\delta,\rho)}$ of representatives by orthogonal projection onto $T' \subset \{d_1\}^\perp$:

$$\tilde{d} \longmapsto d' = \tilde{d}' = \tilde{d} - \frac{\langle \tilde{d}, d_1 \rangle}{\langle d_1, d_1 \rangle} d_1$$

With (15) estimate the maximal distance of a representative $\tilde{d}$ to $\{d_1\}^\perp$. To do that we estimate $0 \ge \langle d_1, \tilde{d} \ominus d_1 \rangle \ge \langle d_1, \tilde{d} \rangle - \langle d_1, d_1 \rangle - 2\kappa|d_1|$, and with $\delta \le |d_1|$ we finally obtain $\langle d_1, \tilde{d} \rangle \le |d_1|^2 + 2\kappa|d_1| \le |d_1|^2 \left(1 + 2\,\frac{\kappa}{\delta}\right)$, and therefore

$$|\tilde{d} - d'| = \frac{\langle d_1, \tilde{d} \rangle}{\langle d_1, d_1 \rangle} |d_1| \le |d_1| + 2\kappa \le |d_1| \left(1 + 2\,\tfrac{\kappa}{\delta}\right). \tag{16}$$

Now we can generalise Prop. 4.1.3 in [4] as follows.

Lemma 11.6. *Let $\tilde d \in \tilde T_{(\delta,\rho)}$. Then $|d'| \le |\tilde d| \le \lambda\,|d'|$, with $\lambda \le (\frac12 - 20\frac{\kappa}{\delta})^{-\frac12} \le 2$. If $\tilde c, \tilde d \in \tilde T_{(\delta,\rho)}$ with $\tilde c \ne \tilde d$ then*

$$\cos\measuredangle(\tilde c - \tilde d, d_1) \le \sqrt{\tfrac12 + 20\tfrac{\kappa}{\delta}}.$$

This shows that the projection $\,' : \tilde T_{(\delta,\rho)} \to T'$ is injective, since $\frac{\kappa}{\delta} \le \frac{1}{80}$.

Proof. Our aim is to estimate $\langle \tilde c - \tilde d, d_1\rangle$. From the definition of the representatives (15) we conclude that $\tilde c \ominus \tilde d = d_1$ is not possible. Thus the minimality of d_1 and $\tilde c \ne \tilde d$ imply that $\tilde c \ominus \tilde d$ and $(\tilde c \ominus \tilde d) \ominus d_1$ are bigger than d_1. Therefore by Lem. 11.4 (a) and (c) we obtain $\delta \le |d_1| \le |\tilde c \ominus \tilde d| \le |\tilde c - \tilde d| + 2\kappa$ and $\delta \le |d_1| \le |(\tilde c \ominus \tilde d) \ominus d_1| \le |(\tilde c - \tilde d) - d_1| + 4\kappa$, which implies $|d_1|^2 - 8\kappa|d_1| + 16\kappa^2 \le |(\tilde c - \tilde d) - d_1|^2 = |\tilde c - \tilde d|^2 + |d_1|^2 - 2\langle \tilde c - \tilde d, d_1\rangle$. We can, after renaming, assume that

$$0 < \langle \tilde d, d_1\rangle \le \langle \tilde c, d_1\rangle, \tag{17}$$

then the above and Cor. 11.5 imply

$$\begin{aligned} 0 \le 2\,\langle \tilde c - \tilde d, d_1\rangle &\le |\tilde c - \tilde d|^2 + 8\,\kappa|d_1| \le |\tilde c - \tilde d|^2 + 8\,\kappa(|\tilde c - \tilde d| + 2\,\kappa) \\ &= (|\tilde c - \tilde d| + 4\,\kappa)^2 \le |\tilde c - \tilde d|^2\left(1 + 8\,\tfrac{\kappa}{\delta}\right)^2. \end{aligned} \tag{18}$$

Only the case where $|\tilde c - \tilde d|$ is not too large might cause problems. The inequalities (15), which define the representatives $\tilde c$ and $\tilde d$, and assumption (17) imply that $0 < \langle \tilde d, d_1\rangle \le \langle \tilde c, d_1\rangle \le |d_1|^2\left(1 + 2\,\frac{\kappa}{\delta}\right)$, in other words

$$0 \le \langle \tilde c - \tilde d, d_1\rangle = \langle \tilde c, d_1\rangle - \langle \tilde d, d_1\rangle \le \langle \tilde c, d_1\rangle \le |d_1|^2\left(1 + 2\,\tfrac{\kappa}{\delta}\right). \tag{19}$$

Therefore we obtain from (18) and (19)

$$\cos^2\measuredangle(\tilde c - \tilde d, d_1) = \frac{\langle \tilde c - \tilde d, d_1\rangle^2}{|\tilde c - \tilde d|^2|d_1|^2} \le \tfrac12\left(1 + 8\,\tfrac{\kappa}{\delta}\right)^2\left(1 + 2\,\tfrac{\kappa}{\delta}\right) \le \tfrac12 + 20\,\tfrac{\kappa}{\delta},$$

thus

$$\frac{1}{\sin\measuredangle(\tilde d, d_1)} = \frac{1}{\sqrt{1 - \cos^2\measuredangle(\tilde d, d_1)}} \le \frac{1}{\sqrt{\frac12 - 20\,\frac{\kappa}{\delta}}} \le 2,$$

since $\frac{\kappa}{\delta} \le \frac{1}{80}$ is easily satisfied. Therefore $\lambda \le 2$, since the above holds for all $\tilde d \in \tilde T_{(\delta,\rho)}$. ∎

Next we want to look more precisely at the set T' and settle some of its important properties. It is indeed an almost translational set in the sense of Def. 11.2:

Lemma 11.7. (Denseness of the set T' in $\{d_1\}^\perp$) *Let $x \in \{d_1\}^\perp \cong \mathbb{R}^{n-1}$ in $\mathbb{R}^n$ with $|x| \le \frac{\rho}{4}$. Then some $c' \in T' \subset \{d_1\}^\perp$ exists such that $|x - c'| \le \sigma + r$.*

Proof. By Def. 11.2 (II) there is for every $x \in \{d_1\}^\perp \subset \mathbb{R}^n$ with $|x| \le \frac{\rho}{4}$ some $c \in T_{(\delta,\rho)}$ such that $|x - c| \le \sigma$. The representative $\tilde c$ of the d_1-orbit O_c and its projection c' exists. The elements $c, \tilde c$ and c' lie in a tube of radius $\frac{2c_{d_n}\rho}{\delta}\kappa$ and centre-line $c + \mathbb{R}\cdot d_1$. In other words $|x - c'| \le \sigma + \frac{2c_{d_n}\rho}{\delta}\kappa \le \sigma + r$, thus T' is $(\sigma + r)$-dense in $\{d_1\}^\perp$. ∎

Now we introduce a well defined addition for some elements in T':

