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Tensor fields and connections on holomorphic orbit spaces of finite groups

Andreas Kriegl, Mark Losik, and Peter W. Michor *

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Abstract. For a representation of a finite group G on a complex vector space V we determine when a holomorphic $\binom{p}{q}$-tensor field on the principal stratum of the orbit space V/G can be lifted to a holomorphic G-invariant tensor field on V. This extends also to connections. As a consequence we determine those holomorphic diffeomorphisms on V/G which can be lifted to orbit preserving holomorphic diffeomorphisms on V. This in turn is applied to characterize complex orbifolds.
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1. Introduction

Locally, an orbifold Z can be identified with the orbit space B/G, where B is a G-invariant neighborhood of the origin in a vector space V with a finite group $G \subset GL(V)$ and, using this identification, one can easily define local (and then global) tensor fields and other differential geometrical objects in Z as appropriate G-invariant tensor fields and objects on $B \subset V$. In particular, one can naturally define Riemannian orbifolds, Einstein orbifolds, symplectic orbifolds, Kähler-Einstein orbifolds etc.

We study complex orbifolds, that is, orbifolds modeled on orbit spaces V/G, where G is a finite subgroup of $GL(V)$ for a complex vector space V. In particular, the orbit spaces $Z = M/G$ of a discrete proper group G of holomorphic transformations of a complex manifold M are complex orbifolds.

An orbifold X has a structure defined by the sheaf $\mathfrak{F}_X$ of local invariant holomorphic functions in a local uniformizing system. X has also a stratification by strata S which are glued from local isotropy type strata of local uniformizing systems. In particular, the regular stratum X_0 is an open dense complex manifold in X.

Holomorphic geometric objects on X (e.g. tensor fields and connections) are locally defined as invariant objects on the uniformizing system. Their restrictions

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to the regular stratum X_0 are usual holomorphic geometric objects on the complex manifold X_0.

A natural question is to characterize these restrictions, i.e. to describe tensor fields and connections on X_0 which are extendible to X. We look at the lifting problem for connections because this allows a very elegant approach to the lifting problem for holomorphic diffeomorphisms. And the last problem has immediate consequences for characterizing complex orbifolds, i.e., for answering the following question: Which data does one need besides $\mathfrak{F}_X$ and X_0 to characterize a complex orbifold X? The main goal of the paper is to answer these questions.

We have first to investigate the local situation, thus we consider a finite subgroup $G \subset GL(V)$ and the orbit space $Z = V/G$ with the structure given by the sheaf $\mathfrak{F}_{V/G}$ of invariant holomorphic functions on V, and the orbit type stratification. The prime role is played by strata of codimension 1 with the orders of the corresponding stabilizer groups, which are arranged in the *reflection divisor* $D_{V/G}$ which keeps track of all complex reflections in G. It turns out that the union Z_1 of Z_0 and of all codimension 1 strata is a complex manifold, see 3.5. We characterize all G-invariant holomorphic tensor fields and connections on V in terms of the *reflection divisor* of the corresponding meromorphic tensor field and connection on Z_1, see 3.7 and 4.2. Our result gives a generalization 3.9 of Solomon's theorem [10], see 3.10. Using the lifting property of connections we are able to prove that a holomorphic diffeomorphism $Z = V/G \to V/G' = Z'$ between two orbit spaces has a holomorphic lift to V which is equivariant over an isomorphism $G \to G'$ if and only if f respects the regular strata and the reflection divisors, i.e. $f(Z_0) \subset Z'_0$ and $f_*(D_Z) \subset D_{Z'}$. In fact we give two proofs of this result, which in [4] is carried over to the algebraic geometry setting for algebraically closed ground fields of characteristic 0. The related problem of lifting (smooth) homotopies from (general) orbit spaces has been treated in [1] and [9].

Applying the local results we prove that a complex orbifold X is uniquely determined by the sheaf $\mathfrak{F}_X$, the regular stratum X_0, and the reflection divisor D_X alone, see 6.6.

