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On Compactification Lattices of Subsemigroups of SL(2,$\mathbb{R}$)

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Dedicated to Karl Heinrich Hofmann on the occasion of his 70^{th} *birthday*

Abstract. Using the tools introduced in [2] we investigate topological semigroup compactifications of closed connected submonoids with dense interior of $\mathrm{Sl}(2,\mathbb{R})$. In particular, we show that the growth of such a compactification is always contained in the minimal ideal, and describe the subspace of all minimal idempotents (typically a two-cell) and the maximal subgroups (these are always isomorphic with a compactification of $\mathbb{R}$). For a large class of such semigroups we give explicit constructions yielding all possible topological semigroup compactifications and determine the structure of the compactification lattice.

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1. Introduction

Let S be a topologized semigroup and $\kappa: S \to S^\kappa$ a continuous homomorphism into a compact (Hausdorff) topological semigroup S^κ. If $\kappa(S)$ is dense in S^κ then the pair (S^κ, κ) is said to be a *topological semigroup compactification* of S. Given a class $\mathcal{S}$ of such semigroups S we are faced with the following basic tasks:

(i) For any semigroup $S \in \mathcal{S}$ find, up to equivalence, all topological semigroup compactifications, if possible, give explicit constructions. In particular, give a construction yielding the universal topological semigroup compactification, the *Bohr compactification* of S. (This universal object always exists, by the Adjoint Functor Theorem.)

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(ii) With every compact topological semigroup go various special objects: the set of idempotents, the maximal subgroups, the minimal ideal and its subobjects. Describe these for the compactifications of S and relate them to interesting objects connected with S and its structural features.

(iii) It is not difficult to see that the topological semigroup compactifications of a fixed semigroup S form a lattice. Describe the structure of this lattice in terms of "known" lattices.

Success in carrying out this program should benefit the structure theory as well as the harmonic analysis on $\mathcal{S}$. In the following two well known examples the above questions can be answered in a very complete and satisfactory manner:

A. The class of all abelian locally compact topological groups. (Cf., e.g., HEWITT-ROSS [10], p.430ff) Here we have the following simple construction: given an abelian locally compact topological group G we pick any subgroup H_d of the discretization $(\hat{G})_d$ of the dual group $\hat{G}$ and let $G^\kappa = (H_d)\hat{}$. Then G^κ is compact and there is a natural morphism $\kappa\colon G \to G^\kappa$ with dense image. Conversely, every topological semigroup compactification of G can be obtained in this way, and the lattice of all such compactifications is isomorphic with the lattice of all subgroups of $(\hat{G})_d$. The Bohr compactification corresponds to the full group $(\hat{G})_d$. (The objects addressed in (ii) are trivial in the case of groups.)

B. The class of all convex cones in a finite dimensional real vector space. Explicit constructions yielding the Bohr compactification C^b of a finite dimensional (closed) cone C have been given by M. FRIEDBERG [8,9] and, later on, by one of the authors [16]. It has been shown that the idempotents of the Bohr compactification of a finite dimensional cone C are in one-to-one correspondence with the faces of the dual cone C^* and that the 'accessible idempotents' (those in the closure of a one parameter subsemigroup) in the growth of the compactification correspond exactly to the exposed faces of C^*. We cannot enter here into the details of the compactification lattice of C, this lattice can be described in terms of the lattice of subcones of the dual cone and the compactification lattices of finite dimensional vector spaces over $\mathbb{R}$. To give the reader a rough idea of what is going on we only remark that the $\mathcal{H}$-reduced compactifications of C are in one-one correspondence to the subcones of C^*. Similar to the situation in case A the key element in the discussion is the existence of a separating family of semicharacters on C. (A general exposition of various relations between compactifications and dual objects is planned for a forthcoming paper.)

The natural nonabelian analogues of cones in real vector spaces are the closed divisible subsemigroups of connected Lie groups, these are exactly the exponential Lie semigroups. The exponential Lie semigroups are classified in the memoir [12], where also a fairly complete description of their structure is given. A reduced exponential Lie semigroup S (i.e., an exponential Lie semigroup S containing no nontrivial normal subgroup of the Lie group generated by S) can always be dissected into (1) a direct product of one or more exponential Lie semigroups living in $\mathrm{Sl}(2,\mathbb{R})$ and (2) an exponential Lie semigroup living in a group which is an almost direct product of a centerfree diagonally metabelian Lie group with a covering group of a compact Lie group. The compactifications of divisible semigroups of type (2) are quite amenable, special cases such as the "affine triangle" have been studied for decades (cf., e.g., [6],[13],[14], [17]).

Thus the first step for carrying out our program (i)–(iii) for divisible subsemigroups of Lie groups is to investigate the topological semigroup compactifications of divisible subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$. It soon turns out, however, that for this task we need a very detailed knowledge of general structural features such as asymptotic behavior and the rectangular structure of $\mathrm{Sl}(2,\mathbb{R})$ [2], perfectness and aliens [3], congruences in subsemigroups of Lie groups [4]. Also, the appropriate methods and ideas are developped best in a context as general as might be presumed, namely in the class of closed connected proper submonoids S of $\mathrm{Sl}(2,\mathbb{R})$ with dense interior. In the rest of the introduction we now always assume that S belongs to this class.

The central and most general result of these notes is that for every compactification (S^κ,κ) of S we have (a) $S^\kappa=\kappa(S)\cup M(S^\kappa)$, (b) there is a continuous and surjective map ε assigning to every asymptotic direction of S (that is, to every element in the asymptotic rectangular band) a minimal idempotent of S^κ, and (c) all maximal subgroups of S^κ contain a dense homomorphic image of $\mathbb{R}$. If we mod out the maximal subgroups of S^κ then ε becomes a homomorphism, and there are only four possibilities for the kernel congruence.

In the generic case S is contained in a perfect exponential semigroup, hence is perfect itself. For this case we give an explicit construction yielding all possible topological semigroup compactifications of S, we show that the idempotents in the compactifications correspond to the asymptotic directions in S, and we describe the compactification lattice of S as the union of copies of three well known lattices: the lattice of all compactifications of $\mathbb{R}$, the four element diamond lattice, and the lattice formed by the empty set and the closed ideals of S.

Similar results hold if S is not perfect, in this case the compactification lattice is isomorphic with the lattice of all closed ideals containing the alien elements of S. These two cases cover already all divisible subsemigroups S.

In the remaining case, where S itself, but not its exponential hull, is perfect, we still can give fairly detailed information, but one important detail is missing: we do not know whether the above mapping ε can be injective (and if so, why). Up to now we only have examples for such semigroups where the minimal ideal of the Bohr compactification is singleton. (Nevertheless: Under a slight restriction it can be shown that every maximal subgroup in a topological semigroup compactification of such a semigroup S must be trivial.)

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2. A topological Rees extension

2.1. The aim of this section is to introduce a generalization of the usual one point compactification, where instead of a zero element we attach a more general compact semigroup, which acts as an ideal. (A variant of this construction was given already in [15], see 4.11 on page 35–36.) Clearly, this construction will yield a compact topological semigroup only if S is perfect, that is, its multiplication is a *perfect mapping* (a closed continuous map with compact fibers; cf. [7] p.236).

Since every semigroup acts on each of its ideals by left and right translations we have to consider two-sided semigroup actions.

2.2. Actions of semigroups on semigroups. Let X, Y be semigroups. Then a map $\alpha\colon (X \times Y) \cup (Y \times X) \to Y$, $(a, b) \mapsto ab$, is said to be a *two-sided semigroup action of X on Y*, or an *s-action* for short, if the associativity rules

$$(x_1x_2)y = x_1(x_2y) \text{ and } y(x_1x_2) = (yx_1)x_2$$
$$x_1(yx_2) = (x_1y)x_2 \text{ and } (y_1x)y_2 = y_1(xy_2)$$
$$x(y_1y_2) = (xy_1)y_2 \text{ and } y_1(y_2x) = (y_1y_2)x$$

hold for any choice of $x_1, x_2, x \in X$ and $y_1, y_2, y \in Y$.

If X contains an identity $\mathbf{1}$ then we require, in addition, that $\mathbf{1}y = y\mathbf{1} = y$, for all $y \in Y$. If X and Y are topological semigroups then our s-action will be assumed to be continuous.

Clearly, the restriction λ $[\mu]$ of α to $X \times Y$ $[Y \times X]$ is a left [right] action of X on Y.

2.3. Example. (i) Let G_ℓ, G_r be closed subgroups of a topological group G, write L for the homogeneous space G/G_ℓ of left cosets gG_ℓ, $g \in G$, and R for the homogeneous space $G_r\backslash G$ of right cosets G_rg, $g \in G$. Then the product space $Y = L \times R$ is a topological semigroup with respect to the rectangular multiplication $(x, y)(x', y') = (x, y')$, and the natural actions

$$G \times Y \to Y,\ (g, (g'G_\ell, G_rg'')) \mapsto (gg'G_\ell, G_rg''),$$
$$Y \times G \to Y,\ ((g'G_\ell, G_rg''), g) \mapsto (g'G_\ell, G_rg''g)$$

combine to an s-action of G on Y (which is jointly continuous).

(ii) If X is a subsemigroup of G and $Y_1 \subseteq G/G_\ell$, $Y_2 \subseteq G_r\backslash G$ with $XY_1 \subseteq Y_1$ and $Y_2X \subseteq Y_2$ then the s-action of G on Y defined in (i) restricts naturally to an s-action of X on $Y_1 \times Y_2$.

In the next section we shall use an example of type 2.3(ii) in the case where X is a subsemigroup of $G = \mathrm{Sl}(2,\mathbb{R})$, and G_ℓ $[G_r]$ is the subgroup of all upper [lower] triangular matrices.

2.4. Remark. (i) Every s-action of X on Y defines, in the obvious way, an associative multiplication on the disjoint union $Z = X \cup Y$. Pick $a, b \in X \cup Y$. If both a and b belong to either X or Y then their product is defined by the multiplication already given on X or Y, respectively; otherwise it is defined by the s-action of X on Y. Note that in the so defined semigroup S the subset Y is an ideal.

(ii) If X and Y are topological semigroups and the s-action is continuous then the semigroup Z of (i) will be a topological semigroup if it is provided with the sum topology. In order to make Z a compact semigroup we have, however, to use a coarser topology, even if we assume that Y is compact. In the present paper we get this coarser topology with the aid of a 'gluing map' $\varphi: X \to Y$, which is asymptotically an equivariant homomorphism. To express this general idea more precisely we need some preparations.

2.5. Notational conventions. (i) In the following all topological spaces will be assumed to be Hausdorff.

(ii) If X is a locally compact topological space, and $\langle x_i \rangle$ is a net in X, then $\lim x_i = \infty$, or $x_i \to \infty$ for short, will mean that $\langle x_i \rangle$ has no convergent subnet in X. In other words, if we think of X as embedded in its one point compactification $X \cup \{\infty\}$ then $\lim x_i = \infty$ in the usual sense.

2.6. A gluing construction for locally compact spaces. Let X, Y be (disjoint) locally compact spaces, K a compact subset of X, and let $\varphi: X \setminus K \to Y$ be a continuous map.

Then we denote with $X \sqcup_\varphi Y$ the topological space whose underlying set is the disjoint union $Z = X \cup Y$, endowed with the topology a basis of which is given by the sets $U \cup V$, where

(i) U is open in X, V is open in Y,

(ii) the closure of $\varphi^{-1}(V) \setminus U$ in X is compact (or empty).

Note that, by construction, the open sets of the space X are open also in Z and that the intersections of Y with the open sets of Z are exactly the open sets in the original topology of Y. In particular, the inclusions $X \to Z$ and $Y \to Z$ are homeomorphic embeddings. Furthermore, Z is Hausdorff. To see this, it suffices to consider points $x \in X$, $y \in Y$. Pick a relatively compact open neighborhood U of x in X. Then y lies in the open subset $(X \setminus \overline{U}) \cup Y$ of Z and $U \cap ((X \setminus \overline{U}) \cup Y) = \emptyset$. Thus Z is Hausdorff.

2.7. Proposition. *Suppose that in the above gluing construction* 2.6 *the space Y is compact. Then for the topological space $Z = X \sqcup_\varphi Y$ the following assertions hold:*

(i) *Z is compact.*

(ii) *In the given topology of Z a net $\langle x_i \rangle$ with $x_i \in X$ converges to an element $y \in Y$ if and only if the following two conditions hold:*

(a) $\lim x_i = \infty$ *in* X,

(b) $\lim \varphi(x_i) = y$ *in* Y.

(iii) *The following assertions are equivalent:*

(c) *For each non-empty open subset* V *of* Y *the set* $\varphi^{-1}(V)$ *is not relatively compact in* X.

(d) X *is dense in* Z.

Proof. (i) Consider a covering $\mathcal{C} = \{U_i \cup V_i\}_{i\in I}$ of Z by open subsets in the basis defined in 2.6. Since Y is compact, there is a finite set $F \subseteq I$ such that $Y = \bigcup_{f\in F} V_f$. For each $f \in F$ there is a compact set $K_f \subseteq X$ with $\varphi^{-1}(V_f) \setminus U_f \subseteq K_f$. Let $K_F = K \cup \bigcup_{f\in F} K_f$. This set is compact, hence there exists a finite set $J \subseteq I$ such that $K_F \subseteq \bigcup_{j\in J} U_j$. The family $\{U_f \cup V_f\}_{f\in F} \cup \{U_j \cup V_j\}_{j\in J}$ is finite and covers $K_F \cup Y$, by construction. Also, the set $X \setminus K_F$ is covered by $\{U_f\}_{f\in F}$, so we have found a finite subcovering of $\mathcal{C}$. Thus Z is compact.

(ii) Suppose first that $\lim x_i = y$ in Z. Then for every compact subset C of X the union $W = (X \setminus C) \cup Y$ is an open neighborhood of y in Z, hence eventually $x_i \in W \cap X = X \setminus C$, which shows (a). To prove (b), consider an arbitrary open neighborhood V of y in Y. Since $\varphi^{-1}(V) \cup V$ is an open neighborhood of y in Z, we have $x_i \in \varphi^{-1}(V) \cup V$ and hence $\varphi(x_i) \in V$, for all sufficiently large indexes i. Thus $\lim \varphi(x_i) = y$ in Y.

Assume now that the conditions (a) and (b) are satisfied. Consider an open set $U \cup V$ in the basis with $y \in U \cup V$. Then $y \in V$ and condition (b) implies that there exists an index $i_1 \in I$ such that $\varphi(x_i) \in V$ whenever $i \geq i_1$. Thus $x_i \in \varphi^{-1}(V)$, for $i \geq i_1$. On the other hand, we find a compact set $C \subseteq X$ satisfying $\varphi^{-1}(V) \setminus U \subseteq C$. Condition (b) yields the existence of an $i_2 \in I$ such that $x_i \in X \setminus C$, for $i \geq i_2$. Choose an $i_0 \in I$ with $i_0 \geq i_1$ and $i_0 \geq i_2$. If $i \geq i_0$, then x_i belongs to U (because $x_i \in \varphi^{-1}(V)$, $x_i \notin C$, and $\varphi^{-1}(V) \setminus U \subseteq C$). We conclude that $\lim x_i = y$ in $X \sqcup_\varphi Y$.

Assertion (iii) is an immediate consequence of the definition of the topology of Z. ■

2.8. Remark. (i) Note that 2.7(ii) says that the closure of X in Z is the union of X with those points $y \in Y$ for which there exist arbitrarily small compact neighborhoods K_y such that $\varphi^{-1}(K_y)$ is not compact. The restriction of φ to the set $\varphi^{-1}(Y \setminus \mathrm{cl}_Z(X))$ is a perfect map.