Definition 11.8. (Addition in the set T') Let $a', b' \in T'$ with $|a'|, |b'| \leq \frac{1}{2}\rho$ and $|a'+b'| \leq \frac{1}{2}\rho - 7\kappa$. For the unique pre-images $\tilde{a}, \tilde{b} \in \tilde{T}_{(\delta,\rho)}$ of a' and b' we have by Lem. 11.6 the bounds $|\tilde{a}| \leq \lambda\,|a'|$ and $|\tilde{b}| \leq \lambda\,|b'|$ with $\lambda \leq 2$. This signifies that the sum $\tilde{a} \oplus \tilde{b}$ is well defined, and the representative $(\tilde{a} \oplus \tilde{b})^{\tilde{}}$ can be obtained as above. Therefore, the following definition of an addition in T' is justified:

$$a' \oplus' b' = \Big((\tilde{a} \oplus \tilde{b})^{\tilde{}}\Big)', \qquad \text{with } a' \text{ and } b' \text{ as above.}$$

In addition, the set T' with the above $\oplus'$-operation is again an almost translational set:

Lemma 11.9. (Almost translational set T') *The set T' together with the $\oplus'$-operation is a κ'-almost translational and σ'-dense set in $\{d_1\}^\perp$ of radii (δ', ρ'), In other words, for all $a', b' \in T'$ with $|a'|, |b'| \leq \rho'$ and $|a' + b'| \leq \rho' - \kappa'$ the Abelian and associative addition satisfies $|a' \oplus' b' - (a' + b')| \leq \kappa'$. There is also a unique inverse operation. The constants are now given by $\kappa' = 7\kappa$, $\sigma' = \sigma + r$ and $(\delta', \rho') = (\frac{1}{2}\delta, \frac{1}{2}\rho)$.*

Proof. For the trivial element we have $0' = 0 \in T'$. Moreover, let $c' \in T' - \{0\}$ with $|c'| \leq \rho'$. Then $\delta \leq |\tilde{c}| \leq \lambda\,|c'|$ with $\lambda \leq 2$, therefore set $\delta' = \frac{1}{2}\delta$ and thus (I) is verified. For (II) we use Lem. 11.7. Using inequality (16) we estimate:

$$\left|\tilde{a} \oplus \tilde{b} - (\tilde{a} + \tilde{b})'\right| \leq \left|\tilde{a} \oplus \tilde{b} - (\tilde{a} + \tilde{b})\right| + |\tilde{a} - a'| + |\tilde{b} - b'| \leq 2|d_1| + 5\kappa$$

Thus since $\tilde{a}, \tilde{b}$ and $\tilde{a} \oplus \tilde{b}$ lie on the same side of $\{d_1\}^\perp$ as d_1 we obtain $(\tilde{a} \oplus \tilde{b})^{\tilde{}} = \tilde{a} \oplus \tilde{b} \oplus d_1^{\oplus k}$ with $k \in \{-3, -2, -1, 0\}$. In order to show (III) we estimate:

$$|a' \oplus' b' - (a' + b')| \leq \left|\Big(\tilde{a} \oplus \tilde{b} \oplus d_1^{\oplus k} - (\tilde{a} \oplus \tilde{b})\Big)'\right| + \left|(\tilde{a} \oplus \tilde{b})' - (\tilde{a} + \tilde{b})'\right| \leq 7\kappa$$

Every $\ominus$-operation causes an error of at most 2κ compared to ordinary vector sums. There are at most $|k| \leq 3$ operations, this gives the last inequality above. Let $a' \in T'$ with $|a'| \leq \rho' - \kappa'$ then there exists the unique negative $\ominus\tilde{a}$ of the representative $\tilde{a} \in T_{(\delta,\rho)}$. Hence the unique negative of a' is $(\widetilde{\ominus\tilde{a}})'$. Commutativity and associativity of the $\oplus'$-operation in T' follow from the commutativity and associativity of the $\oplus$-operation in $T_{(\delta,\rho)}$, thus (IV) is also true. In short, T' is an almost translational set in the sense of Def. 11.2. ∎

Next we need n inductive steps. Let $k \in \{1, \ldots, n\}$ and set

$$\kappa^{(k-1)} = 7^{k-1}\kappa, \quad \sigma^{(k-1)} = \sigma + (k-1)r, \quad \rho^{(k-1)} = 2^{-k+1}\rho \quad \text{and} \quad \delta^{(k-1)} = 2^{-k+1}\delta.$$

We need in the prove of Lem. 11.6 the fact that $\frac{\kappa^{(k-1)}}{\delta^{(k-1)}} \leq \frac{1}{80}$ for all $k \in \{1, \ldots, n\}$, which is always satisfied since ε in assumption (A_6) is supposed to be small enough. Now we are in the position to prove inductively the following important lemma.

Lemma 11.10. *There exist generators $\delta_1, \ldots, \delta_n \in \Delta_\rho^{1/9}$ such that each element $\delta = (D, d) \in \Delta_\rho^{1/9}$ with $|d| \leq \rho/2^n$ can uniquely be written as $\delta = \delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$ with $|d - (l_1 d_1 + \cdots + l_n d_n)| \leq \varepsilon^{\frac{3}{8}}\rho$.*

Proof. By Lem. 11.3, the set $T_{(\delta,\rho)} = \{a = \mathrm{trans}(\alpha) \mid \alpha \in \Delta_\rho^{1/9}\}$ is an almost translational set, in the sense of Def. 11.2.
First show existence, i.e., the set $T_{(\delta,\rho)}$ has a λ-normal basis $\{d_1, \ldots, d_n\}$ with $\lambda \leq 2$, and $|d_k| \leq 11\, r\cdot w_n$ for all $k \in \{1, \ldots, n\}$. Indeed, the assumption on $\frac{\kappa^{(k-1)}}{\delta^{(k-1)}} \leq \frac{1}{80}$ for all $k \in \{1, \ldots, n\}$ allows us to go through n inductive steps, using the arguments above, to find $d_1, \ldots, d_n$. Lem. 11.6 shows that $\lambda \leq (\frac{1}{2} - 20\frac{\kappa^{(k-1)}}{\delta^{(k-1)}})^{-\frac{1}{2}} \leq 2$, and Lem. 11.4 (a) gives a bound for $|d_k| \leq 2\sigma^{(k)} \leq 2\sigma + 2(n-1)r \leq 11\, r \cdot w_n$ for all $k \in \{1, \ldots, n\}$.
Second show uniqueness: If $|\sum_{j=1}^n l_j d_j| \leq \rho - \frac{2c_{d_n}\rho}{\delta}\kappa$ and if $d = d_1^{\oplus l_1} \oplus \cdots \oplus d_n^{\oplus l_n}$ is defined, then Lem. 11.4 implies