2. Preliminaries

2.1. The orbit type stratification. Let V be an n-dimensional complex vector space, G a finite subgroup of $GL(V)$, and $\pi : V \to V/G$ the quotient projection. The ring $\mathbb{C}[V]^G$ has a minimal system of homogeneous generators $\sigma^1, \dots, \sigma^m$. We will use the map $\sigma = (\sigma^1, \dots, \sigma^m) : V \to \mathbb{C}^m$. Denote by Z the affine algebraic variety in $\mathbb{C}^m$ defined by the relations between $\sigma^1, \dots, \sigma^m$. It is known that $\sigma(V) = Z$.

We consider the orbit space V/G endowed with the quotient topology as a local ringed space defined by the following sheaf of rings $\mathfrak{F}_{V/G}$: if U is an open subset of V/G, $\mathfrak{F}_{V/G}(U)$ is equal to the space of G-invariant holomorphic functions on $\pi^{-1}(U)$. Clearly one may consider sections of $\mathfrak{F}_{V/G}$ on U as functions on U. We call these functions holomorphic functions on U. It is known that the map of the orbit space V/G to Z induced by the map σ is a homeomorphism. Moreover, this homeomorphism induces an isomorphism of the sheaf $\mathfrak{F}_{V/G}(U)$ and the structure sheaf of the complex algebraic variety Z (see [7]). Via the above isomorphism we identify the local ringed spaces V/G and Z. Under this identification the projection π is identified with the map σ. Let G and G' be finite subgroups of $GL(V)$ and let $Z = V/G$ and $Z' = V/G'$ be the corresponding orbit spaces. By definition a holomorphic diffeomorphism of the orbit space Z to the orbit space Z' is an isomorphism of Z to Z' as local ringed spaces.

Let K be a subgroup of G, (K) the conjugacy class of K. Denote by $V_{(K)}$ the set of points of V whose isotropy groups belong to (K) and put $Z_{(K)} = \pi(V_{(K)})$. It is known that $\{Z_{(K)}\}$ is a finite stratification of Z, called the isotropy type stratification, into locally closed irreducible smooth algebraic subvarieties (see [5]). Denote by Z^i the union of the strata of codimension greater than i and put $Z_i = Z \setminus Z^i$. Then Z_0 is the principal stratum of Z, i.e. $Z_0 = Z_{(K)}$ for $K = \{\mathrm{id}\}$. It is known that Z_0 is a Zariski open subset of Z and a complex manifold. It is clear that the restriction of the map σ to the set V_{reg} of regular points of V is an tale map onto Z_0.

In this paper we consider the orbit space $Z = V/G$ with the above structure of local ringed space and the stratification $\{Z_{(K)}\}$.

2.2. The divisor of a tensor field. We shall use divisors of meromorphic functions on a complex manifold X. For technical reasons (see e.g. the last formula of this section) we define $\mathrm{div}(0) = \sum_S \infty.S$, where the sum runs over all complex subspaces of X of codimension 1.

Let f and g be two meromorphic functions on X. Then we have

$$\mathrm{div}(f+g) \geq \min\{\mathrm{div}(f), \mathrm{div}(g)\}, \text{ where } \mathrm{div}(f) \text{ denotes the divisor of } f.$$

Taking the minimum means: For each irreducible complex subspace S of X of codimension 1 belonging to the support of f or g take the minimum of the coefficients in $\mathbf{Z}$ of S in $\mathrm{div}(f)$ and $\mathrm{div}(g)$.