(ii) If K is contained in a compact set K_1 and φ_1 is the restriction of φ to $X \setminus K_1$ then $X \sqcup_\varphi Y = X \sqcup_{\varphi_1} Y$ (as topological spaces).

2.9. Construction. If in the above topological construction the space X is a locally compact topological semigroup acting two-sidedly on a compact topological semigroup Y then $X \sqcup_\varphi Y$ becomes a semigroup if we introduce the natural multiplication described in 2.4(i). In order to guarantee that this multiplication is (jointly) continuous we have to assume in addition that X is a perfect semigroup, that the s-action is continuous, and that φ satisfies the following conditions:

(i) φ is *asymptotically a homomorphism*, that is, if $\langle(s_i,t_i)\rangle$ is a net in $X\times X$ with $s_i\to\infty$ and $t_i\to\infty$ then

$$\lim\varphi(s_it_i)=(\lim\varphi(s_i))(\lim\varphi(t_i))$$

whenever the limits $\lim\varphi(s_i)$ and $\lim\varphi(t_i)$ exist in Y.

(ii) φ is *asymptotically equivariant*, that is, if $\langle(s_i,t_i)\rangle$ is a net in $X\times X$ with $s_i\to\infty$ then

$$\lim\varphi(s_it_i)=(\lim\varphi(s_i))(\lim t_i)$$
$$\lim\varphi(t_is_i)=(\lim t_i)(\lim\varphi(s_i))$$

whenever $\lim t_i$ exists in X and $\lim\varphi(s_i)$ exists in Y.

If X is dense in Z then by the continuous extension theorem (cf., e.g., [1], p. 81, Theorem 1) these conditions indeed guarantee that $Z=X\sqcup_\varphi Y$ becomes a topological semigroup (details are left to the reader). Since no ambiguities are to be feared we also denote this semigroup with $X\sqcup_\varphi Y$.

Obviously, our construction yields a topological semigroup compactification of X. We formally write this compactification as $(X\sqcup_\varphi Y,i)$, where $i\colon X\to Z$ denotes the inclusion.

2.10. Remark. (i) In the simplest version of our construction the semigroup Y is singleton and φ is the constant map $X\to Y$. Here our assumptions boil down to the condition that X is perfect, and we get the familiar one point compactification of X with a zero element at infinity.

(ii) If Y is compact but X is not dense in $Z=X\sqcup_\varphi Y$ then it is not clear that the assumptions in our construction imply the joint continuity of the multiplication in Z.

(iii) The crux of the construction lies in finding an appropriate mapping φ. The existence of a map φ satisfying 2.9(i),(ii) can be interpreted as an "asymptotic property" of the topological semigroup X.

2.11. Induced compactifications. If X_1 is a closed subsemigroup of X then the above compactification $(X\sqcup_\varphi Y,i)$ of X naturally induces the compactification $(\overline{i(X_1)},i_1)$ of X_1, where $i_1\colon X_1\to\overline{i(X_1)}$, $i_1(x)=i(x)$. Set $Y_1=\overline{i(X_1)}\cap Y$. If $\varphi^{-1}(\overline{\varphi(X_1\setminus K)}\setminus Y_1)$ is relatively compact then the induced compactification can be written as $(X_1\sqcup_{\varphi_1}Y_1,i_1)$, where $\varphi_1\colon X_1\setminus K_1\to Y_1$, $\varphi_1(x)=\varphi(x)$, and K_1 is a compact subset of X_1 which contains $K\cap X_1$ as well as $X_1\cap\varphi^{-1}(\overline{\varphi(X_1\setminus K)}\setminus Y_1)$. Note that $\varphi^{-1}(\overline{\varphi(X_1\setminus K)}\setminus Y_1)$ is empty (in fact, $\overline{\varphi(X_1\setminus K)}\setminus Y_1=\emptyset$) if for every open subset V of Y meeting $\overline{\varphi(X_1\setminus K)}$ the set $\varphi^{-1}(V)\cap X_1$ is not relatively compact .

3. A natural s-action of $\mathrm{Sl}(2,\mathbb{R})$

3.1. Our next goal is to apply the $\sqcup$-construction and Example 2.3(ii) to get compactifications of subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$. For this task it is convenient to change to the Lie algebra setting. We also need some preparations from [2].

3.2. Notation and basic facts. (i) Throughout these notes $\mathbb{R}^+$ will denote the set of strictly positive reals, and $\mathbb{R}_0^+$ the set of nonnegative reals.

(ii) We abbreviate $\mathrm{Sl}(2,\mathbb{R})$ to Sl_2. Following [11] we write $H = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $P = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, $Q = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$. Furthermore, we denote the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ by Kill and put $\mathsf{Kill}^+ = \{X \in \mathfrak{sl}(2,\mathbb{R}) \mid \det(X) = -\frac{1}{8}\mathsf{Kill}(X,X) < 0\}$. The set

$$\mathsf{Hyp} = \{X \in \mathfrak{sl}(2,\mathbb{R}) \mid -\det(X) = 1\} = \{gHg^{-1} \mid g \in \mathrm{Sl}_2\}$$

is called the *fundamental hyperboloid.* For a point $X \in \mathsf{Hyp}$ let $\mathfrak{p}_X$ $[\mathfrak{q}_X]$ denote the one-dimensional eigenspace of $\mathrm{ad}\,X$ with eigenvalue 2 [-2]. Then $\mathsf{hor}(X) = X + \mathfrak{p}_X$ is called the *horizontal line through* X, and $\mathsf{vert}(X) = X + \mathfrak{q}_X$ is called the *vertical line through* X. Explicit formulas for the so defined mappings hor and vert can be found in [2], 4.5.

(iii) The set of all horizontal [vertical] lines in Hyp is denoted with Hor [Vert]. We let G_ℓ be the parabolic subgroup $\{\mathbf{1}, -\mathbf{1}\}\exp(\mathbb{R}H + \mathbb{R}P)$, and, similarly, $G_r = \{\mathbf{1}, -\mathbf{1}\}\exp(\mathbb{R}H + \mathbb{R}Q)$. We identify Hor with the flag manifold Sl_2/G_ℓ, and Vert with $G_r\backslash\mathrm{Sl}_2$, so that $\mathsf{hor}(H)$ corresponds to G_ℓ, and $\mathsf{vert}(H)$ to G_r. Note that under this convention $\mathsf{hor}(gHg^{-1})$ $[\mathsf{vert}(gHg^{-1})]$ is identified with gG_ℓ $[G_r g^{-1}]$.

(iv) The map $c\colon \mathsf{Hyp} \to \mathsf{Hor} \times \mathsf{Vert}$, $X \mapsto (\mathsf{hor}(X), \mathsf{vert}(X))$, is a topological embedding and the rectangular multiplication of $\mathsf{Hor} \times \mathsf{Vert}$ restricts to a partial product $\diamond$ on Hyp which we call the *diamond product.* If $\mathsf{hor}(X)$ meets $\mathsf{vert}(Y)$, equivalently: if $\mathsf{hor}(X) \neq -\mathsf{vert}(Y)$, then $\mathsf{hor}(X) \cap \mathsf{vert}(Y) = \{X \diamond Y\}$. For example, it can be checked easily that $H \diamond (\alpha H + \beta P + \gamma Q) = H + \frac{2\beta}{1+\alpha}P$ whenever $\alpha \neq -1$.

(v) The restriction of the exponential map to Kill^+ is a diffeomorphism onto the open set of all matrices in Sl_2 with trace > 2. We denote with rlog the *reduced logarithm* $\exp(\mathsf{Kill}^+) \to \mathsf{Hyp}$ which to every element $\exp(X)$, $X \in \mathsf{Kill}^+$, assigns the normalized vector $-\det(X)^{-1} \cdot X$. The map rlog will be the essential ingredient in defining the asymptotic homomorphism φ of our construction.

3.3. A partial s-action on Hyp. We now turn to the s-action of 2.3(i) in the special case where $G = \mathrm{Sl}_2$ and G_ℓ, G_r are defined as in 3.2(iii) above. This s-action induces a partial s-action $\circ$ of Sl_2 on Hyp by the rules

$$g \circ X = c^{-1}(gc(X)), \text{ and } X \circ g = c^{-1}(c(X)g),$$

whenever $gc(X) \in c(\mathsf{Hyp})$, $c(X)g \in c(\mathsf{Hyp})$, respectively. In terms of the diamond product:

$$g \circ X = (gXg^{-1}) \diamond X \text{ and } X \circ g = X \diamond (g^{-1}Xg),$$

whenever the diamond products involved exist. Note that our partial s-action is continuous.

We are interested in asymptotic properties of the action $\circ$. These involve, however, the application of the function rlog, which implies that we have to restrict our partial s-action to a suitable open domain. Moreover, we need a little lemma enabling us to simplify the necessary calculations.

3.4. Lemma.

(i) *If* U *is a neighborhood of the identity in* Sl_2 *then* $\{gHg^{-1} \mid g \in U\}$ *is a neighborhood of* H *in* Hyp.

(ii) *If* $\langle X_n \rangle$ *is a sequence in* Hyp *converging to* H *then there exist a subsequence* $\langle X_m \rangle$ *of it and a sequence* $\langle g_m \rangle$ *in* Sl_2 *such that* $\lim g_m = \mathbf{1}$ *and* $g_m X_m g_m^{-1} = H$ *for every index* m.

Proof. (i) By [2], 4.6(ii), we know that the homogeneous space $\mathrm{Sl}_2/Z\exp(\mathbb{R}H)$ (where $Z = \{\mathbf{1}, -\mathbf{1}\}$ is the center of Sl_2) is homeomorphic with Hyp via the map $g \cdot Z\exp(\mathbb{R}H) \mapsto gHg^{-1}$. The assertion follows now from the openness of the canonical quotient map $\mathrm{Sl}_2 \mapsto \mathrm{Sl}_2/Z\exp(\mathbb{R}H)$.

(ii) We choose a compatible metric d on Sl_2 and write $U_m = \{g \in \mathrm{Sl}_2 \mid d(\mathbf{1}, g) < \frac{1}{m}\}$. By (i) we find a subsequence $\langle X_m \rangle$ of $\langle X_n \rangle$ such that $X_m \in \{gHg^{-1} \mid g \in U_m\}$ for every m. Then for suitable $g_m \in U_m$ we have $X_m = g_m H g_m^{-1}$ and the assertion follows. ■

3.5. The standard domain. We let

$$\mathrm{dom}_\ell \stackrel{\mathrm{def}}{=} \{(g, X) \in \mathrm{Sl}_2 \times \mathsf{Hyp} \mid \exists T > 0 :\ g\exp(tX) \in \exp(\mathsf{Kill}^+) \text{ whenever } t > T\},$$

$$\mathrm{dom}_r \stackrel{\mathrm{def}}{=} \{(X, g) \in \mathsf{Hyp} \times \mathrm{Sl}_2 \mid \exists T > 0 :\ \exp(tX)g \in \exp(\mathsf{Kill}^+) \text{ whenever } t > T\},$$

and define the *standard domain* of $\circ$ to be the union $\mathrm{dom} = \mathrm{dom}_\ell \cup \mathrm{dom}_r$. The following assertions hold:

(i) $(g, X) \in \mathrm{dom}_\ell$ if and only if $(-X, g^{-1}) \in \mathrm{dom}_r$.

(ii) If $(g, X) \in \mathrm{dom}_\ell$ then $(hgh^{-1}, hXh^{-1}) \in \mathrm{dom}_\ell$, for every $h \in \mathrm{Sl}_2$.

(iii) $(g, H) \in \mathrm{dom}_\ell$ if and only if $g \in O = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}_2 \mid a > 0\}$.

(iv) If $(g, X) \in \mathrm{dom}_\ell$ then the diamond product $(gXg^{-1}) \diamond X$ exists. In particular, dom_ℓ is a domain of the partial left action defined in 3.3.

(v) All of the three sets dom_ℓ, dom_r and dom are open.

Proof. Assertion (i) is immediate, (ii) holds by the invariance of $\exp(\mathsf{Kill}^+)$ under inner automorphisms, (iii) follows from 8.8 of [2].

(iv) In view of (ii) we can assume without loss of generality that $X = H$. Thus $g \in O$, by (iii). Using the formula 4.5 of [2] for $\mathsf{hor}(gHg^{-1})$ we see that $\mathsf{hor}(gHg^{-1}) \neq -\mathsf{vert}(H)$, so the diamond product $(gHg^{-1}) \diamond H$ exists.

(v) By (i) and (ii) it suffices to show that dom_ℓ is a neighborhood of (g, H) if $(g, H) \in \mathrm{dom}_\ell$. For an element $(g, H) \in \mathrm{dom}_\ell$ let U be a $\mathbf{1}$-neighborhood in Sl_2 and V a neighborhood of g such that $h^{-1}Vh \subseteq O$ for every $h \in U$. Hence $V \subseteq \bigcap_{h \in U} hOh^{-1}$. For arbitrary $x \in V$ and $h \in U$ there exist $y \in O$ and $T > 0$ such that $x = hyh^{-1}$ and $y \exp(tH) \in \exp(\mathsf{Kill}^+)$, for every $t > T$. Hence $x \exp(thHh^{-1}) = h(y \exp(tH))h^{-1} \in \exp(\mathsf{Kill}^+)$, for every $t > T$. Thus, by 3.4(i), the set $V \times \{hHh^{-1} \mid h \in U\}$ is a neighborhood of (g, H) contained in dom_ℓ. ■

3.6. Remark. The set dom does not contain *all* points where our partial s-action is defined. For example, let $g = -\mathbf{1}$. Then the diamond products $(gHg^{-1}) \diamond H = H \diamond H = H$ and $H \diamond (g^{-1}Hg) = H$ exist, but neither of (g, H), (H, g) lies in the standard domain.

3.7. Elementary properties of our partial s-action. The following assertions are immediate consequences of the definition of $\circ$, proofs are left to the reader.

(i) *If* $(g, X) \in \mathrm{dom}_\ell$ *then* $g \circ X \in \mathsf{vert}\, X$, *if* $(X, g) \in \mathrm{dom}_r$ *then* $X \circ g \in \mathsf{hor}\, X$.

(ii) $h(g \circ X)h^{-1} = (hgh^{-1}) \circ (hXh^{-1})$ *for every* $(g, X) \in \mathrm{dom}_\ell$ *and every* $h \in \mathrm{Sl}_2$.

(iii) $X \circ g = -(g^{-1} \circ (-X))$ *for every* $(X, g) \in \mathrm{dom}_r$. ■

3.8. Asymptotic formulas.

(i) *For any* $(g, X) \in \mathrm{dom}_\ell$ *we have the equality*

$$g \circ X = \lim_{\substack{g' \to g \\ t \to +\infty \\ X' \to X}} \mathrm{rlog}(g' \exp(tX')).$$

In particular, $g \circ X = \lim_{t \to +\infty} \mathrm{rlog}(g \exp(tX))$.

(ii) *For any* $(X, g) \in \mathrm{dom}_r$ *the equality*

$$X \circ g = \lim_{\substack{t \to +\infty \\ X' \to X \\ g' \to g}} \mathrm{rlog}(\exp(tX')g')$$

holds. In particular, $X \circ g = \lim_{t \to +\infty} \mathrm{rlog}(\exp(tX)g)$.

Proof. (i) We first show that

$$(*) \qquad g \circ X = \lim_{\substack{g' \to g \\ t \to +\infty}} \mathrm{rlog}(g' \exp(tX)).$$

For this we may assume (by 3.7(ii)) that $X = H$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have seen in 3.5(iii) that $a > 0$. Remark 8.8 of [2] implies that

$$\lim_{\substack{g' \to g \\ t \to +\infty}} \operatorname{rlog}(g' \exp(tH)) = H + \frac{2c}{a} Q.$$

On the other hand, a straightforward calculation yields $(gHg^{-1}) \diamond H = H + \frac{2c}{a} Q$, so $(*)$ holds.