$$|d - (l_1 d_1 + \cdots + l_n d_n)| \leq \frac{2c_{d_n}\rho}{\delta}\kappa \leq \frac{25n^3}{a_n}\varepsilon^{\frac{3}{8}}\rho \leq \varepsilon^{\frac{3}{8}}\rho,$$

since $\frac{r}{2^n} \geq \delta \geq 2\, a_n c_{d_n} \varepsilon^{\frac{1}{8}}\rho$. There is $\kappa = 25n^3\varepsilon^{\frac{1}{2}}\rho$. Now suppose there are different presentations of the same element

$$d = d_1^{\oplus l_1} \oplus \cdots \oplus d_n^{\oplus l_n} = d_1^{\oplus l_1'} \oplus \cdots \oplus d_n^{\oplus l_n'}$$

with $|\sum_{j=1}^n l_j' \cdot d_j| \leq \rho - \frac{2c_{d_n}\rho}{\delta}\kappa$, such that $d_1^{\oplus l_1'} \oplus \cdots \oplus d_n^{\oplus l_n'}$ is defined with $l_j \neq l_j'$ for at least one $j \in \{1, \ldots, n\}$. Then by commutativity we obtain

$$0 = d_1^{\oplus(l_1 - l_1')} \oplus \cdots \oplus d_n^{\oplus(l_n - l_n')}$$

with $|(l_1 - l_1')d_1 + \cdots + (l_n - l_n')d_n| \leq \varepsilon^{\frac{3}{8}}\rho \leq \delta$. Since $\{d_1, \ldots, d_n\}$ is a λ-normal basis with $\lambda \leq 2$ and $|d_j| \geq \delta^{(j-1)}$ for all $j \in \{1, \ldots, n\}$, it follows that $l_j = l_j'$ for all $j \in \{1, \ldots, n\}$.
The mapping $\mathrm{trans} : \Delta_\rho^{1/9} \to T_{(\delta,\rho)}$ is bijective, thus the elements

$$\mathrm{trans}^{-1}(d_1) = \delta_1, \ \ldots\ , \mathrm{trans}^{-1}(d_n) = \delta_n$$

are generators for $\Delta_\rho^{1/9}$. For any $\delta = \delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$ in $\Delta_\rho^{1/9}$ with $\sum_{j=1}^n |l_j| \cdot |d_j| \leq R - \zeta$, we have the estimate $|d - (l_1 d_1 + \cdots + l_n d_n)| \leq \frac{2c_{d_n}\rho}{\delta}\kappa \leq \varepsilon^{\frac{3}{8}}\rho$, which completes the proof. ■

The remaining sections consist of the construction of a nearby crystallographic group $G \subset E(n)$ and the embedding of $\Gamma_{\frac{\rho}{2^n}}$ into G using Thm. 4.3 twice.

12. Construction of a Lattice L in $\mathbb{R}^n$

Lem. 11.10 provides us with a basis of a lattice, whose inner part differs very slightly from the translational parts in the almost translational set $\Delta_\rho^{1/9}$.

Corollary 12.1. *There exists a lattice $L \subset \mathbb{R}^n$ and an embedding*

$$t : \{\alpha = (A, a) \in \Delta_\rho^{1/9} \mid |a| \leq \rho/2^n\} \longrightarrow L$$

such that $|a - t(\alpha)| \leq \varepsilon^{\frac{3}{8}}\rho$ for all $\alpha = (A, a) \in \Delta_\rho^{1/9}$ with $|\mathrm{trans}(\alpha)| \leq \frac{\rho}{2^n}$.

Proof. By Lem. 11.10 every element $\delta = (D, d) \in \Delta_\rho^{1/9}$ with translational part $|\mathrm{trans}(\delta)| \leq \frac{\rho}{2^n}$ can be written uniquely as $\delta = \delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$ with $|d - (l_1 d_1 + \cdots + l_n d_n)| \leq \varepsilon^{\frac{3}{8}}\rho$. Hence define a lattice $L \subset \mathbb{R}^n$ by

$$L = \{l_1 d_1 + \cdots + l_n d_n \mid l_1, \ldots, l_n \in \mathbb{Z}\},$$

i.e., L is a finitely generated free Abelian subgroup of $\mathbb{R}^n$.
The homomorphism t is indeed injective: let $\delta, \delta' \in \Delta_\rho^{1/9}$ with $\delta \neq \delta'$ and $|d|, |d'| \leq \rho/2^n$, then by Lem. 11.10 they have two different unique representations $\delta = \delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$ and $\delta' = \delta_1^{*l'_1} * \cdots * \delta_n^{*l'_n}$. Therefore $t(\delta) \neq t(\delta')$, since $\{d_1, \ldots, d_n\}$ is a basis of L. ∎

13. Construction of a Finite Group H in $O(n)$

In order to construct the crystallographic group G, which has to contain the slightly deformed lattice L in the translational lattice $G \cap \mathbb{R}^n$, use twice the fact that almost homomorphisms are near homomorphisms, cf. Thm. 4.3. First apply this technique to the orthogonal group $O(n)$ and then to the isometry group of the flat torus with injectivity radius at least π. To satisfy the assumptions of Thm. 4.3, for a moment choose the following bi-invariant distance function $d^{\measuredangle}_{O(n)}(A, B) = 2 \max\{|\measuredangle(A\,v, B\,v)| \mid v \in \mathbb{R}^n\}$ on $O(n)$ instead[3] of $\|\ .\ \|$. So define the map $r_0 : \mathcal{H} \to O(n)$ from the group of equivalence classes $\mathcal{H}$ to the orthogonal group by

$$r_0([\alpha]) = \mathrm{rot}(\alpha), \tag{20}$$

where $\alpha \in [\alpha]$ is a short representative in the equivalence class $[\alpha]$, i.e. an element with $|\mathrm{trans}(\alpha)| \leq 4r \cdot w_n$. The short representative is chosen arbitrarily but kept fixed in what follows. We get $d^{\measuredangle}_{O(n)}(r_0([\alpha] * [\beta]), r_0([\alpha]) \cdot r_0([\beta])) \leq 100 n^3 \varepsilon^{\frac{1}{2}} \leq \frac{\pi}{6}$:

Corollary 13.1. *The map $r_0 : \mathcal{H} \to O(n)$ defined in (20) is a q_1-almost homomorphism with the constant $q_1 = 100 n^3 \varepsilon^{\frac{1}{2}}$.*

Now apply Thm. 4.3 and conclude that there is a homomorphism $r : \mathcal{H} \to O(n)$ near the map r_0 which satisfies

$$d^{\measuredangle}_{O(n)}(r_0([\alpha], r([\alpha])) \leq 2q_1 = 200 n^3 \varepsilon^{\frac{1}{2}} \tag{21}$$

for all $[\alpha] \in \mathcal{H}$. The homomorphism r is injective: let $[\alpha_1] \neq [\alpha_2] \in \mathcal{H}$, thus the definition of the equivalence classes implies $d^{\measuredangle}_{O(n)}(r([\alpha_1], r([\alpha_2])) \geq \frac{4}{19}$. Therefore the homomorphism r is indeed a monomorphism.