Let P be a meromorphic tensor field (i.e., with meromorphic coefficient functions in local coordinates) on X. In local holomorphic coordinates $y^1, \dots, y^n$ on an open subset $U \subset X$ the tensor field P can be written as

$$P|_U = \sum_{i_1,\dots,i_p,j_1,\dots,j_q} P^{i_1\dots i_p}_{j_1\dots j_q} \frac{\partial}{\partial y^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_p}} \otimes dy^{j_1} \otimes dy^{j_q}.$$

and we define the *divisor* of P on U as the minimum of all divisors $\mathrm{div}(P^{i_1\dots i_p}_{j_1\dots j_q}) \in \mathrm{Div}(U)$ for all coefficient functions of P. The resulting coefficient of the complex subspace S of codimension 1 in $\mathrm{div}(P) \in \mathrm{Div}(U)$ does not depend on the choice of the holomorphic coordinate system; e.g., for a vector field $\sum_i X^i \frac{\partial}{\partial y^i} = \sum_{i,k} X^i \frac{\partial u^k}{\partial y^i} \frac{\partial}{\partial u^k}$ we have

$$\mathrm{div}\Big(\sum_i X^i \frac{\partial u^k}{\partial y^i}\Big) \geq \min_i \mathrm{div}\Big(X^i \frac{\partial u^k}{\partial y^i}\Big) = \min_i \Big(\mathrm{div}(X^i) + \mathrm{div}\Big(\frac{\partial u^k}{\partial y^i}\Big)\Big) \geq \min_i \mathrm{div}(X^i).$$

Finally we define the divisor of P on X by gluing the local divisors for any holomorphic atlas of X. Note that a tensor field P is holomorphic if and only if $\mathrm{div}(P) \geq 0$.

3. Invariant tensor fields

3.1. Let P be a G-invariant holomorphic tensor field of type $\binom{p}{q}$ on V. Since σ is an tale map on V_{reg}, there is a unique holomorphic tensor field Q on Z_0 of type $\binom{p}{q}$ such that the pullback $\sigma^*(Q)$ coincides with the restriction of P to V_{reg}. It is clear that the tensor field P is uniquely defined by Q.

Consider a holomorphic tensor field Q of type $\binom{p}{q}$ on Z_0 and its pullback $\sigma^*(Q)$ which is a G-invariant holomorphic tensor field on V_{reg}. Then by the Hartogs extension theorem, $\sigma^*(Q)$ has a G-invariant holomorphic extension to V iff it has a holomorphic extension to $\sigma^{-1}(Z_1)$.

Denote by $\mathfrak{H}$ the set of all reflection hyperplanes corresponding to all complex reflections in G and, for each $H \in \mathfrak{H}$, by e_H the order of the cyclic subgroup of G fixing H. It is clear that $\sigma(\cup_{H\in\mathfrak{H}} H)$ contains all strata of codimension 1. This implies immediately the following

3.2. Proposition. *If $\mathfrak{H} = \emptyset$, for each holomorphic tensor field P_0 on Z_0 the pullback $\sigma^*(P_0)$ has a G-invariant holomorphic extension to V.* ■

3.3. The reflection divisor of the orbit space. Consider the set R_Z of all hyper surfaces $\sigma(H)$ in Z, where H runs through all reflection hyperplanes in V. Note that $\sigma(H)$ is a complex subspace of Z_1 of codimension 1. We endow each $S = \sigma(H) \in R_Z$ with the label e_H of the hyperplane H. It is easily seen that this label does not depend on the choice of H, we denote it by e_S and we consider $e_S.\,S$ as an effective divisor on Z and we consider the effective divisor in Z_1

$$D = D_{V/G} = D_Z = \sum_{S\in R_Z} e_S.\,S,$$

which we call the *reflection divisor*.

3.4. Basic example. Let the cyclic group $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$ with generator $\zeta_r = e^{2\pi i/r}$ act on $\mathbb{C}$ by $z \mapsto e^{2\pi ik/r}z$ for $r \geq 2$. The generating invariant is $\tau(z) = z^r$.

We consider first a holomorphic tensor field $P = f(z)(dz)^{\otimes q} \otimes (\frac{\partial}{\partial z})^{\otimes p}$ on $\mathbb{C}$. It is invariant, $\zeta_r^* P = P$, if and only if $f(\zeta_r z) = \zeta_r^{p-q} f(z)$, so that in the expansion $f(z) = \sum_{k\geq 0} f_k z^k$ at 0 of f the coefficient $f_k \neq 0$ at most when $k \cong p-q \mod r$. Writing $p-q = rs+t$ with $s \in \mathbb{Z}$ and $0 \leq t < r$ we see that P is invariant if and only if $f(z) = z^t g(z^r)$ for holomorphic g.