We prove now the asserted equality assuming again that $X = H$. Consider the sequences $\langle g_n \rangle$ in Sl_2, $\langle t_n \rangle$ in $\mathbb{R}^+$, and $\langle X_n \rangle$ in Hyp such that $g_n \to g$, $t_n \to +\infty$, and $X_n \to H$. Applying 3.4(ii) there is a subsequence $\langle X_m \rangle$ of $\langle X_n \rangle$ and a sequence $\langle h_m \rangle$ in Sl_2 converging to $\mathbf{1}$ such that $h_m X_m h_m^{-1} = H$, for every index m. Then

$$\operatorname{rlog}(g_m \exp(t_m X_m)) = h_m^{-1} \operatorname{rlog}(h_m g_m h_m^{-1} \exp(t_m H)) h_m$$

converges to $g \circ H$ by $(*)$, which finishes the proof.

Assertion (ii) follows from (i) above and 3.7(iii). ∎

3.9. Remark. In the proof of 3.8 we encountered the special formula $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \circ H = H + \frac{2c}{a} Q$, which is valid for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}_2$ with $a > 0$. This formula has other interesting consequences, which we record for later use:

(i) *Let* $(g, X) \in \mathrm{dom}_\ell$ $[(X, g) \in \mathrm{dom}_r]$. *Then* $g \circ X = X$ $[X \circ g = X]$ *if and only if* $g \in \exp(\mathbb{R}\, \mathsf{hor}(X))$ $[g \in \exp(\mathbb{R}\, \mathsf{vert}(X))]$.

(ii) *Suppose that the diamond products* $X \diamond Y$ *and* $Y \diamond X$ *exist. Then*

$$\lim_{\substack{t \to \infty \\ X' \to X}} \exp(tX') \circ Y = \lim_{\substack{t \to \infty \\ Y' \to Y}} X \circ \exp(tY') = X \diamond Y.$$

If $X \neq Y \diamond X$ $[Y \neq X \diamond Y]$ *then* $\exp(\mathbb{R}^+ Y) \circ X$ $[Y \circ \exp(\mathbb{R}^+ X)]$ *is the open line segment between* X *and* $Y \diamond X$ $[Y$ *and* $Y \diamond X]$.

(iii) *If* $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), H + \gamma Q) \in \mathrm{dom}_\ell$ *then* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \circ (H + \gamma Q) = H + \frac{4c + 2d\gamma}{2a + b\gamma} Q$.

(iv) *If* $(H + \beta P, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in \mathrm{dom}_r$ *then* $(H + \beta P) \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = H + \frac{4b + 2d\beta}{2a + c\beta} P$.

Proof. (i) follows from the formula $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \circ H = H + \frac{2c}{a} Q$ after conjugating the element X to the element H.

(ii) The equality $\lim_{\substack{t \to \infty \\ X' \to X}} \exp(tX') \circ Y = X \diamond Y$ can be checked by a straightforward calculation after conjugating Y to H. In view of 3.7(iii) this equality yields the second one. For the last assertion of (ii) we now have only to observe that $t \mapsto \exp(tY) \circ X$ is injective.

(iii) An easy calculation shows that

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \circ (H + \gamma Q) = e^{\gamma/2 \cdot \mathrm{ad} Q} \left(e^{-\gamma/2 \cdot \mathrm{ad} Q} \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \right) \circ H \right) = H + \frac{4c + 2d\gamma}{2a + b\gamma} Q,$$

and this establishes the assertion. Assertion (iv) follows from (iii) and formula 3.7(iii). ∎

4. Compact Rees extensions of subsemigroups of $\mathbf{Sl(2,\mathbb{R})}$

4.1. In the present section we apply the general construction of Section 2 to produce topological semigroup compactifications of subsemigroups of Sl_2. This construction will yield, in particular, explicit descriptions for all injective compactifications for a large class of perfect subsemigroups of Sl_2 (including the exponential ones).

The following theorems summarize some results about exponential subsemigroups of Sl_2 and about the diamond product, needed for our constructions below. (Cf. [2] 3.8, 5.5, 7.12, 8.9, 8.10(i), 9.4(i) and [3] 6.13.)

4.2. Theorem. *Let W be the Lie wedge of a closed three dimensional exponential subsemigroup Σ of Sl_2. Then W is a three dimensional Lie semialgebra in $\mathfrak{sl}(2,\mathbb{R})$ and $\Sigma = \exp W$ and the following assertions are equivalent:*

- (i) *Σ is perfect.*
- (ii) *W is contained in $\{0\} \cup \mathsf{Kill}^+$.*
- (iii) *$W \cap \mathsf{Hyp} = \mathrm{rlog}(\Sigma \setminus \{\mathbf{1}\})$ is a compact connected rectangular band semigroup with respect to the diamond product $\diamond$.* ■

4.3. Theorem. *Suppose that $X, Y \in \mathsf{Hyp}$ such that both diamond products $X \diamond Y$ and $Y \diamond X$ exist. Then*

$$X \diamond Y = \lim_{\substack{(s,t)\to(\infty,\infty)\\(X',Y')\to(X,Y)}} \mathrm{rlog}\left(\exp(sX')\exp(tY')\right).$$ ■

4.4. Conventions I. Throughout the rest of this section we assume that W is the Lie wedge of a closed perfect exponential subsemigroup Σ of Sl_2 with $\dim \Sigma = 3$. We write (slightly at variance with the notation in [2]) $D = W \cap \mathsf{Hyp}$, also, D is considered as a semigroup with respect to $\diamond$. In the terminology of [2] (7.2), D is the closure of a type 0 rectangular domain. By 3.8 the partial s-action $\circ$ restricts to an s-action of Σ on D, which we also denote with $\circ$.

4.5. Proposition. *Let $\varphi: \Sigma \setminus \{\mathbf{1}\} \to D$ be defined by $\varphi(s) = \mathrm{rlog}(s)$. Then the following assertions hold:*

- (i) *The map φ is asymptotically a homomorphism and asymptotically equivariant.*
- (ii) *For every $X \in D$ the set $\varphi^{-1}(X) = \exp(\mathbb{R}^+ X)$ is not relatively compact in Σ.*

Proof. (i) That φ is asymptotically a homomorphism is immediate from 4.3. That φ is asymptotically equivariant follows from 3.8.

Assertion (ii) is obvious. ■

4.6. Remark. It is also possible (though less convenient) to define φ in terms of a global function $\varphi^*\colon (\overline{\exp(\mathsf{Kill}^+)} \setminus \{\mathbf{1}\}) \to (G/G_\ell) \times (G_r \backslash G)$, where $G = \mathrm{Sl}_2$ and G_ℓ, G_r are as in 3.2(iii). The map φ^* is defined by $\varphi^*(g) = (g_\ell G_\ell, G_r g_r)$, where g_ℓ, g_r are elements with $g \in g_\ell \exp(\mathbb{R}_0^+ H + \mathbb{R}P) g_\ell^{-1} \cap g_r^{-1} \exp(\mathbb{R}_0^+ H + \mathbb{R}Q) g_r$. (Note that the cosets $g_\ell G_\ell$, $G_r g_r$ are uniquely defined.) The functions φ and φ^* are related to each other via the map $c\colon \mathsf{Hyp} \to (G/G_\ell) \times (G_r \backslash G)$ of 3.2(iv), by the formula $c(\varphi(s)) = \varphi^*(s)$, for all $s \in \Sigma \setminus \{\mathbf{1}\}$.

4.7. The semigroup Σ^D. By 4.5 Construction 2.9 yields a (topological) semigroup compactification $(\Sigma \sqcup_\varphi D, i)$ of Σ, with inclusion map i. We henceforth abbreviate $\Sigma \sqcup_\varphi D$ to Σ^D.

4.8. Remark. (i) Note that Σ^D is a nonabelian compact uniquely divisible (UDC) topological semigroup. It is a three dimensional analogue of the so-called "affine triangle"

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in M(2, \mathbb{R}) \mid\ 0 \le x,\ 0 \le y,\ x + y \le 1 \right\}.$$

The semigroup Σ^D is a foliated union of copies of the affine triangle.

(ii) In particular, if $X \in D$ then $\overline{i(\exp(\mathbb{R}_0^+ X))} = i(\exp(\mathbb{R}_0^+ X)) \cup \{X\}$. Thus if $X \neq Y \in D$ then $\overline{i(\exp(\mathbb{R}_0^+ X))} \cap \overline{i(\exp(\mathbb{R}_0^+ Y))} = \{\mathbf{1}\}$.

(iii) The minimal ideal D of Σ^D is a rectangular band, so all maximal subgroups of Σ^D are trivial.

(iv) Let ρ be a finite dimensional representation of Σ^D, such that $\rho(\mathbf{1})$ is the identity. By Theorem 2.1 of [4] any congruence on Σ is either a Rees congruence or the identity, so $\det \circ \rho(\Sigma) = \{\mathbf{1}\}$. This means that the matrices in $\rho(\Sigma)$ have determinant 1, hence are invertible. It follows that $\rho(\Sigma^D)$ is a compact topological group, which implies $\rho(\Sigma^D) = \{\mathbf{1}\}$, since every continuous homomorphism of Σ into a compact topological group is trivial (see Proposition 6.2 below). Thus all finite dimensional representations of Σ^D are trivial.

(v) The so-called ABC-theorem of BROWN and FRIEDBERG [5] states, among other things, that if S satisfies *all* of the following conditions (a)–(e) then S has a faithful real n-dimensional representation:

(a) $E(S) = \{\mathbf{1}\} \cup M(S)$.
(b) $M(S)$ is homeomorphic to an $(n-1)$-cell.
(c) The centralizer of each idempotent $e \in M(S)$ is isomorphic with the (multiplicative) semigroup $[0, 1]$.
(d) $S \setminus M(S)$ admits left cancellation.
(e) $M(S)$ is a left zero semigroup.

Since conditions (a)–(d) are obviously satisfied for $S = \Sigma^D$ and $n = 3$ the above observation (iv) shows that the ABC-theorem is no longer valid if condition (e) is omitted.

We next pass to a construction where the $\mathcal{H}$-classes are nontrivial.

4.9. Conventions II. To avoid tedious calculations we henceforth also assume that $D = W \cap \mathsf{Hyp}$ contains the element H. This can always be enforced by applying a suitable inner automorphism, so we do not lose generality. We write

$$\mathsf{h} := \mathsf{hor}(H) \cap D, \quad \mathsf{v} := \mathsf{vert}(H) \cap D.$$

Note that $D = \mathsf{v} \diamond \mathsf{h}$.

4.10. The paragroup $M(D, K)$. Let (K, k) be a topological group compactification of the additive group of real numbers. Then we turn the compact space $M(D, K) = \mathsf{v} \times K \times \mathsf{h}$ into a paragroup, with multiplication

$$(X, g, Y)(X', g', Y') = (X, g\sigma(Y, X')g', Y'),$$

where, for any $Y = H + \beta P \in \mathsf{h}$, $X' = H + \gamma Q \in \mathsf{v}$ we let

$$\sigma(H + \beta P, H + \gamma Q) = k(\log(1 + \frac{\beta\gamma}{4})).$$

To justify the definition of the sandwich map σ, note first that for Y, X' as above we always have $4 + \beta\gamma \neq 0$. This follows from [2] 4.15 and the fact that the diamond product $X' \diamond Y$ exists. By 4.2 the set D is connected, so we conclude that $4 + \beta\gamma$ has the same sign for all pairs $(H + \beta P, H + \gamma Q) \in \mathsf{h} \times \mathsf{v}$, this sign must be positive since $(H, H) \in \mathsf{h} \times \mathsf{v}$.

4.11. The s-action of Σ on $M(D, K)$. With the aid of the s-action $\circ$ of Σ on D we next define an s-action of Σ on $M(D, K)$: For $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$ and $(H + \gamma Q, g, H + \beta P) \in \mathsf{v} \times K \times \mathsf{h}$ we put

$$s(H + \gamma Q, g, H + \beta P) = (s\circ(H + \gamma Q), k(\log(a + \frac{b\gamma}{2}))g, H + \beta P)$$
$$(H + \gamma Q, g, H + \beta P)s = (H + \gamma Q, gk(\log(a + \frac{c\beta}{2})), (H + \beta P)\circ s).$$

Note that the condition $s\exp(t(H + \gamma Q)) \in \exp(\mathsf{Kill}^+)$ $[\exp(t(H + \beta P))s \in \exp(\mathsf{Kill}^+)]$ for every $t \in \mathbb{R}^+$ implies that $a + \frac{b\gamma}{2} > 0$ $[a + \frac{c\beta}{2} > 0]$. It is readily verified by direct calculation that our definition indeed yields an s-action of Σ on $M(D, K)$.

4.12. Example. Suppose that $s = \exp(tX)$, where $X = (H + \gamma Q) \diamond (H + \beta P)$ and $t > 0$. Then we get

$$s(H + \gamma Q, g, H + \beta P) = (H + \gamma Q, k(t)g, H + \beta P).$$

A similar formula holds for $(H + \gamma Q, g, H + \beta P)s$.

For the next proposition we recall that by 3.5(iii) we have $a > 0$ for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma$.

4.13. Definition and Proposition. Let $M(D,K)$ be the paragroup of 4.10. Then we define a continuous map $\Phi\colon \Sigma\setminus\{\mathbf{1}\}\to M(D,K)$,

$$s=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto \Phi(s)=(\varphi(s)\diamond H, k(\log(a)), H\diamond\varphi(s)).$$

(Since the choice of D,K will always be clear from the context we shall omit any reference to D,K in the notation for Φ.)

The following assertions hold:

(i) *The map Φ is asymptotically a homomorphism and asymptotically equivariant.*

(ii) *For every non-empty open subset V of $M(D,K)$ the set $\Phi^{-1}(V)$ is not relatively compact in Σ.*

Proof. (i) To verify that Φ is asymptotically a homomorphism choose nets $\langle s_i=\left(\begin{smallmatrix} a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\rangle$ and $\langle s_i'=\left(\begin{smallmatrix} a_i' & b_i'\\ c_i' & d_i'\end{smallmatrix}\right)\rangle$ in Σ such that $s_i\to\infty$, $s_i'\to\infty$, $\Phi(s_i)\to(H+\gamma Q,g,H+\beta P)$, and $\Phi(s_i')\to(H+\gamma' Q,g',H+\beta' P)$. We have to show that the net $\langle\Phi(s_is_i')\rangle$ converges to $(H+\gamma Q, gk(\log(1+\frac{\beta\gamma'}{4}))g', H+\beta'P)$. By 4.5(i) the map φ is asymptotically a homomorphism, so $\varphi(s_i)\to(H+\gamma Q)\diamond(H+\beta P)$ and $\varphi(s_i')\to(H+\gamma' Q)\diamond(H+\beta' P)$ imply that $\varphi(s_is_i')\to(H+\gamma Q)\diamond(H+\beta' P)$. Thus it remains to verify that

$$k(\log(a_ia_i'+b_ic_i'))\to gk(\log(1+\tfrac{\beta\gamma'}{4}))g'. \tag{$*$}$$

To accomplish this, observe first that the equality $a_ia_i'+b_ic_i'=a_i\left(1+\frac{b_i}{a_i}\frac{c_i'}{a_i'}\right)a_i'$ implies that

$$k(\log(a_ia_i'+b_ic_i'))=k(\log(a_i))k\left(\log\left(1+\tfrac{b_i}{a_i}\tfrac{c_i'}{a_i'}\right)\right)k(\log(a_i')).$$

Since $\operatorname{rlog}(s_i)=\varphi(s_i)\to(H+\gamma Q)\diamond(H+\beta P)=\frac{1}{4+\beta\gamma}((4-\beta\gamma)H+4\beta P+4\gamma Q)$ we obtain (using the formula for the exponential function) that

$$\lim\frac{b_i}{a_i}=\frac{\frac{4\beta}{4+\beta\gamma}}{1+\frac{4-\beta\gamma}{4+\beta\gamma}}=\frac{\beta}{2}.$$

Similarly, $\operatorname{rlog}(s_i')=\varphi(s_i')\to(H+\gamma'Q)\diamond(H+\beta'P)$ implies that $\lim\frac{c_i'}{a_i'}=\frac{\gamma'}{2}$. So, $(*)$ holds, hence Φ is asymptotically a homomorphism.