Corollary 13.2. *The group $(\mathcal{H}, *)$ of order $|\mathcal{H}| \leq w_n$ is isomorphic to a subgroup $r(\mathcal{H}) = H$ of $O(n)$. Two different elements $A, B \in H$ have a pairwise distance $\|A^{-1}B\| \geq \frac{1}{10}$. Especially if $A \in H$ and $\|A\| < \frac{1}{10}$ then $A = I$.*

Now let us show that the definition of r is independent of the choice of the representatives: let α and α' be short representatives of $[\alpha]$. Then $\alpha \sim \alpha'$ mod

[3]Note that $\|A\| = 2\sin\left(\frac{1}{4} d^{\measuredangle}_{O(n)}(I, A)\right) \in [0, 2]$ and $d^{\measuredangle}_{O(n)}(I, A) = 4 \arcsin\left(\frac{1}{2}\|A\|\right) \in [0, 2\pi]$.

$\Delta_\rho^{1/9}$, i.e. $\alpha = \mu * \alpha'$ with $\mu \in \Delta_\rho^{1/9}$. Since r is a homomorphism and $r([\mu]) = I$ for $\mu \in [id]$ we obtain:

$$r([\alpha]) = r([\mu * \alpha']) = r([\mu] * [\alpha']) = r([\mu]) \cdot r([\alpha']) = r([\alpha']) \tag{22}$$

Thus the definition of the homomorphism r is independent of the choice of the representatives.

14. Adjusting the Lattice L

In the case where $r(\mathcal{H}) = H$ is the group consisting of all rotational parts of a crystallographic group, we know that each element $A \in H \subset O(n)$ acts on the translational lattice L such that $A \cdot L = L$. This is not necessarily true for H and L. Therefore, we will slightly deform a big enough neighbourhood of $0 \in L$ into a part of a lattice $\widehat{L}$, which is invariant by all elements of H. Then extend this part of the lattice linearly to $\widehat{L}$.

Lemma 14.1. *Let $c \in L$ with $|c| \leq \rho/2^{n+1}$ and $\alpha \in \Gamma_{\rho/3}$, i.e. $r([\alpha]) \in H$. Then there exists a unique $c' \in L$ with $|c'| \leq \rho/2^n$ and $|r([\alpha])c - c'| \leq 225n^3\varepsilon^{3/8}\rho$.*

We denote this unique element c' in the lattice L by $\widetilde{Ac}$.

Proof. Since $\{d_1, \dots, d_n\}$ is a basis of the lattice L we have $c = l_1 d_1 + \cdots + l_n d_n$ with unique coefficients $l_1, \dots, l_n \in \mathbb{Z}$. Set $\delta = \delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$, where $\delta_1, \dots, \delta_n$ are the generators of $\Delta_\rho^{1/9}$. Lem. 11.10 tells us that

$$|\text{trans}(\delta_1^{*l_1} * \cdots * \delta_n^{*l_n}) - (l_1 d_1 + \cdots + l_n d_n)| = |d - c| \leq \varepsilon^{\frac{3}{8}}\rho,$$

hence $|d| \leq |c| + \varepsilon^{\frac{3}{8}}\rho \leq \frac{\rho}{2^{n+1}} + \varepsilon^{\frac{3}{8}}\rho$. Look now at $\alpha * \delta * \overline{\alpha}$. The rotational part satisfies $\|\text{rot}(\alpha * \delta * \overline{\alpha})\| \leq \frac{1}{9}$ and the translational part $|\text{trans}(\alpha * \delta * \overline{\alpha})| \leq \frac{1}{2^n}\rho$. Therefore Lem. 11.10 is applicable to $\alpha * \delta * \overline{\alpha}$. Hence write $\alpha * \delta * \overline{\alpha} = \delta_1^{*l'_1} * \cdots * \delta_n^{*l'_n}$ with unique $l'_1, \dots, d'_n \in \mathbb{Z}$. In other words $c' = l'_1 d_1 + \cdots + l'_n d_n \in L$. Abbreviate $r_0([\alpha]) = \text{rot}(\alpha) = A$ and, using Lem. 11.10, estimate $|Ac - c'| \leq 25n^3\varepsilon^{\frac{3}{8}}\rho$. Hence using inequality (21) estimate $|r([\alpha]) \cdot c - c'| \leq 225n^3\varepsilon^{\frac{3}{8}}\rho$, which completes the proof. ■

Now apply the following construction to the inner part of the lattice L to obtain a slightly deformed lattice $\widehat{L}$. Let $b \in L$ with $|b| \leq \frac{\rho}{2^{n+1}}$ and $\widetilde{Ab} \in L$ as described in Lem. 14.1. Then define

$$\widehat{b} = \frac{1}{m} \sum_{A \in H} A^{-1} \widetilde{Ab}, \tag{23}$$

where $m = |H| = |\mathcal{H}| \leq w_n$. Now we are able to show that the set of all $\widehat{b}$ generates a lattice which is invariant under the group $H \subset O(n)$:

Lemma 14.2. *Let $b \in L$ with $|b| \leq \frac{\rho}{2^{n+1}}$. Then $|\widehat{b} - b| = 225n^3\varepsilon^{\frac{3}{8}}\rho$.*

Proof. Estimate the difference between $\widehat{b}$ and b

$$|\widehat{b}-b| \le \frac{1}{m}\sum_{A\in H}|A^{-1}\widetilde{Ab}-b| \le \frac{1}{m}\sum_{A\in H}|\widetilde{Ab}-Ab| \le 225n^3\varepsilon^{\frac{3}{8}}\rho,$$

using Lem. 14.1. ∎

We show that the map $\widehat{\ }: L \to \mathbb{R}^n$ is not only almost the identity but also $\mathbb{Z}$-linear. Lem. 14.2 implies that

$$\min\{|\widehat{d}| \mid \widehat{d} \in \widehat{L}-\{0\}\} \ge \delta - 225n^3\varepsilon^{\frac{3}{8}}\rho \ge \tfrac{\delta}{2},$$

since ε in assumption (A_6) is supposed to be small enough.