We use the coordinate $y = \tau(z) = z^r$ on $\mathbb{C}/\mathbb{Z}_r = \mathbb{C}$, $\tau^* dy = rz^{r-1}dz$ and $\tau^*(\frac{\partial}{\partial y}|_{\mathbb{C}\setminus 0}) = \frac{1}{rz^{r-1}}\frac{\partial}{\partial z}|_{\mathbb{C}\setminus 0}$, and we write

$$\begin{aligned}
P|_{\mathbb{C}\setminus 0} &= g(z^r)z^t(dz)^{\otimes q} \otimes (\tfrac{\partial}{\partial z})^{\otimes p} \\
&= g(y)z^t(rz^{r-1})^{p-q}(dy)^{\otimes q} \otimes (\tfrac{\partial}{\partial y})^{\otimes p} \\
&= g(y)z^{-rs}(rz^r)^{p-q}(dy)^{\otimes q} \otimes (\tfrac{\partial}{\partial y})^{\otimes p} \\
&= g(y)r^{p-q}y^{p-q-s}(dy)^{\otimes q} \otimes (\tfrac{\partial}{\partial y})^{\otimes p}
\end{aligned}$$

(we omitted τ^*). Thus a holomorphic tensor field P of type $\binom{p}{q}$ on $\mathbb{C}$ is $\mathbb{Z}_r$-invariant if and only if $P|_{\mathbb{C}\setminus 0} = \tau^* Q$ for a meromorphic tensor field

$$Q = g(y)y^m(dy)^{\otimes q} \otimes (\tfrac{\partial}{\partial y})^{\otimes p}$$

on $\mathbb{C}$ with g holomorphic with $g(0) \neq 0$ and with

$$m \geq p - q - s.$$

It is easily checked that the above inequality is equivalent to the following one

$$mr + (q - p)(r - 1) \geq 0.$$

3.5. Suppose $\mathfrak{H} \neq \emptyset$. Let $z \in Z_1 \setminus Z_0$ and $v \in \sigma^{-1}(z)$. Then there is a unique hyperplane $H \in \mathfrak{H}$ such that $v \in H$ and the isotropy group G_v is isomorphic to a cyclic group. It is evident that the order $r_z = e_H$ of G_v depends only on $z = \sigma(v)$ and is locally constant on $Z_1 \setminus Z_0$.

By the holomorphic slice theorem (see [5], [6]) there is a G_v-invariant open neighborhood U_v of v in V such that the induced map $U_v/G_v \to V/G$ is a local biholomorphic map at v.

Choose orthonormal coordinates $z^1, \dots, z^n$ in V with respect to a G-invariant Hermitian inner product on V, so that $H = \{z^n = 0\}$. Then the ring $\mathbb{C}[V]^{G_v}$ is generated by $z^1, \dots, z^{n-1}, (z^n)^r$, where $r = r_z$.

Put $\tau^1 = z^1, \dots, \tau^{n-1} = z^{n-1}$, $\tau^n = (z^n)^r$, and $\tau = (\tau^1, \dots, \tau^n) : U_v \to \mathbb{C}^n$. Then there are holomorphic functions f^i $(i = 1, \dots, n)$ in an open neighborhood W_z of $z \in \mathbb{C}^m$ such that $\tau^a = f^a \circ \sigma|_{U_v}$. On the other hand, we know that in an open neighborhood of v all σ^a for $(a = 1, \dots, m)$ are holomorphic functions of the τ^i. We denote by y^i the holomorphic function on Z such that $\tau^i = y^i \circ \sigma$. Then we can use y^i as coordinates of Z defined in the open neighborhood $W_z \subseteq \mathbb{C}^m$ of z. Note that we found holomorphic coordinates near each point of Z_1, so we have:

Corollary. *The union Z_1 of all codimension ≤ 1 strata, with the restriction of the sheaf $\mathfrak{F}_{V/G}$, is a complex manifold.* ■

3.6. The reflection divisor of a meromorphic tensor field on Z_1. Let $\Gamma_{\mathcal{M}}(T^p_q(Z_1))$ be the space of meromorphic tensor fields (i.e. with meromorphic coefficient functions in local holomorphic coordinates on the complex manifold Z_1), and let $P \in \Gamma_{\mathcal{M}}(T^p_q(Z_1))$.