We now show that Φ is asymptotically equivariant. Pick nets $\langle s_i=\left(\begin{smallmatrix} a_i & b_i\\ c_i & d_i\end{smallmatrix}\right)\rangle$ and $\langle s_i'=\left(\begin{smallmatrix} a_i' & b_i'\\ c_i' & d_i'\end{smallmatrix}\right)\rangle$ in Σ such that $s_i\to s=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in S$, $s_i'\to\infty$, and $\Phi(s_i')\to(H+\gamma'Q,g,H+\beta'P)$. We show that $\Phi(s_is_i')\to s(H+\gamma'Q,g,H+\beta'P)$.

The convergence $\Phi(s_i')\to(H+\gamma'Q,g,H+\beta'P)$ implies that $\varphi(s_i')\to(H+\gamma'Q)\diamond(H+\beta'P)$, and, by 4.5(i), that $\varphi(s_is_i')\to s\circ(H+\gamma'Q)\diamond(H+\beta'P)$. Thus it remains to prove that

$$k(\log(a_ia_i'+b_ic_i'))\to k\left(\log\left(a+\tfrac{b\gamma'}{2}\right)\right)g. \tag{$**$}$$

The equality $a_i a'_i + b_i c'_i = (a_i + b_i \frac{c'_i}{a'_i}) a'_i$ implies that

$$k(\log(a_i a'_i + b_i c'_i)) = k\left(\log\left(a_i + b_i \frac{c'_i}{a'_i}\right)\right) k(\log(a'_i)).$$

We have seen above that $\lim \frac{c'_i}{a'_i} = \frac{\gamma'}{2}$, so we obtain $(**)$, and thence $\Phi(s_i s'_i) \to s(H+\gamma' Q, g, H+\beta' P)$. By the same arguments $\Phi(s'_i s_i) \to (H+\gamma' Q, g, H+\beta' P)s$, thus Φ is asymptotically equivariant.

(ii) Let V be a non-empty open subset of $M(D,K)$. Consider an arbitrary element $(Y,g,Z) \in V$ and an open subset U of K containing g such that

$$\{Y\} \times U \times \{Z\} \subseteq V.$$

Let $X = Y \diamond Z = \alpha H + \beta P + \gamma Q$. We know from 8.2 of [2] that $1 + \alpha > 0$. The properties of the compactifications of the additive group $\mathbb{R}$ imply that there exists a sequence $\langle r_n \rangle$ of positive reals converging to $+\infty$ such that $k(r_n) \in U$ for every $n \in \mathbb{N}$. Since $1 + \alpha > 0$ we find for every $n \in \mathbb{N}$ a positive real t_n such that $\cosh(t_n) + \alpha \sinh(t_n) = e^{r_n}$. Then $t_n \to \infty$. Since $\Phi(\exp(t_n X)) = (Y, k(r_n), Z) \in V$ for every $n \in \mathbb{N}$ we conclude that $\Phi^{-1}(V)$ is not relatively compact in Σ. ■

4.14. The semigroup $\Sigma^{D,K}$. By 4.13 Construction 2.9 yields a (topological) semigroup compactification $(\Sigma \sqcup_\Phi M(D,K), j)$ of Σ with inclusion map j. We henceforth abbreviate $\Sigma \sqcup_\Phi M(D,K)$ to $\Sigma^{D,K}$.

4.15. Connections between Σ^D and $\Sigma^{D,K}$. The map $\Sigma^{D,K} \to \Sigma^D$ defined by

$$\begin{aligned} a &\mapsto a && \text{for } a \in S \\ (X,g,Y) &\mapsto X \diamond Y && \text{for } (X,g,Y) \in M(D,K) \end{aligned}$$

is a continuous and surjective homomorphism whose kernel congruence is the $\mathcal{H}$-relation on $M(D,K)$.

4.16. Induced compactifications of a subsemigroup of Σ. If S is a closed subsemigroup of Σ then our compactifications (Σ^D, i) and $(\Sigma^{D,K}, j)$ naturally induce compactifications of S, namely $(\overline{i(S)}, i_S)$ and $(\overline{j(S)}, j_S)$, where $i_S(x) = i(x)$ and $j_S(x) = j(x)$ for all $x \in S$.

4.17. Proposition. *Let S be a closed subsemigroup of our exponential semigroup Σ. We write φ' for the restriction of φ to $S \setminus \{\mathbf{1}\}$, and $D' = \overline{\varphi'(S \setminus \{\mathbf{1}\})}$. Then $\overline{i(S)} = i(S) \cup D'$ and the compact semigroup $\overline{i(S)}$ can be written as the $\sqcup$-product $S \sqcup_{\varphi'} D'$.*

Proof. Since $\operatorname{rlog}(s^n) = \operatorname{rlog}(s)$, for every $s \in S \setminus \{\mathbf{1}\}$, $n \in \mathbb{N}$, and $s^n \to \infty$ this assertion follows by a straightforward application of the definition of the $\sqcup$-product (cf. 2.11). ■

4.18. For the next proposition we recall from [2] 10.1, that the *umbrella set* $\mathrm{Umb}(A)$ of a subset A in a Lie group G is defined to be the set of all elements X in the Lie algebra of G such that $\exp(tX) \in A$ for all suitably large positive reals t, say, $t \geq T$. (Cf. also 5.2 below.)

4.19. Proposition. *We retain the assumptions and the terminology of* 4.17*, and suppose, in addition, that* S *has dense and connected interior and that* $\overline{\mathrm{Umb}}(S) = \mathbb{R}_0^+ D$*. Then*

(i) $D' = D$ *and* $\overline{j(S)} = j(S) \cup M(D,K)$*;*

(ii) $\overline{\Phi(S \setminus \{\mathbf{1}\})} = M(D,K)$*, and the compact semigroup* $\overline{j(S)}$ *can be written as the* $\sqcup$*-product* $S \sqcup_{\Phi'} M(D,K)$*, where* Φ' *denotes the restriction of* Φ *to* $S \setminus \{\mathbf{1}\}$.

Proof. (i) The equality $D' = D$ follows from the fact (cf. [2], 10.5) that the algebraic interior of $\overline{\mathrm{Umb}}(S)$ coincides with $\mathrm{Umb}(\mathrm{Int}\, S)$.

If X is an interior point of D (in Hyp) then $\exp([T,\infty[X) \subseteq \mathrm{Int}\, S$ for some $T > 0$, thus the proof of 4.13(ii) shows that the closure of $j(\exp([T,\infty[X))$ contains the set $\{X \diamond H\} \times K \times \{H \diamond X\}$. This implies that $M(D,K) \subseteq \overline{j(S)}$ and the assertion follows.

(ii) The proof of (i) yields the inclusion $M(D,K) \subseteq \overline{\Phi(S \setminus \{\mathbf{1}\})}$. Assertion (ii) follows now from 2.11. ∎

5. Compactifications of subsemigroups of Sl(2,ℝ): Basic Facts

5.1. Notation. (i) Unless stated explicitly otherwise, throughout this section we let S be a closed proper (i.e., $S \neq \mathrm{Sl}_2$) subsemigroup of Sl_2 with dense and connected interior. Note that the interior of a connected subsemigroup of Sl_2 is always connected if it clusters at the identity (cf., e.g., [2], 3.2).

(ii) We use the standard notation for semigroups. In particular,

— $E(\ )$ stands for the set of idempotents,

— $M(\ )$ for the minimal ideal (if it exists),

— $H(\ )$ for $\mathcal{H}$-classes.

(iii) We consider a (topological semigroup) compactification of S, for which we write (S^κ, κ), or S^κ for short. The universal such compactification, the Bohr compactification of S, is denoted by (S^b, b_S), or S^b.

(iv) Since no ambiguities are to be feared we denote both the left and the right natural action of S on its compactification S^κ by the dot ".". Thus $s.x = \kappa(s)x$ and $x.s = x\kappa(s)$ for every $x \in S^\kappa$ and $s \in S$.

(v) If $0 \neq X \in \mathfrak{sl}(2,\mathbb{R})$ with $\exp(\mathbb{R}^+X) \cap S \neq \emptyset$ then e_X denotes the (unique) idempotent in the minimal ideal of the compact abelian semigroup $\overline{\kappa(\exp(\mathbb{R}^+X) \cap S)}$. Since there is no danger of confusion we usually do not specify the particular compactification.

5.2. Frequently used facts about umbrella sets. (Cf. [2] 10.3, 3.4, 10.5)

(i) If A is a subsemigroup of G and $\exp X$ lies in the interior of A then $X \in \mathrm{Umb}(A)$.

(ii) If $G = \mathrm{Sl}_2$ and A is an open and connected proper subsemigroup of G then

(a) $A \subset \exp(\mathsf{Kill}^+)$,

(b) $\mathrm{Umb}(A) = \mathbb{R}^+ \operatorname{rlog}(A)$ and $\mathrm{Umb}(A) \subseteq \mathrm{Umb}(\overline{A}) \subseteq \overline{\mathrm{Umb}}(A)$,

(c) $\mathrm{Umb}(A)$ is the interior of an exponential Lie wedge, i.e., a wedge in $\overline{\mathsf{Kill}^+}$ which is the intersection of at most four half spaces, each bounded by a Borel algebra.

(d) If $X \in \overline{\mathrm{Umb}}(A) \cap \mathsf{Hyp}$ and $a \in \overline{A}$ then the pairs (X, a) and (a, X) lie in the standard domain and $\{X \circ a, a \circ X\} \subset \overline{\mathrm{Umb}}(A)$.

5.3. Proposition. *Let $x = \exp(X)$ be an interior point of S and suppose that $\langle n_i \rangle$ is a net of positive real numbers with $\lim n_i = +\infty$ and such that the limit $\ell = \lim \kappa(\exp(n_i X))$ exists. Then*

(i) *$\ell S^\kappa \ell = M_X$, where M_X is the minimal ideal of the compact abelian semigroup $\overline{\kappa(\exp(\mathbb{R}_0^+ X) \cap S)}$;*

(ii) *ℓ lies in the minimal ideal $M(S^\kappa)$ of S^κ, and $M_X = H(e_X)$;*

(iii) *there is a unique continuous homomorphism $k\colon \mathbb{R} \to H(e_X)$ with $k(t) = e_X . \exp(tX)$ whenever $\exp(tX) \in S$, under this homomorphism the image of $S \cap \exp(\mathbb{R}X)$ is dense in $H(e_X)$;*

(iv) *if $y = \exp Y \in S$ and X, Y lie in the span of a single horizontal [vertical] line then $e_X e_Y = e_Y$ [$e_X e_Y = e_X$], that is, e_X and e_Y lie in the same minimal right [left] ideal.*

Proof. Let $S_X := \exp(\mathbb{R}_0^+ X) \cap S$. We observe first that if $\langle t_i \rangle$ is a net of positive real numbers such that $\lim t_i = +\infty$ and the limit $m = \lim \kappa(\exp(t_i X))$ exists, then $m \in M_X$. To see this, consider an arbitrary element $\exp(aX) \in S_X$. The equality $\exp(t_i X) = \exp(aX) \exp((t_i - a)X)$ holds in S_X for sufficiently large t_i. We conclude that $m \in \exp(aX).\overline{\kappa(S_X)}$. This implies that $m \in M_X$.

(i) We know that $\operatorname{Int} S \subseteq \exp(\mathsf{Kill}^+)$, so there exists an inner automorphism of $\mathfrak{sl}(2, \mathbb{R})$ taking $\operatorname{rlog}(x)$ to H. Thus it suffices to show the assertion under the assumption that $X = tH$, where $t > 0$. Pick $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S$. Then $a > 0$, by 3.5(iii), and we find

$$\begin{aligned}
\exp(n_i X) s \exp(n_i X) &= \begin{pmatrix} e^{tn_i} & 0 \\ 0 & e^{-tn_i} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{tn_i} & 0 \\ 0 & e^{-tn_i} \end{pmatrix} \\
&= \begin{pmatrix} e^{2tn_i} a & b \\ c & de^{-2tn_i} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \frac{c}{a} e^{-2tn_i} & 1 \end{pmatrix} \begin{pmatrix} ae^{2tn_i} & 0 \\ 0 & \frac{e^{-2tn_i}}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} e^{-2tn_i} \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Since $x \in \operatorname{Int} S$ we conclude that for n_i sufficiently large the elements

$$u_i = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} e^{-2tn_i} & 1 \end{pmatrix} x \quad \text{and} \quad v_i = x \begin{pmatrix} 1 & \frac{b}{a} e^{-2tn_i} \\ 0 & 1 \end{pmatrix}$$

lie in $\operatorname{Int} S$ as well. Note that $\lim u_i = \lim v_i = x$.

Now we pass to a suitable subnet such that the elements

$$y_i \stackrel{\text{def}}{=} \begin{pmatrix} ae^{2n_i - 2t} & 0 \\ 0 & \frac{e^{-2n_i + 2t}}{a} \end{pmatrix}$$

lie in S and such that $y = \lim \kappa(y_i)$ exists. Then $y \in M_X$ and therefore

$$\ell.s.\ell = \lim \kappa(\exp(n_i X) s \exp(n_i X)) = \lim u_i.\kappa(y_i).v_i = x.y.x \in M_X.$$

Thus we have shown that $\ell.S.\ell \subseteq M_X$. We conclude that

$$M_X = \ell M_X \ell \subseteq \ell S^\kappa \ell = \overline{\ell.S.\ell} \subseteq M_X.$$

Hence $\ell S^\kappa \ell = M_X$.

(ii) By (i) $e_X S^\kappa e_X = \ell S^\kappa \ell = M_X$ is a group. We know from the general theory of compact semigroups that an idempotent e of a compact topological semigroup C lies in the minimal ideal of C if and only if eCe is a group. Thus $e_X \in M(S^\kappa)$ and $\ell \in M_X = e_X S^\kappa e_X = H(e_X) \subseteq M(S^\kappa)$.

(iii) By (ii) we have a continuous homomorphism $k_0 \colon [t_0, \infty[\to H(e_X)$, $t \mapsto e_X.\exp(tX)$, where t_0 is a positive real such that $\exp([t_0,\infty[\cdot X) \subset S$. Define

$$k(t) = \begin{cases} e_X.\exp(tX) & \text{if } t \geq t_0, \\ \exp(2t_0 X).(e_X.\exp(2t_0 - t)X)^{-1} & \text{if } t < t_0. \end{cases}$$

Then k is a homomorphism and its restriction to $]t_0, \infty[$ is continuous, so it is continuous everywhere. It is easily checked that $k(t) = e_X.\exp(tX)$ whenever $\exp(tX) \in S$.