Lemma 14.3. *Let $b, c \in L$ with $|b|, |c|$ and $|b+c| \le \frac{\rho}{2^{n+1}}$. Then the map $\widehat{\ }: L \to \mathbb{R}^n$ is $\mathbb{Z}$-linear, i.e. $\widehat{b+c} = \widehat{b}+\widehat{c}$.*

Proof. There are unique elements $\widetilde{Ab}, \widetilde{Ac} \in L$ which satisfy $|Ab-\widetilde{Ab}|$ and $|Ac-\widetilde{Ac}| \le 225n^3\varepsilon^{\frac{3}{8}}\rho$, thus $|\widetilde{Ab}+\widetilde{Ac}-A(b+c)| \le 2\cdot 225n^3\varepsilon^{\frac{3}{8}}\rho$. By Lem. 14.1 a unique $(b+c)' \in L$ exists such that $|A(b+c)-(b+c)'| \le 225n^3\varepsilon^{\frac{3}{8}}\rho$. By uniqueness and $3\cdot 225n^3\varepsilon^{\frac{3}{8}}\rho \le \frac{\delta}{2}$, conclude $\widetilde{Ab}+\widetilde{Ac} = (A(b+c))^{\sim}$. Therefore the claim follows using the linearity in (23). ∎

In other words $\widehat{L}$ is a lattice which is spanned by the image under the map $\widehat{\ }: L \to \mathbb{R}^n$ of the inner part of L. Next we show that $\widehat{L}$ is in fact invariant under all $A \in H$.

Lemma 14.4. *Let $A \in H$ and $\widehat{b} \in \widehat{L}$ with $|\widehat{b}| \le \frac{\rho}{2^{n+1}}$. Then $A\widehat{b} = \widehat{\widetilde{Ab}} \in \widehat{L}$.*

Proof. We have

$$\begin{aligned} A\widehat{b} &= \frac{1}{m}\sum_{B\in H} AB^{-1}\widetilde{Bb} = \frac{1}{m}\sum_{B\in H}(BA^{-1})^{-1}(BA^{-1}Ab)^{\sim} \\ &= \frac{1}{m}\sum_{B\in H}(BA^{-1})^{-1}(BA^{-1}\widetilde{Ab})^{\sim} = \frac{1}{m}\sum_{C\in H} C^{-1}(C\widetilde{Ab})^{\sim} = \widehat{\widetilde{Ab}} \in \widehat{L}. \end{aligned}$$ ∎

The inner part of the lattice $\widehat{L}$, in other words all $\widehat{b} \in \widehat{L}$ with $|\widehat{b}| \le \frac{\rho}{2^{n+1}}$, is invariant under all $A \in H$. Since H acts linearly on the left it follows that the lattice $\widehat{L}$ is invariant under H.

15. Replacing each Element in $\mathcal{H}$ by one in $E(n)$

In this section we replace each element $[\alpha] \in \mathcal{H}$ by a certain element $(r([\alpha]), a^\star) \in E(n)$. To do so we distinguish two cases depending on weather $r([\alpha])$ is the identity I in the orthogonal group $O(n)$.

(i) If $r([\alpha]) = I$ then we take $a^\star = \widehat{a} = (\mathrm{trans}(\alpha))^{\widehat{\ }}$ as explained in Sec. 14.

(ii) If $r([\alpha]) \ne I$ then we have to properly adjust the translational parts. This again will be done with Thm. 4.3. This time, instead of the orthogonal group, we use the isometry group $M = \mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ of the flat torus $\mathbb{R}^n/\widehat{L}$:

The flat torus: We investigate the flat torus $\mathbb{R}^n/\widehat{L}$. The shortest closed geodesic on $\mathbb{R}^n/\widehat{L}$ is only known to be longer than $\min\{|\widehat{d}_1|,\ldots,|\widehat{d}_n|\}-225n^3\varepsilon^{\frac{3}{8}}\rho$. Therefore the injectivity radius is $\mathrm{inj}(\mathbb{R}^n/\widehat{L}) \geq \frac{1}{2}\min\{|\widehat{d}_1|,\ldots,|\widehat{d}_n|\} - \frac{225}{2}n^3\varepsilon^{\frac{3}{8}}\rho \geq \frac{\delta}{4}$, where $\widehat{d}_1,\ldots,\widehat{d}_n$ are the generators of the lattice $\widehat{L}$. Notice that $\{d_1,\ldots,d_n\}$ is a λ-normal basis of L with $\lambda \leq 2$. The diameter of the flat torus satisfies

$$\mathrm{inj}(\mathbb{R}^n/\widehat{L}) \leq \mathrm{diam}(\mathbb{R}^n/\widehat{L}) \leq \tfrac{1}{2}(|\widehat{d}_1|^2+\cdots+|\widehat{d}_n|^2)^{\frac{1}{2}} \leq \sqrt{n}\sigma \leq \rho.$$

The isometry group of the flat torus: The group of Euclidean motions $E(n)$ acts on $\mathbb{R}^n$ from the left $E(n)\times\mathbb{R}^n \to \mathbb{R}^n$ by $(\alpha,x)\mapsto \alpha(x) = A\,x + a$. The normaliser of the lattice $\widehat{L}\in\mathbb{R}^n$ in the group $E(n)$ is defined by the expression $\mathrm{Norm}_{E(n)}(\widehat{L}) = \{\alpha\in E(n) \mid (\alpha,\widehat{L}) = (\widehat{L},\alpha)\}$, thus $\widehat{L}\triangleleft\mathrm{Norm}_{E(n)}(\widehat{L})$. The isometry group of the flat torus $\mathbb{R}^n/\widehat{L}$ is known to be $\mathrm{Iso}(\mathbb{R}^n/\widehat{L}) = \mathrm{Norm}_{E(n)}(\widehat{L})/\widehat{L}$. Let $U\in O(n)$ be the maximal group which leaves invariant the lattice $\widehat{L}$. We have $H\subseteq U$. Since U is a discrete subgroup in the compact group $O(n)$ the number $\theta = \min\{d^{\measuredangle}_{O(n)}(A,B) \mid A,B\in U \text{ with } A\neq B\}$ exists and is bounded away from zero. The isometry group of the flat torus can be represented by the compact Lie group

$$\mathrm{Iso}(\mathbb{R}^n/\widehat{L}) = \{\alpha\widehat{L} = (A, a+\widehat{L}) \mid A\in U \text{ and } a\in\mathbb{R}^n\}.$$

The group structure in the compact Lie group $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ is given by:

neutral element:	$id\,\widehat{L} = (I,\widehat{L})$
inverse:	$(\alpha\widehat{L})^{-1} = (A,a+\widehat{L})^{-1} = (A^{-1},-A^{-1}a+\widehat{L})$
multiplication:	$\alpha\widehat{L}\cdot\beta\widehat{L} = (A,a+\widehat{L})\cdot(B,b+\widehat{L}) = (AB,Ab+a+\widehat{L})$

Topologically $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ is a compact disjoint sum of $k=|U|$ flat tori $\mathbb{R}^n/\widehat{L}$.