Let S be an irreducible component of $Z_1 \setminus Z_0$ and let $z \in S$. Local coordinates $y^1, \dots, y^n$ on $U \subset Z_1$, centered at z, are called adapted to the stratification of Z_1 if $S = \{y^n = 0\}$ near z. By definition the coordinates $y^1, \dots, y^n$ from 3.5 have this property. Denote by $\mathcal{O}_z$ the ring of germs of holomorphic functions and by $\mathcal{M}_z$ the field of germs of meromorphic functions, both at $z \in Z_1$.

Let $y^1, \dots, y^n$ be local coordinates on $U \subset Z_1$, centered at z, adapted to the stratification of Z_1 . Then on U the meromorphic tensor field P is given by

$$P|_U = \sum_{i_1,\dots,i_p,j_1,\dots,j_q} P^{i_1\dots i_p}_{j_1\dots j_q} \frac{\partial}{\partial y^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_p}} \otimes dy^{j_1} \otimes dy^{j_q},$$

where the $P^{i_1\dots i_p}_{j_1\dots j_q}$ are meromorphic on U. Let us fix one nonzero summand of the right hand side: for the coefficient function we have $P^{i_1\dots i_p}_{j_1\dots j_q} = (y^n)^m f$ for some integer m such that the germs at z of y^n, g, and h are pairwise relatively prime in $\mathcal{O}_z$ where $f = g/h \in \mathcal{M}_z$. Suppose that the factor $\frac{\partial}{\partial y^n}$ appears exactly p' times and the factor dy^n appears exactly q' times in this summand. The integer

$$\mu = mr + (q' - p')(r - 1),$$

a priori depending on z, is constant along an open dense subset of S and it is called the *reflection residuum* of the summand at S. Finally let $\mu_S(P)$ be the minimum of the reflection residua at S of all summands of P in the representation of P.

Let $\tilde{y}^1, \dots, \tilde{y}^n$ be arbitrary local coordinates on $U \subset Z_1$, centered at z, adapted to the stratification of Z_1. In a neighborhood of z we have $y^n = f\tilde{y}^n$, where f is a holomorphic function such that $f(z) \neq 0$. Remark that $\tilde{y}^n$ divides $\frac{\partial y^n}{\partial \tilde{y}^i}$ and $\frac{\partial \tilde{y}^n}{\partial y^i}$ $(i = 1, \dots, n)$ in $\mathcal{O}_z$. A straightforward calculation using the above remark shows that the values of $\mu_S(P)$ calculated in the coordinates $\tilde{y}^i$ and in the coordinates y^i are the same. Then $\mu_S(P)$ does not depend on the choice of the system of local coordinates adapted to the stratification of Z_1. For details see [4]: there we checked this in the algebraic geometry setting where the use of tensor fields is less familiar.

We now can define the *reflection divisor*

$$\operatorname{div}_D(P) = \operatorname{div}_{D_{V/G}}(P) \in \operatorname{Div}(U)$$

as follows: take the divisor $\operatorname{div}(P)$, and for each irreducible component S of $Z_1 \setminus Z_0$ do the following: if S appears in the support of $\operatorname{div}(P) \in \operatorname{Div}(U)$, replace its coefficient by $\mu_S(P)$; if it does not appear, add $\mu_S(P).S$ to it. If S is not contained in $Z_1 \setminus Z_0$, we keep its coefficient in $\operatorname{div}(P)$.