(iv) As in the proof of (i) we assume that $X = tH$. We show the assertion for X, Y lying in the span of the same horizontal line, the other case follows by taking transposes. Then $Y = t'(H + \beta P)$ for suitable reals t', β. Replacing y by y^n for some $n \in \mathbb{N}$ we enforce that $x' := \exp t'H$ lies in the interior of S. Then

$$yx^{n_i} = x^{n_i} \exp(t'(H + \beta e^{-2n_i t} P))$$

and, since x' is an inner point of S, for large indexes i the point $\exp(t'(H + \beta e^{-2n_i t} P)$ lies in S. Applying κ and passing to limits we thus find that $y.\ell = \ell.x' \in H(\ell)$ and the assertion follows. ∎

5.4. Lemma. *Let $\langle x_i \rangle$ be a net in $\operatorname{Int} S$ with $\lim x_i = \infty$ and such that the limits $\ell = \lim \kappa(x_i)$ and $X = \lim \operatorname{rlog}(x_i)$ exist. Suppose that $X \in \operatorname{Umb}(\operatorname{Int} S)$. Then the following assertions hold:*

(i) $\ell \in e_X S^\kappa e_X$.

(ii) *ℓ lies in the minimal ideal $M(S^\kappa)$ of S^κ.*

(iii) $\ell \in H(e_X)$.

Proof. (i) Since $X \in \operatorname{Umb}(\operatorname{Int} S)$ we find, by virtue of 10.3(ii) in [2], a neighborhood U of X in $\mathfrak{sl}(2,\mathbb{R})$ and a positive number t_0 such that for all $Y \in U$ and all $t \geq t_0$ the exponential image $\exp(tY)$ is contained in $\operatorname{Int} S$. For convenience, we write $X_i = \operatorname{rlog}(x_i)$. Then $x_i = \exp(t_i X_i)$ for some $t_i \in \mathbb{R}^+$. Since $\lim x_i = \infty$, we have that $\lim t_i = +\infty$. Fix some number $t > t_0$. Then for sufficiently large indexes i we have $X_i \in U$ and $t_i - t \geq t_0$, so the elements $\exp(tX_i)$ and $\exp((t_i - t)X_i)$ lie in S. Passing to a suitable subnet we enforce that $y = \lim \kappa(\exp((t_i - t)X_i))$ exists. Then

$$\exp(tX).y = \lim \kappa(\exp(tX_i))\kappa(\exp((t_i - t)X_i)) = \lim \kappa(\exp(t_i X_i)) = \ell.$$

Similarly $y.\exp(tX) = \ell$. Thus $\ell \in \exp(tX).S^\kappa \cap S^\kappa.\exp(tX)$ for all $t > t_0$. Plugging in a net $\langle t_j \rangle$ with $\kappa(\exp(t_j X)) \to e_X$ and passing to limits we therefore see that $\ell \in e_X S^\kappa e_X$, as asserted.

Assertions (ii) and (iii) follow from (i) and Proposition 5.3. ∎

5.5. Remark. For the next lemma note that for every net $\langle X_i \rangle$ in Hyp with $\lim X_i = \infty$ there exist a subnet $\langle X_j \rangle$ and positive reals λ_j such that $\langle \lambda_j X_j \rangle$ converges to a nonzero nilpotent matrix.

We also recall that Sl_2^+ denotes the set of all matrices in Sl_2 with nonnegative entries. It is known from [2] 10.5 that every proper subsemigroup of Sl_2 with dense and connected interior is contained in an exponential subsemigroup, hence, by [2] 3.8, is conjugate to a subsemigroup of Sl_2^+.

5.6. Lemma. *Suppose that S is contained in Sl_2^+ and let $\langle x_i \rangle$ be a net in $\operatorname{Int} S$ such that $x_i \to \infty$ and $\operatorname{rlog}(x_i) \to \infty$ in $\overline{\operatorname{Umb}}(S)$. We assume that $\lim \lambda_i \operatorname{rlog}(x_i) = P$ for a suitable net $\langle \lambda_i \rangle$ of positive reals. Then for every $x = \begin{pmatrix} v & w \\ y & z \end{pmatrix} \in \operatorname{Int} S$*

(i) $\lim \operatorname{rlog}(x x_i) = -H + 2\frac{v}{y} P$,

(ii) $\lim \operatorname{rlog}(x_i x) = H + 2\frac{z}{y} P$.

Proof. Put $\operatorname{rlog}(x_i) = \alpha_i H + \beta_i P + \gamma_i Q$. Then β_i, γ_i are nonnegative and thus the relation $\alpha_i^2 + \beta_i\gamma_i = 1$ implies that $\gamma_i \leq 1/\beta_i$. Since $\lambda_i \beta_i \to 1$ we must have $\beta_i \to \infty$ and $\gamma_i \to 0$.

Write $x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \exp(t_i \operatorname{rlog}(x_i))$ for some positive t_i. We claim that $b_i \to +\infty$. If this is not the case, then $\langle b_i \rangle$ has a convergent subnet $\langle b_\ell \rangle$. The formula for the exponential function (see, for example, 2.5 of [2]) implies then that $t_\ell \to 0$, hence $\langle a_\ell \rangle$, $\langle c_\ell \rangle$, and $\langle d_\ell \rangle$ are bounded, which contradicts the assumption that $x_i \to \infty$. Thus $b_i \to +\infty$.

The formula for rlog (see 2.5 of [2]) says that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\operatorname{rlog}(g) = \begin{pmatrix} \frac{a-d}{u} & \frac{2b}{u} \\ \frac{2c}{u} & -\frac{a-d}{u} \end{pmatrix} \quad \text{with} \quad u = \sqrt{(a+d)^2 - 4}.$$

Thus $\beta_i = \frac{2b_i}{\sqrt{(a_i+d_i)^2-4}}$. Since $\beta_i \to +\infty$ and $b_i \to +\infty$ we therefore get

$\frac{a_i+d_i}{b_i} \to 0$, hence

$$\frac{a_i}{b_i} \to 0 \quad\text{and}\quad \frac{d_i}{b_i} \to 0. \quad\text{Also}\quad \frac{c_i}{b_i} = \frac{\gamma_i}{\beta_i} \to 0.$$

Now $\operatorname{rlog}(xx_i) = \begin{pmatrix} \alpha_i^* & \beta_i^* \\ \gamma_i^* & -\alpha_i^* \end{pmatrix}$ with

$$\begin{aligned}
\alpha_i^* &= \frac{va_i + wc_i - (yb_i + zd_i)}{\sqrt{(va_i + wc_i + yb_i + zd_i)^2 - 4}} \longrightarrow -1 \\
\beta_i^* &= 2\frac{vb_i + wd_i}{\sqrt{(va_i + wc_i + yb_i + zd_i)^2 - 4}} \longrightarrow 2\frac{v}{y} \\
\gamma_i^* &= 2\frac{ya_i + zc_i}{\sqrt{(va_i + wc_i + yb_i + zd_i)^2 - 4}} \longrightarrow 0.
\end{aligned}$$

This establishes (i). The proof of (ii) is left to the reader. ■

5.7. Notation. In the following we use the abbreviation (slightly at variance with the notation in [2]) $D := \overline{\operatorname{Umb}}(S) \cap \mathsf{Hyp} = \overline{\operatorname{rlog}(S \cap \exp(\mathsf{Kill}^+))}$. The set D is called the *set of regular directions* in S.

5.8. Theorem. *Let S be a closed proper subsemigroup of Sl_2 with dense and connected interior. Let $\langle x_n \rangle$ be a net in $\operatorname{Int} S$ with $\lim x_n = \infty$ and such that the limit $\ell = \lim \kappa(x_n)$ exists. Then there exists a subnet $\langle x_i \rangle$ of $\langle x_n \rangle$ such that one of the following assertions holds:*

(i) $X = \lim \operatorname{rlog}(x_i)$ *exists and lies in the interior of D in Hyp. Then $\ell \in H(e_X) \subseteq M(S^\kappa)$, where e_X is the unique minimal idempotent in the closure of $\kappa(\exp(\mathbb{R}^+ X) \cap S)$.*

(ii) $X = \lim \operatorname{rlog}(x_i)$ *exists and lies in the interior of one of the horizontal line segments belonging to the boundary of D in Hyp. Then $S^\kappa \ell \subseteq M(S^\kappa)$.*

(iii) $X = \lim \operatorname{rlog}(x_i)$ *exists and lies in the interior of one of the vertical line segments belonging to the boundary of D in Hyp. Then $\ell S^\kappa \subseteq M(S^\kappa)$.*

(iv) *Either $\langle \operatorname{rlog}(x_i) \rangle$ converges to a corner point of D or $\operatorname{rlog}(x_i) \to \infty$. Then $S^\kappa \ell S^\kappa = M(S^\kappa)$.*

Proof. We first pass to a subnet $\langle x_i \rangle$ such that either $X = \lim \operatorname{rlog}(x_i)$ exists or $\operatorname{rlog}(x_i) \to \infty$.

(i) This follows from Lemma 5.4 and the fact that $\operatorname{Int} D = \operatorname{Umb}(\operatorname{Int} S) \cap \mathsf{Hyp}$ (cf. [2] 10.5(ii)).

(ii) We conclude from 3.9(i),(ii) that $s \circ X$ lies in the interior of D whenever s lies in the dense subset $\operatorname{Int} S \setminus \exp(\mathbb{R}\,\mathsf{hor}(X))$ of S. For such an element s we have $\lim sx_i = \infty$ and, by 3.8(i), $\operatorname{rlog}(sx_i) = s \circ X$. Applying (i) we see that $(\operatorname{Int} S \setminus \exp(\mathbb{R}\,\mathsf{hor}(X))).\ell \subseteq M(S^\kappa)$. Since $M(S^\kappa)$ is closed and $\kappa(\operatorname{Int} S \setminus \exp(\mathbb{R}\,\mathsf{hor}(X)))$ is dense in S^κ this implies the assertion.

(iii) follows from (ii) by applying the anti-automorphism $x \mapsto x^{-1}$.

(iv) By minimality it obviously suffices to show $S^\kappa \ell S^\kappa \subseteq M(S^\kappa)$.

Assume first that $X = \lim \operatorname{rlog}(x_i)$ is a corner point of D. Then by 3.9(i),(ii) the point $s \circ X$ lies in the interior of one of the vertical line segments belonging to the boundary of D in Hyp, provided $s \in \operatorname{Int} S \setminus \exp(\mathbb{R}\,\mathsf{hor}(X))$. Thus, by (iii) we see that $(\operatorname{Int} S \setminus \exp(\mathbb{R}\,\mathsf{hor}(X))).\ell S^\kappa \subseteq M(S^\kappa)$, and, consequently, $S^\kappa \ell S^\kappa \subseteq M(S^\kappa)$.

Assume now that $\operatorname{rlog}(x_i) \to \infty$. Then, not losing generality, we also assume that $S \subseteq \mathrm{Sl}_2^+$ and that we can find positive numbers λ_i such that $\lambda_i \operatorname{rlog}(x_i) \to P$. By Lemma 5.6(i) we know that for every s lying in a dense subset of S the limit $\lim \operatorname{rlog}(sx_i)$ lies in the interior of the line segment $\mathsf{vert}(-H) \cap D$. For all such points s we infer from (iii) that $s.\ell S^\kappa \subseteq M(S^\kappa)$, so the assertion follows from the compactness of $M(S^\kappa)$. ■

5.9. Large elements. In the following an element $s \in S^\kappa$ is said to be S-*large*, or *large* for short, if it is the limit $s = \lim \kappa(x_n)$ where $\langle x_n \rangle$ is a net in S with $\lim x_n = \infty$. Note that the set of large elements is the intersection $\bigcap \overline{\kappa(A)}$, where each A is a cocompact set in S. Since, by our general assumption, S is a subsemigroup of a group it follows without difficulty that the S-large elements form a closed ideal in S^κ which contains the growth $S^\kappa \setminus \kappa(S)$ of the compactification S^κ.

5.10. Theorem. *Let S be a closed proper subsemigroup of* Sl_2 *with dense and connected interior. Then an S-large element $m \in S^\kappa$ lies in the minimal ideal $M(S^\kappa)$ if and only if $m \in S^\kappa m S^\kappa$. In particular, if S contains the identity then every S-large element m belongs to $M(S^\kappa)$.*

Proof. By 5.8 we know that, in any case, $S^\kappa m S^\kappa \subseteq M(S^\kappa)$. This establishes our assertion. ■

5.11. Example. If in the above theorem S is not a monoid then the growth of a compactification of S need not be contained in the minimal ideal.

(i) We first remark that for any topological semigroup with constant multiplication $xy = z_0$ the Bohr compactification is equivalent with the Stone-Čech compactification, with constant multiplication $x'y' = \beta(z_0)$. In this case the minimal ideal of the Bohr compactification never meets the growth of the Bohr compactification, and the growth is nonvoid whenever S is completely regular but noncompact.

(ii) If $\mathcal{S}$ is any topological semigroup on a locally compact normal space such that the Rees quotient $\mathcal{S}/\overline{\mathcal{S}^2}$ is not compact then $\mathcal{S}$ has a topological semigroup compactification whose growth is nonvoid and does not meet the minimal ideal. This follows from (i) and the fact that $\mathcal{S}_1 = \mathcal{S}/\overline{\mathcal{S}^2}$ is a semigroup with constant multiplication. (Since $\mathcal{S}$ is defined on a normal space the Rees quotient $\mathcal{S}/\overline{\mathcal{S}^2}$ is completely regular, hence its Stone-Čech compactification is an embedding, with nonvoid growth since $\mathcal{S}/\overline{\mathcal{S}^2}$ is noncompact.) By the universality of the Bohr compactification this implies that the growth of $\mathcal{S}^b$ cannot be contained entirely in its minimal ideal.

(iii) Let S be the semigroup consisting of the matrices in Sl_2^+ where both diagonal entries are ≥ 2. This semigroup is connected and has dense interior, since it can be written also as the set of all matrices of the form

$$s(a,b,u) = \begin{pmatrix} a & b \\ (u-1)/b & u/a \end{pmatrix}, \quad u \geq 2a \geq 4, b > 0.$$

Note that $s(a,b,u) \in S^2$ always implies $a \geq 4$, thus the Sl_2-closed set $C = \{s(2,b,4) \mid b > 0\}$ is properly contained in $S \setminus \overline{S^2}$, so C is not relatively compact in S. Thus by (ii) the growth of S in its Bohr compactification is not contained in the minimal ideal. In fact, $(b_S \mid C, \overline{b_S(C)})$ is equivalent with the Stone-Čech compactification of the reals and $\overline{b_S(C)}$ does not meet the minimal ideal.

6. Injectivity and Noninjectivity of Compactifications

6.1. Theorem. *Suppose that (S^κ, κ) is a topological semigroup compactification of a closed connected proper submonoid S of Sl_2 with dense interior, and write $I \stackrel{\text{def}}{=} \kappa^{-1}(M(S^\kappa))$.*

(i) *An element $s \in S$ lies in I if and only if there exists an $s' \neq s$ with $\kappa(s) = \kappa(s')$. If I is nonempty then $\kappa(I) = \{z\}$, where z is the zero element of S^κ.*

(ii) *The restriction of κ to $S \setminus I$ is a homeomorphic embedding.*

(iii) *The following assertions are equivalent:*

  (a) *κ is injective.*

  (b) *κ is a homeomorphic embedding.*

  (c) *I is empty.*

  (d) *κ is not surjective.*

Proof. (i) It is shown in [4] 2.1 that every closed congruence on a closed connected submonoid with dense interior of Sl_2 is either the identity or a Rees congruence. Thus assertion (i) holds whenever the kernel congruence of κ is not the identity, that is, if κ is not injective.

Suppose now that κ is injective and assume that $x \in I$. If $x = \mathbf{1}$ then $S^\kappa = M(S^\kappa)$ and $M(S^\kappa)$ contains a central idempotent, so S^κ is a group. Now the following general proposition, recorded also for later use, shows that S^κ is singleton, in contradiction to the injectivity of κ.