Bi-invariant norm on the isometry group of the flat torus: Now we equip $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ with the flat metric of $\mathbb{R}^n/\widehat{L}$ on the identity component and by left translation on all other components. Our norm has to be such that the injectivity radius is at least π, therefore multiply the distance function on $\mathbb{R}^n/\widehat{L}$ by $2\pi/\mathrm{inj}(\mathbb{R}^n/\widehat{L})$. Let $\alpha\widehat{L}$ be an isometry in $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$. Then define its norm by

$$\|\alpha\widehat{L}\|_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})} = \tfrac{2\pi}{\mathrm{inj}(\mathbb{R}^n/\widehat{L})}\max\left\{\tfrac{\mathrm{diam}(\mathbb{R}^n/\widehat{L})}{\theta}\,d^{\measuredangle}_{O(n)}(I,A)\,,\min\{|\widehat{d}-a| \mid \widehat{d}\in\widehat{L}\}\right\}.$$

This norm induces a bi-invariant distance function on the isometry group of the flat torus:

Lemma 15.1. *Let $\alpha\widehat{L},\beta\widehat{L}\in\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$. Then*

$$d_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})}(\alpha\widehat{L},\beta\widehat{L}) = \|(\alpha\widehat{L})^{-1}\cdot\beta\widehat{L}\|_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})}$$

is bi-invariant.

Proof. The left-invariance of the distance function d_{Iso} follows from the definition. To prove the right-invariance of d_{Iso} we distinguish two cases: First consider isometries $\alpha\widehat{L}$ and $\beta\widehat{L}$ which are in the same component of $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$, in other words the rotational parts A and B are equal. Now let $\gamma\widehat{L}$ be any element in $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$. Then

$$\begin{aligned}
&d_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})}(\alpha\widehat{L}\cdot\gamma\widehat{L},\beta\widehat{L}\cdot\gamma\widehat{L})\\
&=\tfrac{2\pi}{\mathrm{inj}}\max\left\{\tfrac{\mathrm{diam}}{\theta}\,d^{\measuredangle}_{O(n)}(A\,C,B\,C)\,,\min\{|\widehat{d}-((Ac+a)-(Bc+b)|\mid\widehat{d}\in\widehat{L}\}\right\}\\
&=\tfrac{2\pi}{\mathrm{inj}}\max\left\{\tfrac{\mathrm{diam}}{\theta}\,d^{\measuredangle}_{O(n)}(A,B)\,,\min\{|\widehat{d}-(a-b)|\mid\widehat{d}\in\widehat{L}\}\right\}\\
&=d_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})}(\alpha\widehat{L},\beta\widehat{L}).
\end{aligned}$$

Secondly the isometries $\alpha\widehat{L}$ and $\beta\widehat{L}$ are supposed to be in different components, in other words the rotational parts A and B are not equal. Discreteness of the rotational subgroup U gives $\theta\le d^{\measuredangle}_{O(n)}(A,B)=d^{\measuredangle}_{O(n)}(A\,C,B\,C)$. Let $\gamma\widehat{L}$ be any element in $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$. Then compactness of the torus $\mathbb{R}^n/\widehat{L}$ implies

$$\min\{|\widehat{d}-((Ac+a)-(Bc+b)|\mid\widehat{d}\in\widehat{L}\}\le\tfrac{\mathrm{diam}}{\theta}\,d^{\measuredangle}_{O(n)}(A\,C,B\,C).$$

Hence the rotational part dominates the translational part in the maximum. So the bi-invariance of the distance function $d^{\measuredangle}_{O(n)}(I,\,\cdot\,)$ on $O(n)$ implies the right-invariance of d_{Iso} also in this second case. Notice that the compactness of the torus and discreteness of the rotational subgroup are crucial in the above reflections. ■

With this new distance function a shortest closed geodesic in $\mathbb{R}^n/\widehat{L}$ is longer than 2π, hence the injectivity radius of the exponential map is at least π. Since $\frac{2\pi}{\theta}\frac{\mathrm{diam}}{\mathrm{inj}}\ge 1$, it follows that the distance function d_{Iso} on the isometry group of the flat torus satisfies the assumptions of Thm. 4.3. Therefore it is time to apply Thm. 4.3 with $M=\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ and the above distance function: for each $[\alpha]\in\mathcal{H}$ we take the short representative $\alpha=(A,a)$ in the equivalence class of $[\alpha]$ which we selected in Sec. 13. Then we define the map $\omega_0:\mathcal{H}\to\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ by

$$\omega_0([\alpha])=(r([\alpha]),a+\widehat{L}).\tag{24}$$

A sufficiently good estimation gives $d_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})}(\omega_0([\alpha]*[\beta]),\omega_0([\alpha])\cdot\omega_0([\beta]))\le\varepsilon^{\frac{1}{4}}$:

Corollary 15.2. *The map $\omega_0:\mathcal{H}\to\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ defined in* (24) *is a q_2-almost homomorphism with the constant $q_2=\varepsilon^{1/4}$.*

We apply Thm. 4.3 and get: there is a homomorphism $\omega:\mathcal{H}\to\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ near ω_0, i.e., for all $[\alpha]\in\mathcal{H}$ we have

$$d_{\mathrm{Iso}(\mathbb{R}^n/\widehat{L})}\,(\omega_0([\alpha]),\omega([\alpha]))\le 2q_2=2\,\varepsilon^{\frac{1}{4}}.\tag{25}$$

Furthermore $\mathrm{rot}(\omega([\alpha]))=\mathrm{rot}(\omega_0([\alpha]))$, which is even equal to $r([\alpha])$: indeed, if the rotational parts were in different components of the isometry group of the flat torus, i.e. $\theta<d^{\measuredangle}_{O(n)}(\mathrm{rot}(\omega_0([\alpha])),\mathrm{rot}(\omega([\alpha])))$, then this would contradict the estimation (25).

We can even show more: the homomorphism $\omega:\mathcal{H}\to\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ is an embedding, i.e., it is injective: let $[\alpha_1]\ne[\alpha_2]\in\mathcal{H}$ and so the definition of the equivalence relation implies that $d^{\measuredangle}_{O(n)}(r([\alpha_1]),r([\alpha_2]))>0$, so $\omega([\alpha_1])$ and $\omega([\alpha_2])$ are in different components of $\mathrm{Iso}(\mathbb{R}^n/\widehat{L})$, hence are different.

Now we show that the definition of ω is independent of the choice of the representatives: let α and α' be short representatives of $[\alpha]$. Then $\alpha \sim \alpha' \mod \Delta_\rho^{1/9}$, i.e. $\alpha = \mu * \alpha'$ with $\mu \in \Delta_\rho^{1/9}$. Since ω is a homomorphism and $\omega([\mu]) = id\,\widehat{L}$ for $\mu \in [id]$ we obtain:

$$\omega([\alpha]) = \omega([\mu * \alpha']) = \omega([\mu] * [\alpha']) = \omega([\mu]) \cdot \omega([\alpha']) = id\,\widehat{L} \cdot \omega([\alpha']) = \omega([\alpha']) \quad (26)$$

Thus the definition of the homomorphism ω is independent of the choice of the representatives.

Set $\omega([\alpha]) = (r([\alpha]), \mathrm{trans}(\omega([\alpha])) + \widehat{L})$. Define the map $f : \omega(\mathcal{H}) \subset \mathrm{Iso}(\mathbb{R}^n/\widehat{L}) \to E(n)$ given by $\omega([\alpha]) \mapsto (r([\alpha]), a^\star)$, where $a^\star \in \mathrm{trans}(\omega([\alpha])) + \widehat{L}$ is the element which is closest to the translational part a of the short representative α of the equivalence class $[\alpha]$. The map f lifts $\omega(\mathcal{H})$ to the isometry group $E(n)$, i.e., every element in $\omega(\mathcal{H})$ is considered as an Euclidean motion with a special choice of its translational part. Equation (25) implies:

$$|a - \mathrm{trans} \circ f \circ \omega([\alpha])| = |a - a^\star| = 2\, q_2\, \tfrac{\mathrm{inj}}{\pi} \leq \tfrac{1}{\pi}\, \varepsilon^{\frac{1}{4}} \rho \quad (27)$$

Thus for all $[\alpha] \in \mathcal{H}$ the translational part of the short representative α of $[\alpha]$ is only very slightly changed under the map $f \circ \omega$.

16. Generating the Crystallographic Group G in $E(n)$

We define the subgroup $G \subset E(n)$ generated by the lattice $\widehat{L}$ and the finite group $f \circ \omega(\mathcal{H}) = \{f \circ \omega([\alpha]) \mid [\alpha] \in \mathcal{H}\}$. We obtain

$$G = \{(r([\alpha]), a^\star + d) \in E(n) \mid [\alpha] \in \mathcal{H} \text{ and } d \in \widehat{L}\}.$$

By Cor. 15.2 and Thm. 4.3 the map $\omega : \mathcal{H} \to \mathrm{Iso}(\mathbb{R}^n/\widehat{L})$ is a homomorphism. In other words, for all $[\alpha], [\beta] \in \mathcal{H}$ we have

$$\omega([\alpha]^{-1}) = (\omega([\alpha]))^{-1} = ((r([\alpha]))^{-1}, -(r([\alpha]))^{-1} a^\star + \widehat{L}) \quad (28)$$

$$\omega([\alpha] * [\beta]) = \omega([\alpha]) \cdot \omega([\beta]) = (r([\alpha]) \cdot r([\beta]), r([\alpha]) \cdot b^\star + a^\star + \widehat{L}), \quad (29)$$

thus $(r([\overline{\alpha}]), -r([\overline{\alpha}]) \cdot a^\star)$ and $(r([\alpha * \beta]), r([\alpha]) \cdot b^\star + a^\star)$ are elements of G. The set G together with the usual product in $E(n)$ is a group which contains $\widehat{L}$ and $f \circ \omega(\mathcal{H})$: the neutral element $(I, 0)$ is in G. Furthermore let $(r([\alpha]), a^\star + d), (r([\beta]), b^\star + d') \in G$. Then investigate the inverse of $(r([\alpha]), a^\star + d) \in E(n)$:

$$((r([\alpha]))^{-1}, -(r([\alpha]))^{-1} \cdot a^\star - (r([\alpha]))^{-1} \cdot d) = (r([\overline{\alpha}]), -r([\overline{\alpha}]) \cdot a^\star - r([\overline{\alpha}]) \cdot d)$$

And then investigate the product of $(r([\alpha]), a^\star + d)$ and $(r([\beta]), b^\star + d') \in E(n)$:

$$(r([\alpha]) \cdot r([\beta]), r([\alpha]) \cdot b^\star + a^\star + r([\alpha]) \cdot d' + d) = (r([\alpha * \beta]), r([\alpha]) \cdot b^\star + a^\star + r([\alpha]) \cdot d' + d)$$

Since $\omega(\mathcal{H}) = H$ leaves invariant the lattice $\widehat{L}$ we obtain that $-r([\overline{\alpha}]) \cdot d$ and $r([\alpha]) \cdot d' + d$ are elements in the lattice $\widehat{L}$. Therefore (28) and (29) imply that the inverse and the product are in G. Thus G is generated by $\widehat{L}$ and $f \circ \omega(\mathcal{H})$.

Lemma 16.1. *The group G generated by $\widehat{L}$ and $f \circ \omega(\mathcal{H})$ is a crystallographic group with normal lattice $\widehat{L}$ and rotational group $\omega(\mathcal{H}) = H$.*

Proof. The lattice $\widehat{L}$ generated by n linearly independent vectors $\widehat{l_1}, \dots, \widehat{l_n}$ is a free Abelian subgroup of G, cf. Lem. 14.3. In addition, the subgroup $\widehat{L}$ is normal in G: Indeed, let $(r([\alpha]), a^\star + d)$ be any element in G. The product

$$(r([\alpha]), a^\star + d) \cdot (I, d') \cdot (r([\alpha]), a^\star + d)^{-1} = (I, r([\alpha]) \cdot d')$$

is in $\widehat{L}$ for all elements $(I, d') \in \widehat{L}$, since the finite group $\omega(\mathcal{H})$ leaves invariant the lattice $\widehat{L}$, cf. Lem. 14.4. The index of $\widehat{L}$ in G is bounded by $[\widehat{L} : G] = |H| = |\omega(\mathcal{H})| \leq w_n = (1 + 20\sqrt{n})^{n^2}$, cf. Lem. 9.4 (c). Thus G is a discrete group with compact fundamental domain and therefore a crystallographic group with lattice $\widehat{L}$ and rotational group H. ∎

17. Embedding $\Gamma_{\rho/2^{n+1}}$ into the Crystallographic Group G

The set $\Gamma_{\rho/3}$ is partitioned into equivalence classes by the equivalence relation $\sim$ mod $\Delta_\rho^{1/9}$. Therefore every $\gamma \in \Gamma_{\rho/2^{n+1}}$ is a product $\gamma = \delta * \alpha$, where α is the chosen short representative of $[\gamma]$ and δ an element in the almost translational set $\Delta_\rho^{1/9}$ with $|d| \leq \rho/2^n$, i.e. $\delta = \gamma * \overline{\alpha} \in \Delta_\rho^{1/9}$. Therefore define the map $\Phi : \Gamma_{\rho/2^{n+1}} \to G$ by

$$\gamma = \delta * \alpha \longmapsto (I, l_1\widehat{d_1} + \dots + l_n\widehat{d_n}) \cdot f \circ \omega([\alpha]) = (r([\alpha]), a^\star + l_1\widehat{d_1} + \dots + l_n\widehat{d_n}).$$

In what follows we want to derive several properties of this map Φ:

Lemma 17.1. (Statement (2) of Thm. 2.5) *The map $\Phi : \Gamma_{\rho/2^{n+1}} \to G$ is an embedding with $d_{E(n)}(\gamma, \Phi(\gamma)) \leq \varepsilon^{\frac{1}{4}}$ for all $\gamma \in \Gamma_{\rho/2^{n+1}}$.*