6.2. Proposition. *Let S be a connected subsemigroup with dense interior of a connected noncompact simple Lie group G. Then every continuous homomorphism of S into a compact topological group is trivial.*

Proof. By [11] VII.3.28 the free topological group on S is a covering group of G. Since every connected noncompact simple Lie group is minimally almost periodic this implies that the free topological group on S has no nontrivial continuous homomorphisms into a compact topological group. By the definition of the free topological group on S this finishes the proof. ■

6.3. Proof of 6.1(i) (ctd). Next we suppose that $\kappa(x) \in M(S^\kappa)$ with $x \neq \mathbf{1}$. Then the centralizer of x is nowhere dense in Sl_2 and we can find a point $s \in S$ with $x^2sx \neq xsx^2$. But $\kappa(x).S.\kappa(x) \subseteq H(\kappa(x))$, hence, since the maximal subgroups of S^κ are abelian, $\kappa(x^2sx) = \kappa(x)\kappa(xsx) = \kappa(xsx)\kappa(x) = \kappa(xsx^2)$. Since this contradicts the injectivity of κ we conclude that I must be empty whenever κ is injective and the assertion follows.

(ii) Suppose that $\langle s_n \rangle$ is a net in $S \setminus I$ such that $\lim \kappa(s_n) = \kappa(s)$, for some $s \in S \setminus I$.

If $\langle s_{n'} \rangle$ were a subnet of $\langle s_n \rangle$ with $\lim s_{n'} = \infty$ then, by Theorem 5.10, $\kappa(s) \in M(S^\kappa)$, a contradiction to $s \in S \setminus I$.

By (i) every convergent subnet of $\langle s_n \rangle$ must converge to s, so we conclude that $\lim s_n = s$ and the proof is finished.

(iii) The proof of (i) also shows the implication (a) $\Longrightarrow$ (c). The implication (c) $\Longrightarrow$ (b) is a consequence of (ii), and (b) $\Longrightarrow$ (a) is trivial. From (i) and the inclusion $S^\kappa \setminus \kappa(S) \subseteq M(S^\kappa)$ we infer that κ is surjective if $I \neq \emptyset$, so (d) $\Longrightarrow$ (c). Since S does not contain any idempotent except $\mathbf{1}$ we also have (a) $\Longrightarrow$ (d). ■

The following corollary is only a handy reformulation of some of the assertions in 6.1, its proof is therefore omitted.

6.4. Corollary. *Let S be a closed connected proper submonoid of* Sl_2 *with dense interior and let (S^κ, κ) be a (topological) semigroup compactification of S. Then the following assertions are equivalent and imply that S^κ contains a zero element:*

(i) *The compactification map $\kappa: S \to S^\kappa$ is not injective.*

(ii) *There exists a closed (nonvoid) ideal I of S such that κ is constant on I.*

(iii) *$\kappa: S \to S^\kappa$ is surjective.*

(iv) *$\kappa(S)$ meets the minimal ideal $M(S^\kappa)$.* ■

6.5. Aliens. Recall from [3] that an element s in a locally compact semigroup S is called an *alien* if it is the limit of a net $\langle x_n y_n \rangle$, where $x_n \to \infty$ or $y_n \to \infty$. The set of all aliens in S is denoted by $\mathrm{Al}(S)$. A locally compact topological semigroup is perfect if and only if it contains no aliens. If z is an alien then for any compactification map κ the image $\kappa(z)$ is a large element in the sense of Section 5.9.

6.6. Corollary. *Let S be a closed connected submonoid of* Sl_2 *with dense interior. Then S contains aliens if and only if none of its compactification maps is injective. Equivalently: S is perfect if and only if there exists a compactification (S^κ, κ) such that $\kappa(S)$ does not meet the minimal ideal $M(S^\kappa)$.*

Proof. This follows from 5.10 and 6.4. ∎

6.7. Corollary. *Let S be a closed subsemigroup of* Sl_2 *with dense and connected interior. Then the following assertions are equivalent:*

(i) $S^b = M(S^b)$.

(ii) *All elements of S are aliens in S.*

Proof. The assertion is trivial in the case $S = \mathrm{Sl}_2$, so we assume $S \neq \mathrm{Sl}_2$. (i) $\Longrightarrow$ (ii) By [3], 5.2, 5.5, and 5.6 (cf. also 6.9 below), we know that the large ideal compactification maps the non-aliens of S onto non-minimal elements, so (i) implies (ii).

(ii) $\Longrightarrow$ (i) Note first that by definition, every point $s \in S$ is the limit of products $s_n t_n$ with $\lim s_n = \lim t_n = \infty$. Applying this fact to the factors s_n we see that every $s \in S$ is also the limit of triple products $x_n y_n z_n$ with $\lim x_n = \lim y_n = \lim z_n = \infty$. Passing to suitable subnets we enforce that the limits $u = \lim b_S(x_n)$, $v = \lim b_S(y_n)$, and $w = \lim b_S(z_n)$ exist. We conclude that $b_S(s) = uvw \in S^b v S^b$, hence, by Theorem 5.8, $S^b = M(S^b)$. ∎

6.8. Example. (Cf. [3], 7.6) The set

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2^+ \;\middle|\; a + c \geq b + d,\ c \geq d \right\}$$

is a closed subsemigroup of Sl_2 with dense and connected interior, and all of its elements are aliens, thus, by the above corollary, its Bohr compactification is a paragroup. In fact, its Bohr compactification is a singleton set: We have seen in [3],7.6 that for every element s in a dense subset of the semigroup and all $n \in \mathbb{N}$ there exist elements y_n, z_n in the semigroup such that

$$s = \lim \begin{pmatrix} n & 0 \\ 1/n & 1/n \end{pmatrix} y_n = \lim z_n \begin{pmatrix} 1/n & 0 \\ n & n \end{pmatrix}.$$

This means that the Bohr compactification map maps s onto $e.s = s.f$, where e and f are fixed idempotents. It follows that the Bohr compactification of our semigroup is a left group as well as a right group, hence is a group. But since Sl_2 is simple and noncompact this group must be singleton, by 6.2.

6.9. Large ideals. The following definitions and facts are taken from [3], section 5.

(i) Let S be a noncompact locally compact topological semigroup. An ideal I of S is said to be a *large ideal* if $\overline{S \setminus I}$ is compact. Note that for closed large ideals I the Rees quotient S/I is always a compact topological semigroup.

(ii) If the intersection of all closed large ideals in S is nonempty then it is an ideal in S. If S is a subsemigroup of a topological group (more generally, if S is c-perfect ([3], 2.4)) then every element in this intersection is an alien.

(iii) Every non-alien element in a noncompact locally compact subsemigroup of a topological group G has a neighborhood which is the complement of a large ideal.

6.10. Large ideal compactifications. In the following assertions (i)–(iii) we let S be a noncompact locally compact subsemigroup of a topological group G (or, more generally, a noncompact c-perfect locally compact topological semigroup).

(i) Let A be a (possibly void) subset of S. Then the set $\mathcal{I}_{\mathcal{A}}$ of all closed large ideals of S containing A is directed under $\supseteq$ since for $I, J \in \mathcal{I}_{\mathcal{A}}$ the intersection $I \cap J$ also lies in $\mathcal{I}_{\mathcal{A}}$. The intersection $\bigcap \mathcal{I}_{\mathcal{A}}$ is the closed ideal generated by $A \cup \mathrm{Al}(S)$.

(ii) The limit S^{ℓ_A} of the projective system $\{S/I\}_{I \in \mathcal{I}_{\mathcal{A}}}$ exists and is a compact semigroup (cf. Theorem 2.22 of [6]), we write ℓ_A for the corresponding morphism $S \to S^{\ell_A}$. If A is empty then we call the ensuing compactification the (universal) *large ideal compactification* of S and denote it by (S^ℓ, ℓ) or S^ℓ for short.

(iii) The compactifications S^{ℓ_A} with $A \neq \emptyset$ are in 1-1-correspondence with the closed ideals of S which contain $\mathrm{Al}(S)$. In particular our discussion shows that for every closed ideal I of S containing $\mathrm{Al}(S)$ there exists a compactification (S^κ, κ) such that the restriction of κ to I is constant and the restriction of κ to $S \setminus I$ is a topological embedding.

6.11. Notation. (i) We write $\mathrm{NONINJ}(S)$ for the set of equivalence classes (S^κ, κ) of compactifications of S with κ noninjective, augmented by the large ideal compactification (S^ℓ, ℓ). (Note that ℓ is injective if and only if $\mathrm{Al}(S) = \emptyset$.) We provide NONINJ with the usual order of compactifications.

(ii) We write $\mathrm{ALID}(S)$ for the set of all closed ideals of S which contain $\mathrm{Al}(S)$, augmented by the empty set if $\mathrm{Al}(S) = \emptyset$. The set $\mathrm{ALID}(S)$ is endowed with the usual order $\subseteq$.

6.12. Theorem. *Let S be a closed connected submonoid of Sl_2 with dense interior. Then the following assertions hold:*

(i) *The map $\mathrm{NONINJ}(S) \to \mathrm{ALID}(S)$, assigning to (S^κ, κ) the set $I = \kappa^{-1}(M(S^\kappa))$, is an order anti-isomorphism.*

(ii) *If $\mathrm{Al}(S) \neq \emptyset$ then the Bohr compactification of S is equivalent with the large ideal compactification.*

Note that the ordered sets $\mathrm{NONINJ}(S)$ and $\mathrm{ALID}(S)$ form complete lattices.

Proof. (i) follows from 6.4 and the remarks in 6.10.

(ii) The assertion trivially holds if $S = \mathrm{Sl}_2$. So let us suppose that $S \neq \mathrm{Sl}_2$. We already know by 5.10 that $S^b = b_S(S) \cup M(S^b)$ and that $b_S(\mathrm{Al}(S)) \subseteq$

$M(S^b)$. Since $\mathrm{Al}(S)$ is nonvoid we know by 6.1 and 6.6 that $M(S^b)$ is a singleton. So the natural morphism $S^b \to S^\ell$ separates the points of S^b and the assertion follows. ∎

7. Idempotents and Directions

7.1. We start this section with the observation that the map of 5.1(v), assigning to every $X \in \mathrm{rlog}(S \cap \exp(\mathsf{Kill}^+))$ the idempotent e_X, is continuous and can be extended continuously to the set $D = \overline{\mathrm{rlog}(S \cap \exp(\mathsf{Kill}^+))}$ of regular directions in S. A further extension, dealing also with 'non-regular directions,' will be given in 7.5 below.

7.2. Proposition. *Assume that S is a closed connected proper submonoid of* Sl_2 *with dense interior. Then there exists a unique continuous map $\varepsilon\colon D \to E(M(S^\kappa))$ with the following property* (a):

(a) *If $\langle x_i \rangle$ is a net in $S \cap \exp(\mathsf{Kill}^+)$ with*

$$\lim x_i = \infty, \quad \lim \mathrm{rlog}(x_i) = X, \quad \lim \kappa(x_i) = m \in S^\kappa,$$

then $m \in H(\varepsilon(X))$.

Proof. We first pick a point $X \in D$ and claim that there exists a unique idempotent $\varepsilon(X) \in M(S^\kappa)$ such that (a) holds.

The definition of Umb immediately implies that there exists a net $\langle x_i \rangle$ in $S \cap \exp(\mathsf{Kill}^+)$ such that $\lim x_i = \infty$, $\lim \mathrm{rlog}(x_i) = X$, we may also assume that $\lim \kappa(x_i) = m \in S^\kappa$ exists. By 5.10, $m \in M(S^\kappa)$, so there exists a minimal idempotent $\varepsilon(X)$ with $m \in H(\varepsilon(X))$. To show that the definition of $\varepsilon(X)$ does not depend on the choice of $\langle x_i \rangle$, suppose that $\ell = \lim \kappa(y_j)$ for another net $\langle y_j \rangle$ in $S \cap \exp(\mathsf{Kill}^+)$ with $\lim y_j = \infty$, $\lim \mathrm{rlog}(y_j) = X$. Then, by 5.10, $\ell \in M(S^\kappa)$, so we have to prove that $\ell \in mS^\kappa m$. Pick $(a,b) \in \mathrm{Int}\, S \times \mathrm{Int}\, S$ such that $a \circ X \circ b \in \mathrm{Int}\, D$, the set of all such pairs is dense in $S \times S$ (cf. 3.9(i),(ii)). By the asymptotic formulas 3.8 and by Lemma 5.4, we have $\{a.\ell.b, a.m.b\} \subseteq H(e_{a \circ X \circ b})$ and therefore $a.\ell.b \in (a.m.b)S^\kappa(a.m.b)$. Taking limits $a \to \mathbf{1}, b \to \mathbf{1}$ proves our claim.

We now show that ε is continuous. Consider a net $\langle X_i \rangle$ of $\mathrm{Umb}(\mathrm{Int}\, S) \cap \mathsf{Hyp}$ converging to $X \in D$. Then there exists a net $\langle t_i \rangle$ of positive reals with $t_i \to \infty$ and such that $\exp(t_i X_i) \in \mathrm{Int}\, S$ for every index i. Clearly, $\lim \exp(t_i X_i) = \infty$ and we have

$$\varepsilon(X_i) = e_{X_i} \in \exp(t_i X_i).H(e_{X_i}).\exp(t_i X_i).$$

Then, by (a), every limit of a convergent subnet of the net $\langle e_{X_i} \rangle$ must be equal to $\varepsilon(X)$, hence every convergent subnet of $\langle e_{X_i} \rangle$ has the limit $\varepsilon(X)$. This shows that $\lim \varepsilon(X_i) = \varepsilon(X)$. The asserted continuity now follows from the fact that $\mathrm{Umb}(\mathrm{Int}\, S) \cap \mathsf{Hyp}$ is dense in D. ∎

7.3. Basic properties of the map ε. We retain the assumptions of 7.2.

(i) *If* $x = \exp(tX) \in S$ *with* $X \in \mathsf{Hyp}$, $t > 0$, *then* $X \in D$ *and* $e_X = \varepsilon(X)$.

(ii) *If two elements* X, X' *of* D *lie on the same horizontal [vertical] line then the corresponding idempotents* $\varepsilon(X), \varepsilon(X')$ *lie in the same minimal right [left] ideal of* S^κ.

(iii) *If* $X, Y \in D$ *and the diamond products* $X \diamond Y$ *and* $Y \diamond X$ *exist then* $\varepsilon(X)\varepsilon(Y) \in H(\varepsilon(X \diamond Y))$, *or, equivalently,* $\varepsilon(X \diamond Y)\varepsilon(Y) = \varepsilon(X)\varepsilon(X \diamond Y) = \varepsilon(X \diamond Y)$.

(iv) *If* $X \in D$ *and* $s \in S$ *then* $s.\varepsilon(X) \in H(\varepsilon(s \circ X)))$ *and* $\varepsilon(X).s \in H(\varepsilon(X \circ s)))$.

(v) *If* $X \in D$ *then* $\mathsf{vert}(X) \cap D = \overline{S \circ X}$, *and* $E(S^\kappa \varepsilon(X)) = \overline{\varepsilon(\mathsf{vert}(X) \cap D)}$. *Similarly,* $\mathsf{hor}(X) \cap D = \overline{X \circ S}$, *and* $E(\varepsilon(X) S^\kappa) = \overline{\varepsilon(\mathsf{hor}(X) \cap D)}$.

(vi) *If* X *is an interior point of* D *in* Hyp *then the sets* $\mathsf{hor}(X) \cap D$ *and* $\mathsf{vert}(X) \cap D$ *are compact line segments and the map*

$$(\mathsf{vert}(X) \cap D) \times (\mathsf{hor}(X) \cap D) \to E(M(S^\kappa)),$$
$$(A, B) \mapsto (\varepsilon(A)\varepsilon(B))^{-1}\varepsilon(A)\varepsilon(B),$$

is continuous and surjective.