Proof. Notice that every $\delta \in \Delta_\rho^{1/9}$ with $|d| \leq \rho/2^n$ has a unique representation $\delta = \delta_1^{*l_1} * \dots * \delta_n^{*l_n}$, cf. Lem. 11.10. Furthermore, it is well defined in the essential crystallographic set of isometries Γ_R if $\sum_{j=1}^n |l_j| \cdot |d_j| \leq R - \zeta \leq 2c_{d_n}\rho$. Since all non-trivial elements in $\Delta_\rho^{1/9}$ have a translational part which is bigger than δ, we can conclude that $\delta \cdot \sum_{j=1}^n |l_j| \leq 2c_{d_n}\rho$. We want to estimate $d_{E(n)}(\gamma, \Phi(\gamma))$, first the rotational part and then the translational part: For the rotational part we have

$$\|(\mathrm{rot}(\gamma))^{-1} \cdot \mathrm{rot}(\Phi(\gamma))\| \leq 131 n^3 \varepsilon^{\frac{1}{2}}, \tag{30}$$

using Lem. 10.1 and estimation (21). Secondly using Lem. 10.1 and estimation (27) we consider the translational part

$$|\mathrm{trans}(\gamma) - \mathrm{trans}(\Phi(\gamma))| \leq 9\,\rho\,\varepsilon^{\frac{1}{4}}. \tag{31}$$

Since ε in assumption (A_6) is supposed to be small enough the estimations for the rotational part and translational part now imply $d_{E(n)}(\gamma, \Phi(\gamma)) \leq \varepsilon^{\frac{1}{4}}$. This implies the statement (2) of Thm. 2.5.

In addition we show that $\Phi : \Gamma_{\rho/2^{n+1}} \to G$ is an embedding: indeed, we assume that $\Phi(\gamma_1) = \Phi(\gamma_2)$ for any two elements $\gamma_1, \gamma_2 \in \Gamma_{\rho/2^{n+1}}$. Let us abbreviate $\gamma_i = (C_i, c_i)$ and $\Phi(\gamma_i) = (\hat{C}_i, \hat{c}_i)$ for $i \in \{1, 2\}$. Using (30) and (31) we estimate $\|C_1^{-1}C_2\| \leq \mu$ and $|c_1 - c_2| \leq \delta$. Therefore Lem. 6.1 implies that $\gamma_1 = \gamma_2$, hence Φ is injective. ∎

Lemma 17.2. (Statement (1) of Thm. 2.5) *The map* $\Phi : \Gamma_{\rho/2^{n+1}} \to G$ *is a homomorphism in the following sense: let* γ_1 *and* γ_2 *be elements in* $\Gamma_{\rho/2^{n+1}}$ *with* $|c_1| + |c_2| \leq \rho/2^{n+1}$. *Then* $\Phi(\gamma_1 * \gamma_2) = \Phi(\gamma_1) \cdot \Phi(\gamma_2)$ *and* $\Phi(id) = id$.

Proof. We abbreviate $\gamma_i = (C_i, c_i)$ and $\Phi(\gamma_i) = (\hat{C}_i, \hat{c}_i)$ for $i \in \{1, 2\}$. Using Cor. 13.2 and estimation (31) we can give the analogue of property (III) in Def. 2.3 for elements in G: If $\Phi(\gamma_1), \Phi(\gamma_2) \in G$ with $\Phi(\gamma_1) \neq \Phi(\gamma_2)$ satisfy $\|\hat{C}_1^{-1}\hat{C}_2\| \leq \frac{1}{10}$ then $|\hat{c}_1 - \hat{c}_2| > \delta - 18\,\rho\,\varepsilon^{\frac{1}{4}}$. We estimate:

$$\|(\mathrm{rot}(\Phi(\gamma_1) \cdot \Phi(\gamma_2)))^{-1} \cdot \mathrm{rot}(\Phi(\gamma_1 * \gamma_2))\| \leq \tfrac{1}{10}$$
$$|\mathrm{trans}(\Phi(\gamma_1) \cdot \Phi(\gamma_2)) - \mathrm{trans}(\Phi(\gamma_1 * \gamma_2))| \leq \delta - 18\,\rho\,\varepsilon^{\frac{1}{4}}$$

Hence $\Phi(\gamma_1 * \gamma_2) = \Phi(\gamma_1) \cdot \Phi(\gamma_2)$. This completes the proof of Thm. 2.5 (1). ■

Lemma 17.3. (Statement (3) of Thm. 2.5) *There exists for every element in the group* G *with translational part smaller than* $\rho/2^{n+1} - 9\,\rho\,\varepsilon^{\frac{1}{4}}$ *a corresponding Euclidean motion in* $\Gamma_{\rho/2^{n+1}}$. *In other words*

$$G \cap \{(A, a) \in E(n) \mid |a| \leq \tfrac{\rho}{2^{n+1}} - 9\,\rho\,\varepsilon^{\frac{1}{4}}\} \subseteq \Phi(\Gamma_{\rho/2^{n+1}}).$$

Proof. Let $(r([\alpha]), a^\star + d)$ be an element in G with $|a^\star + d| \leq \rho/2^{n+1} - 9\,\rho\,\varepsilon^{\frac{1}{4}}$. We take the chosen short representative $\alpha \in [\alpha]$. Now we represent $d \in \widehat{L}$ uniquely in the basis $\{\widehat{d}_1, \ldots, \widehat{d}_n\}$ of the lattice $\widehat{L}$, i.e. $d = l_1\widehat{d}_1 + \cdots + l_1\widehat{d}_n$. Set $\delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$ where $\delta_1, \ldots, \delta_n$ are the generators for $\Delta_\rho^{1/9}$. Using (27) and (31) we estimate $|\mathrm{trans}(\delta_1^{*l_1} * \cdots * \delta_n^{*l_n})| \leq \frac{\rho}{2^n}$. Therefore, applying Lem. 11.10 tells us that the representation $\delta_1^{*l_1} * \cdots * \delta_n^{*l_n}$ is unique. We have $|\mathrm{trans}(\delta_1^{*l_1} * \cdots * \delta_n^{*l_n} * \alpha)| \leq |a^\star + d| + 9\,\rho\,\varepsilon^{\frac{1}{4}} \leq \rho/2^{n+1}$. So we find the unique element $\delta_1^{*l_1} * \cdots * \delta_n^{*l_n} * \alpha$ in $\Gamma_{\rho/2^{n+1}}$ that corresponds to $(r([\alpha]), a^\star + d)$ under the map Φ. This finally completes the proof of Thm. 2.5 (3). ■

18. Conclusion

Lem. 16.1 shows that G is a crystallographic group. Furthermore, Lem. 17.2, Lem. 17.1 and Lem. 17.3 settle the statements (1) to (3) of Thm. 2.5, and equations (22) and (26) show that the whole construction of G and $\Phi : \Gamma_{\rho/2^{n+1}} \to G$ does not depend on the choice of the short representatives. This finishes the proof.

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