(vii) *If* $\varepsilon(X) = \varepsilon(Y)$ *then for every* $Z \in D$ *we also have* $\varepsilon(X \diamond Z) = \varepsilon(Y \diamond Z)$ *and* $\varepsilon(Z \diamond X) = \varepsilon(Z \diamond Y)$, *provided that the diamond products exist.*

(viii) *If* $\varepsilon(X) = \varepsilon(Y)$ *then for every* $s \in S$ *we also have* $\varepsilon(s \circ X) = \varepsilon(s \circ Y)$ *and* $\varepsilon(X \circ s) = \varepsilon(Y \circ s)$.

Proof. Assertion (i) is left to the reader, (ii) follows from 5.3(iv) since ε is continuous, and (iii) is a direct consequence of (ii) and the definition of the $\diamond$-product.

(iv) Let $\langle x_i \rangle$ be a net in $S \cap \exp(\mathsf{Kill}^+)$ such that $x_i \to \infty$, $\mathrm{rlog}(x_i) \to X$, and $\lim \kappa(x_i) = \varepsilon(X)$. Then $\mathrm{rlog}(s x_i) \to s \circ X$ and $\mathrm{rlog}(x_i s) \to X \circ s$, by 3.8, so the assertion follows now from 7.2.

(v) The assertions 3.7(i) and 3.9(ii) yield that $\mathsf{vert}(X) \cap D = \overline{S \circ X}$ and $\mathsf{hor}(X) \cap D = \overline{X \circ S}$. Combined with (iv), these equalities finally imply that $E(S^\kappa \varepsilon(X)) = \overline{\varepsilon(\mathsf{vert}(X) \cap D)}$ and $E(\varepsilon(X) S^\kappa) = \overline{\varepsilon(\mathsf{hor}(X) \cap D)}$.

(vi) The compactness of $\mathsf{hor}(X) \cap D$ and $\mathsf{vert}(X) \cap D$ follows from the classification of rectangular domains in section 7 and example 5.3 of [2], it is also not difficult to devise a direct proof. Now (v) implies that $E(S^\kappa \varepsilon(X)) = \varepsilon(\mathsf{vert}(X) \cap D)$ and $E(\varepsilon(X) S^\kappa) = \varepsilon(\mathsf{hor}(X) \cap D)$, and therefore the continuity as well as the surjectivity of the map in (v) follows from the general theory of compact topological semigroups and the continuity of ε.

Assertion (vii) is an immediate consequence of (iii), (viii) follows from (iv). ∎

7.4. A useful formula. *We retain the assumptions of* 7.2 *and suppose that the matrix H is an interior point of D in* Hyp. *We write k for the unique continuous homomorphism $\mathbb{R} \to H(\varepsilon(H))$ such that $k(t) = \varepsilon(H).\exp(tH)$ whenever* $\exp(tH) \in S$ (cf. 5.3(iii)). *Then for every matrix $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S$ we have the formula*

$$\varepsilon(H)\kappa(s)\varepsilon(H) = k(\log(a)).$$

This formula will be applied in 7.9.

Proof. Let $\langle t_i \rangle$ be a net in $\mathbb{R}^+$ with $t_i \to \infty$ and $\lim \kappa(\exp(t_i H)) = \varepsilon(H)$. Then

$$(*) \qquad \exp(t_i H) s \exp(t_i H) = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} e^{-2t_i} & 1 \end{pmatrix} \begin{pmatrix} a e^{2t_i} & 0 \\ 0 & \frac{e^{-2t_i}}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} e^{-2t_i} \\ 0 & 1 \end{pmatrix}.$$

For any inner point u of S and all sufficiently large indexes i the points

$$u \begin{pmatrix} 1 & 0 \\ \frac{c}{a} e^{-2t_i} & 1 \end{pmatrix}, \quad \begin{pmatrix} a e^{2t_i} & 0 \\ 0 & \frac{e^{-2t_i}}{a} \end{pmatrix} = \exp((2t_i + \log(a))H), \quad \begin{pmatrix} 1 & \frac{b}{a} e^{-2t_i} \\ 0 & 1 \end{pmatrix} u$$

are contained in S. Note that $\varepsilon(H)\kappa(\exp((2t_i + \log(a))H)) = k(2t_i + \log(a))$. Now, applying κ to $(*)$ and passing to limits, we see that $u.\varepsilon(H)\kappa(s)\varepsilon(H).u = u.\varepsilon(H)k(\log(a))\varepsilon(H).u$. Since u can be chosen arbitrarily near the identity this implies the assertion. ∎

The next proposition shows that ε is compatible with the map c of 3.2(iv), in that it can be extended to a continuous map on the set of asymptotic directions $\mathrm{Asy}(S) = \overline{c(D)}$.

7.5. Proposition. *Under the assumptions of* 7.2 *we always have a continuous and surjective map $\bar{\varepsilon}\colon \mathrm{Asy}(S) \to E(M(S^\kappa))$ which extends ε, that is, $\bar{\varepsilon}(c(X)) = \varepsilon(X)$, for all $X \in D$. This map has the following properties:*

(i) *If $\langle x_i \rangle$ is a net in $S \cap \exp(\mathsf{Kill}^+)$ with*

$$\lim x_i = \infty, \quad \lim c(\mathrm{rlog}(x_i)) = (h, v), \quad \lim \kappa(x_i) = m \in S^\kappa,$$

then $m \in H(\bar{\varepsilon}(h, v))$.

(ii) *If $(h, v), (h', v') \in \overline{c(D)}$ then $\bar{\varepsilon}(h, v)\bar{\varepsilon}(h', v') \in H(\bar{\varepsilon}(h, v'))$.*

(iii) *$\bar{\varepsilon}$ is injective if and only if ε is injective.*

Proof. We pick an interior point $Z \in D$ and claim that $\overline{c(D)} = \mathsf{hor}(D \diamond Z) \times \mathsf{vert}(D \diamond Z)$. Indeed, for $X \in D$ we have $\mathsf{hor}(X) = \mathsf{hor}(X \diamond Z)$ and $\mathsf{vert}(X) = \mathsf{vert}(Z \diamond X)$, thus $\overline{c(D)} \subseteq \mathsf{hor}(D \diamond Z) \times \mathsf{vert}(D \diamond Z)$. On the other hand, if $(h, v) \in \mathsf{hor}(D \diamond Z) \times \mathsf{vert}(D \diamond Z)$ with $h \cap v \neq \emptyset$ then $(h, v) \in c(D)$. Since $h \cap v = \emptyset$ implies that neither h nor v can meet the interior of D, we conclude that $c(D)$ misses at most two points of $\mathsf{hor}(D \diamond Z) \times \mathsf{vert}(D \diamond Z)$. (In fact, the missing points must be corner points.)

We define $\bar{\varepsilon}(h, v) = e$, where e is the unique idempotent in the $\mathcal{H}$-class of $\varepsilon(U)\varepsilon(V)$ and $h \cap \mathsf{vert}(Z) = \{U\}$, $v \cap \mathsf{hor}(Z) = \{V\}$. Obviously, $\bar{\varepsilon}$ extends

ε. By Corollary 7.3(vi), $\bar{\varepsilon}$ is continuous and surjective. By continuity, assertion (ii) follows from 7.3(iii).

(i) Let $\langle x_i \rangle$ be a net with the properties of (i). Then we may and do assume that the limits $A = \lim \operatorname{rlog}(x_i) \diamond Z$ and $B = \lim Z \diamond \operatorname{rlog}(x_i)$ exist, note that then $h \cap \mathsf{vert}(Z) = \{A\}$ and $\mathsf{hor}(Z) \cap v = \{B\}$. Now, by 7.3(iii), $\varepsilon(\operatorname{rlog}(x_i) \diamond Z) = \varepsilon(\operatorname{rlog}(x_i))\varepsilon(\operatorname{rlog}(x_i) \diamond Z) \in \kappa(x_i)S^\kappa$, so, by continuity, $\varepsilon(A) \in mS^\kappa$. In the same way we see that $\varepsilon(B) \in S^\kappa m$, thus $\varepsilon(A)\varepsilon(B) \in mS^\kappa \cap S^\kappa m = H(m)$, which implies $m \in H(\varepsilon(A)\varepsilon(B)) = H(\bar{\varepsilon}(h,v))$.

(iii) It is obvious that ε must be injective if $\bar{\varepsilon}$ is injective Suppose now that ε is injective but $\bar{\varepsilon}$ is not. Then we have $\bar{\varepsilon}(h,v) = \bar{\varepsilon}(h',v')$, for some $(h,v) \in \overline{c(D)} \setminus c(D)$, $(h',v') \in \overline{c(D)}$. Let X be an inner point of D in Hyp. If $h \neq h'$ then $(h, \mathsf{vert}(X))$ and $(h', \mathsf{vert}(X))$ are distinct and lie in $c(D)$, so $\bar{\varepsilon}(h, \mathsf{vert}(X)) \neq \bar{\varepsilon}(h', \mathsf{vert}(X))$, and (ii) implies a contradiction. Similarly, if $v \neq v'$ then $\bar{\varepsilon}(\mathsf{hor}(X), v) \neq \bar{\varepsilon}(\mathsf{hor}(X), v')$ leads to a contradiction. ∎

For the next result recall that a semigroup is called *left* [*right*] *simple* if it contains no proper left [right] ideals.

7.6. Theorem. *Let (S^κ, κ) be a topological semigroup compactification of a closed connected proper submonoid S of Sl_2 with dense interior, and suppose that the map $\varepsilon\colon D \to E(M(S^\kappa))$ of 7.2 is not injective. Then the minimal ideal of S^κ is either left or right simple and all of its maximal subgroups are singleton. More specifically, the following assertions are equivalent:*

(i) *$\varepsilon(A) = \varepsilon(B)$ for two distinct elements $A, B \in D$ which lie on the same horizontal [vertical] line;*

(ii) *ε is constant along any horizontal [vertical] line, i.e., $\varepsilon(\mathsf{hor}(X) \cap D)$ [$\varepsilon(\mathsf{vert}(X) \cap D)$] is singleton for every $X \in D$;*

(iii) *the minimal ideal $M(S^\kappa)$ is a minimal left [right] ideal.*

Proof. If $\varepsilon(A) = \varepsilon(B)$ then $\varepsilon(A) = \varepsilon(A \diamond B) = \varepsilon(B) = \varepsilon(B \diamond A)$, provided that the diamond products exist (by 7.3(vii)), thus if ε is not injective on D then it is not injective either on a horizontal line or on a vertical line. We only treat the case where A, B lie on the same horizontal line.

If (iii) holds then for every idempotent $e \in M(S^\kappa)$ the map $S \to H(e), s \mapsto e.s.e$ is a continuous homomorphism mapping S onto a dense subsemigroup of the compact topological group $H(e)$, so $H(e) = \{e\}$, by 6.2.

(i) $\Longrightarrow$ (ii) We assume that X lies in the interior of D and that $\varepsilon(X) = \varepsilon(Y)$ for some $Y \in D \cap \mathsf{hor}(X) \setminus \{X\}$. By 7.3(vii),(viii) our assumption (i) implies that such pairs X, Y exist.

Since X is in the interior of D there exists a positive real T such that $\exp(tX) \in S$ whenever $t \geq T$. We know from 3.9(ii) that $\lim_{t \to \infty} Y \circ \exp(tX) = Y \diamond X = X$. By 3.9(i) and 7.3(viii), $\varepsilon(X) = \varepsilon(X \circ \exp(tX)) = \varepsilon(Y \circ \exp(tX))$, for all $t \geq T$, therefore ε must be constant on the set $X \cup Y \circ \exp([T, \infty[\cdot X))$, which is a non-degenerate line segment joining X and $Y \circ \exp(TX)$ (see 3.9(ii)). Thus the ε-class $\varepsilon^{-1}(\varepsilon(X)) \cap \mathsf{hor}(X)$ of X has interior points in $\mathsf{hor}(X)$. Now we observe that $\varepsilon^{-1}(\varepsilon(X\circ s)) \supseteq \varepsilon^{-1}(\varepsilon(X))\circ s$, so $\varepsilon^{-1}(\varepsilon(X'))\cap \mathsf{hor}(X)$ has nonvoid interior in $\mathsf{hor}(X)$, for every $X' \in X\circ S$ (recall that $\circ$ is the restriction of a group action). Since $\mathsf{hor}(X)$ is separable this means that ε assumes at most countably many different values on $X\circ S$, since $X\circ S$ is pathwise connected we conclude that ε is constant on $X\circ S$. But $X\circ S$ is dense in $\mathsf{hor}(X)\cap D$ (by 7.3(v)), so ε must be constant on $\mathsf{hor}(X)\cap D$. For arbitrary $Z\in D$ we have $\mathsf{hor}(Z)\cap D = Z\diamond(\mathsf{hor}(X)\cap D)$, so ε is constant also on $\mathsf{hor}(Z)\cap D$, in view of 7.3(vii).

(ii) $\Longrightarrow$ (iii) Since for any X in the interior of D the map ε sends $(\mathsf{hor}(X)\cap D)$ onto the set of all idempotents of the minimal right ideal $\varepsilon(X)S^\kappa$ (by 7.3(v),(vi)), the assertion follows from the general structure theorem about the minimal ideal in a compact topological semigroup.

The implication (iii) $\Longrightarrow$ (i) follows from 7.3(ii). ∎

7.7. Corollary. *If there exist elements $X, Y\in D$, not lying in the same Borel algebra, such that $\varepsilon(X)=\varepsilon(Y)$ then S^κ has a zero element.*

Proof. This follows from 7.6 and the fact that the four points X, $X\diamond Y$, Y, $Y\diamond X$ are distinct if X and Y do not lie in the same Borel subalgebra. ∎

7.8. Notation. Suppose now that D is compact (or, equivalently, that $\overline{\mathrm{Umb}}(S)$ does not contain any nonzero nilpotent elements, cf 4.2). Let $\Sigma = \exp(\mathbb{R}_0^+ D)$ and consider the maps $i_S\colon S\to\overline{i(S)}$ and $j_S\colon S\to\overline{j(S)}$ defined in 4.16. We abbreviate $\overline{i(S)}$ to S^D, and $\overline{j(S)}$ to S^M. Recall that by 4.17 the compactification S^D is equivalent with the $\sqcup$-product $S\sqcup_{\varphi'} D$, and that, by 4.19, the compactification S^M is equivalent with $S\sqcup_{\Phi'} M(D,K)$.

For the next theorem we define the following congruences $\mathcal{L}_M$, $\mathcal{R}_M$ on S^D, which are derived from Green's relations:

$$a\mathcal{L}_M b \quad\text{if}\quad \begin{cases} a=b & \text{or}\\ a,b\in D & \text{and } ab=a;\end{cases}$$

$$a\mathcal{R}_M b \quad\text{if}\quad \begin{cases} a=b & \text{or}\\ a,b\in D & \text{and } ab=b.\end{cases}$$

Note that the join $\mathcal{L}_M\vee\mathcal{R}_M$ is the congruence which collapses the minimal ideal D of S^D to a zero element.

7.9. Theorem. *Let S be a closed connected proper submonoid of* Sl_2 *with dense interior such that the set D of regular directions is compact, and let (S^κ,κ) be a topological semigroup compactification of S with injective compactification morphism κ. We suppose that the interior of D in* Hyp *contains the point H (this can be always enforced by applying a suitable inner automorphism). We write k for the homomorphism, constructed in* 5.3(iii), $\mathbb{R}\to K = H(e_H)$ *with* $k(t) = e_H.\exp(tH)$ *whenever* $\exp(tH)\in S$. *Then the following assertions hold:*

(i) *If the map $\varepsilon\colon D\to E(M(S^\kappa))$, defined as in 7.2, is not injective then*

k is constant and S^κ is isomorphic to S^D/ρ, where ρ is one of the congruences $\mathcal{L}_M$, $\mathcal{R}_M$ or $\mathcal{L}_M \vee \mathcal{R}_M$.

(ii) *If ε is injective then S^κ is isomorphic to S^M, where $M = M(D,K)$. An isomorphism is given by the map $\mu\colon S^M = S \cup M(D,K) \to S^\kappa$, where $\mu(s) = \kappa(s)$ if $s \in S$ and $\mu(X \diamond H, g, H \diamond X) = \varepsilon(X \diamond H) g \varepsilon(H \diamond X)$ if $X \in D, g \in K$. The compactification (S^κ, κ) is equivalent with the Bohr compactification if and only if (K,k) is equivalent with the Bohr compactification of $\mathbb{R}$.*

Proof. Assertion (i) follows from Theorem 7.6 and Corollary 7.7.

(ii) We first show that the map μ is a continuous homomorphism. Since the restriction of μ to S is obviously continuous, it suffices to show that $\kappa(s_n) \to \varepsilon(X \diamond H) g \varepsilon(H \diamond X)$ whenever

$$\langle s_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \rangle$$

is a net in S such that $\lim \kappa(s_n)$ exists and

$$(\dagger) \qquad s_n \to (X \diamond H, g, H \diamond X) \text{ in } S^M.$$

The definition of the topology of S^M implies that $(\dagger)$ is equivalent to the following condition:

$s_n \to \infty$, $\operatorname{rlog}(s_n) \to X$, $k(\log(a_n)) \to g$.

Applying formula 7.4 and taking limits we see that

$$\lim \varepsilon(H)\kappa(s_n)\varepsilon(H) = \lim \varepsilon(H) k(\log(a_n))\varepsilon(H) = g.$$

Let $\langle \kappa(s_i) \rangle$ be a convergent subnet of $\langle \kappa(s_n) \rangle$. Since $\operatorname{rlog}(s_i) \to X$ we have (recall 7.2 and 7.3(iii))

$$\begin{aligned} \lim \kappa(s_i) &= \lim \varepsilon(X \diamond H)\kappa(s_i)\varepsilon(H \diamond X) \\ &= \lim \varepsilon(X \diamond H)\varepsilon(H)\kappa(s_i)\varepsilon(H)\varepsilon(H \diamond X). \\ &= \varepsilon(X \diamond H) g \varepsilon(H \diamond X) \end{aligned}$$

So every convergent subnet of $\langle \kappa(s_n) \rangle$ converges to $\varepsilon(X \diamond H) g \varepsilon(H \diamond X)$, thus $\langle \kappa(s_n) \rangle$ converges itself to this element. It follows that μ is continuous, hence it is also a homomorhism (being a homomorphism on the dense subset S).

By definition, the map μ is injective on S as well as on $M = M(D,K)$, so by 6.1 it is injective on S^M. The surjectivity follows by definition (or since $\kappa(S)$ is dense in S^κ).

The rest of the assertion follows from the universality of the Bohr compactification of $\mathbb{R}$. ∎

7.10. Corollary. *We retain the assumptions and the notation of the above Theorem 7.9. Let $X = H + \beta P \in \mathsf{hor}(H) \cap D$ and $Y = H + \gamma Q \in \mathsf{vert}(H) \cap D$. Then $\varepsilon(X)\varepsilon(Y) = k(\log(1 + \beta\gamma/4))$.*

Proof. This formula follows from Theorem 7.9 and the definition of S^M (7.8, 4.10). ∎

7.11. Proposition. *Let S be a closed connected proper submonoid of* Sl_2 *with dense interior and assume that D is not compact. Assume also that the Lie wedge of S contains a non-nilpotent element. (This assumption is automatically satisfied if S is a Lie semigroup.) Then the minimal ideal of any compactification S^κ of S consists of idempotents.*

Proof. If ε fails to be injective then the assertion follows from 7.6. Suppose now that ε is injective.

Applying a suitable inner automorphism of $\mathfrak{sl}(2,\mathbb{R})$ we enforce that H lies in the interior of D in Hyp. Since D is noncompact there exist $X = H+\gamma Q \in D\cap \mathsf{vert}(H)$ and $Y = H+\beta P \in D\cap \mathsf{hor}(H)$ such that $X\diamond Y$ does not exist, or, equivalently, such that $\beta\gamma=-4$ (cf. [2] 4.15). We know that the $\diamond$ product $A\diamond B$ always exists if one of A, B lies in the interior of D (in Hyp). Thus X and Y must lie on the boundary of D in Hyp, so X is an endpoint of $\mathsf{vert}(H)\cap D$ and Y is an endpoint of $\mathsf{hor}(H)\cap D$. Let $X' = H+\gamma' Q$ be the endpoint $\neq X$ of $\mathsf{vert}(H)\cap D$, and $Y' = H+\beta' P$ the endpoint $\neq Y$ of $\mathsf{hor}(H)\cap D$.

By assumption we can find an element Z of D which also lies in the Lie wedge of S. Since $X\diamond Y$ does not exist, Z cannot live both in $\mathsf{hor}(X)$ and in $\mathsf{vert}(Y)$. Similarly, if also $X'\diamond Y'$ does not exist then $Z\notin \mathsf{hor}(X')\cap\mathsf{vert}(Y')$. We assume that $Z\notin\mathsf{hor}(X)$ and $Z\notin\mathsf{vert}(Y')$, the other cases can be treated analogously and are therefore left to the reader. For $0<\eta\le 1$ we next define

$$X_\eta = H+(1-\eta)\gamma Q, \qquad Y'_\eta = \begin{cases} Y' & \text{if } X'\diamond Y' \text{ exists} \\ H+(1-\eta)\beta' P & \text{otherwise} \end{cases}$$

and note that the four elements X_η, X', Y, Y'_η generate a compact $\diamond$-subsemigroup D_η of D. Thus $\exp(\mathbb{R}_0^+ D_\eta)$ is an exponential subsemigroup of Sl_2. Moreover, if η is sufficiently small, say $\eta\le\eta_0$, then Z is contained in the interior of D_η in D, so $S_\eta = S\cap\overline{\exp(\mathbb{R}_0^+ D_\eta)}$ is a closed submonoid of Sl_2 whose interior clusters at $\mathbf{1}$. Since $S_\eta\subseteq\overline{\exp(\mathsf{Kill}^+)}$ we conclude that S_η is connected ([2] 3.4(iv)). Clearly, our compactification (S^κ,κ) restricts to a topological semigroup compactification of S_η. Moreover, $H(e_H)\subseteq\overline{\kappa(S_\eta)}$.

As in 7.9 we now put $K=H(e_H)$ and write k for the induced compactification morphism $\mathbb{R}\to K$. Pick any $x_0\in K$. We shall show that $x_0=\varepsilon(Y)\varepsilon(X)$, since x_0 was arbitrary this means that K is a singleton and the assertion follows.

The set $k(]-\infty,\log(\eta_0)])$ is dense in K, thus we can find a net $\langle\eta_n\rangle$ in $\mathbb{R}$ with $0<\eta_n\le\eta_0$ and $\lim\eta_n\to 0$, $k(\log(\eta_n))\to x_0$. Now by 7.10 $k(\log(\eta_n)) = \varepsilon(H+\beta P)\varepsilon(H+(1-\eta_n)\gamma Q)\to\varepsilon(Y)\varepsilon(X)$ which finishes our proof. ∎

7.12. Example. (cf. [17] 4.1(ii), [2] 6.5, 10.12) Let S be the semigroup of all 2×2-matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ with nonnegative entries in Sl_2 and $a\ge 1$. Then the Bohr compactification of S is equivalent with the one-point compactification $S_\infty = S\cup\{\infty\}$.

Proof. We use our current notation for the special case $\kappa = b_S$. Recall that S is a perfect Lie semigroup with Lie wedge $W = \mathbb{R}_0^+ H + \mathbb{R}_0^+ P + \mathbb{R}_0^+ Q$

([11], p. 419, [17] 4.1(ii), [3] 6.13) and that $\overline{\mathrm{Umb}}(S) = \mathbb{R}H + \mathbb{R}_0^+P + \mathbb{R}_0^+Q$ ([2] 10.12). Since $D = \overline{\mathrm{Umb}}(S) \cap \mathsf{Hyp}$ is noncompact we conclude from 7.11 that the maximal subgroups in S^κ are singleton. Also, the set $H + \mathbb{R}_0^+P$ is a horizontal line in D, the set $H + \mathbb{R}_0^+Q$ is a vertical line in D. Now the relation $\exp(t(H + \lambda P))\exp(sP) = \exp(e^{2t}sP)\exp(t(H + \lambda P))$, for every $s, t \in \mathbb{R}^+$, implies that $e_X = e_X e_P = e_P e_X = e_P$ for every $X \in H + \mathbb{R}_0^+P$. Similarly, $e_Y = e_Q$ for all $Y \in H + \mathbb{R}_0^+Q$, and the assertion follows from 7.7. ■

8. Summary of the main results

8.1. Theorem. *Let (S^κ, κ) be a topological semigroup compactification of a closed connected proper submonoid S of* Sl_2 *with dense interior. Then the following assertions hold:*

(i) *The S-large elements of S^κ lie in the minimal ideal $M(S^\kappa)$. Thus, in particular, $S^\kappa = \kappa(S) \cup M(S^\kappa)$.*

(ii) *Let $D = \overline{\mathrm{rlog}(S \cap \exp(\mathsf{Kill}^+))}$ and write* $\mathrm{Asy}(S)$ *for the set $\overline{c(D)}$ of all asymptotic directions in S. Then there exists a continuous surjection $\bar{\varepsilon}\colon \mathrm{Asy}(S) \to E(M(S^\kappa))$ which is a homomorphism mod $\mathcal{H}$, that is, $\bar{\varepsilon}(h,v)\bar{\varepsilon}(h',v') \in H(\bar{\varepsilon}(h,v'))$.*

(iii) *Exactly one of the following cases takes place:*

   (a) *$\bar{\varepsilon}$ is injective, hence a homeomorphism. In this case κ is an embedding.*

   (b) *$\bar{\varepsilon}(h,v) = \bar{\varepsilon}(h',v')$ if and only if $v = v'$. Then $M(S^\kappa)$ consists of right zeros of S^κ.*

   (c) *$\bar{\varepsilon}(h,v) = \bar{\varepsilon}(h',v')$ if and only if $h = h'$. Then $M(S^\kappa)$ consists of left zeros of S^κ.*

   (d) *$M(S^\kappa)$ is singleton. In this case S^κ has a zero element.*

(iv) *Every maximal subgroup of S^κ is isomorphic with a compactification of $\mathbb{R}$.*

Proof. Assertion (i) is part of Theorem 5.10 and (ii) is 7.5(i),(ii). Assertion (iii) follows from 7.5(iii), 6.4, and 7.6. The last assertion is a consequence of 5.3(ii),(iii). ■

8.2. Explicit Constructions. As before we let (S^κ, κ) be a topological semigroup compactification of a closed connected proper submonoid S of Sl_2 with dense interior.

(i) *If κ is not injective then there exists a closed ideal I of S such that (S^κ, κ) is equivalent with the projective limit $\lim\limits_{\leftarrow}(S/J, \ell_J)$, where J runs over all closed large ideals of S which contain I, and for every J the map ℓ_J is the quotient morphism $S \to S/J$.*

(ii) *If D is compact and $\bar{\varepsilon}$ is injective then there exists a compactification (K,k) of $\mathbb{R}$ and a conjugate S_1 of S in* Sl_2 *such that (S^κ,κ) is equivalent with a $\sqcup$-product $(S_1^M = S_1 \sqcup M(D,K), j_{S_1})$, as described in* 4.19.

(iii) *If D is compact, κ is injective, and $\bar{\varepsilon}$ is not injective then (S^κ,κ) is equivalent with a quotient $(S^D/\rho, i_\rho)$ of (S^D, i_S), where $S^D = S \sqcup D$ is defined as in* 4.17, *and ρ is one of the congruences $\mathcal{L}_M$, $\mathcal{R}_M$ or $\mathcal{L}_M \vee \mathcal{R}_M$ of* 7.8.

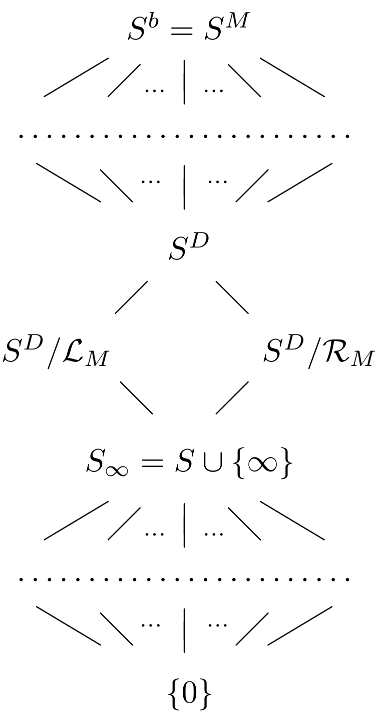
Proof. (i) follows from 6.4 and 6.12, and (ii),(iii) are consequences of 7.9. ■

8.3. Remark. (i) The quotients S^D/ρ can be defined also as $\sqcup$-products. (Exercise)

(ii) Note that D is compact if and only if S is contained in a perfect exponential subsemigroup of Sl_2. At present we do not have explicit constructions for the case where D is noncompact but S is perfect.

8.4. The lattice of all compactifications of S. Let S be a closed connected proper submonoid of Sl_2 with dense interior.

(i) If D is compact then we have complete information about the lattice of all compactifications of S. As sketched in the diagram at the left this lattice consists of three parts:



(1) On top we have the sublattice of those compactifications where ε is injective. These compactifications are characterized by the associated compactification (K,k) of the reals, thus this part of the lattice is isomorphic with the lattice of all topological group compactifications of $\mathbb{R}$, which, in turn, is isomorphic with the lattice of all subgroups of the discretization $\mathbb{R}_d$ of $\mathbb{R}$, ordered by $A \le B$ if $A \subseteq B$. The minimal element in the lattice of injective compactifications is S^D.

(2) Next comes the diamond lattice made up of S^D and its quotients $S^D/\mathcal{L}_M$, $S^D/\mathcal{R}_M$ and $S^D/(\mathcal{L}_M \vee \mathcal{R}_M) = S_\infty$.

(3) On the bottom we have the lattice formed by S_∞ and all topological semigroup compactifications with noninjective compactification map. This lattice is isomorphic with the lattice formed by the closed ideals of S and the empty set, ordered by $A \le B$ if $A \supseteq B$.

(ii) If S is perfect but D is noncompact then the compactifications described in (3) above form a full sublattice of the lattice of all topological semigroup compactifications. At the moment we do not know whether this sublattice is a proper sublattice. If the Lie wedge of S contains at least one regular matrix then by 7.11 all maximal subgroups of S^b are trivial, so part (1) of (i) has no counterpart in this case.

(iii) If S is not perfect then we know from 6.6 and 6.12 that the compactification lattice of S is anti-isomorphic with the $\subseteq$-lattice ALIDS of all closed ideals containing the aliens of S.

(iv) If S is an exponential semigroup then D is compact if and only if S is perfect if and only if the Lie wedge of S contains no nonzero nilpotent elements. Thus for exponential semigroups S with inner points in Sl_2 there are exactly two cases: (a) S is perfect— then D is compact and the compactification lattice has the form described in (i) above; (b) S is nonperfect— then the compactification lattice is as described in (iii). (Al(S) has been explicitly calculated in [3].)

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