Finally we glue the global *reflection divisor* $\operatorname{div}_D(P) \in \operatorname{Div}(Z_1)$ from the local ones, using a holomorphic atlas for Z_1.

3.7. Theorem. *Let $G \subset GL(V)$ be a finite group, with reflection divisor $D = D_{V/G} = D_Z$. Then we have:*

- *Let P be a holomorphic G-invariant tensor field on V. Then the reflection divisor $\operatorname{div}_D(\pi_* P) \geq 0$.*
- *Let $Q \in \Gamma_{\mathcal{M}}(T^p_q(Z_1))$ be a meromorphic tensor field on Z_1. Then the G-invariant meromorphic tensor field $\pi^* Q$ extends to a holomorphic G-invariant tensor field on V if and only if $\operatorname{div}_D(Q) \geq 0$.*

The above remains true for G-invariant holomorphic tensor fields defined in a G-stable open subset of V.

Proof. This follows directly from Hartogs' extension theorem, the basic example 3.4 using $y^1, \dots, y^{n-1}$ as dummy variables, and the definition of the reflection divisor $\operatorname{div}_D(P)$ as explained in 3.6. ∎

3.9. Corollary. *The mapping σ establishes an injective correspondence between the space of holomorphic G-invariant tensor fields of type $\binom{p}{q}$ on V which are skew-symmetric with respect to the covariant entries, and the space of holomorphic tensor fields on Z_1 of the same type and the same skew-symmetry condition. If $p = 0$ the correspondence is bijective.*

The above remains true for G-invariant holomorphic tensor fields defined in a G-stable open subset of V.

Proof. Let P be a holomorphic G-invariant tensor field on V satisfying the conditions of the corollary. For each nonzero decomposable summand of $\pi_* P$ take the integers m, p', and q' defined in 3.6. By skew symmetry of P we have $q' \leq 1$.

By Theorem 3.7 we get $\operatorname{div}_D(\pi_* P) \geq 0$ and thus $mr \geq (p' - q')(r-1) > -r$. So $m \geq 0$ and the summand is holomorphic on Z_1.

If Q is a holomorphic differential form on Z_1 its pullback $\sigma^* Q$ is a G-invariant holomorphic form on $\sigma^{-1}(Z_1)$ and then has a holomorphic extension to the whole of V. ■

3.10. Remarks. Note that Corollary 3.9 is a generalization of Solomon's theorem (see [10]): *If $G \subset GL(V)$ is a finite complex reflection group then every G-invariant polynomial exterior q-form ω on V can be written as $\omega = \sigma^*\varphi$ for a polynomial q-form φ on $\mathbb{C}^n$, where $\sigma = (\sigma^1, \dots, \sigma^n) : V \to \mathbb{C}^n$ is the mapping consisting of a minimal system of homogeneous generators of $\mathbb{C}[V]^G$.*

Actually, in the case of a reflection group $Z = \mathbb{C}^n$ and each holomorphic $\binom{p}{q}$-tensor field Q on Z_1 has a holomorphic extension to Z by Hartogs' extension theorem.

4. Invariant complex connections

4.1. Let Γ be a holomorphic G-invariant complex connection on V. Then the image $\sigma_*\Gamma$ of Γ under the map σ defines a holomorphic complex connection on Z_0.

Let $z \in Z_1 \setminus Z_0$, $v \in \sigma^{-1}(z)$, and r the order of G_v. Consider the coordinates z^i in V defined in 3.5. Denote by Γ^i_{jk} the components of the connection Γ with respect to these coordinates. By assumption, the Γ^i_{jk} are holomorphic functions on V. Recall the standard formula for the image γ of Γ under a holomorphic diffeomorphism $f = (y^a(x^i))$

$$\gamma^a_{bc} \circ f = \frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\frac{\partial x^k}{\partial y^c}\Gamma^i_{jk}(x^l) - \frac{\partial^2 y^a}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial y^b}\frac{\partial x^j}{\partial y^c}.$$

Remark that the similar formula is true for the transformation of the components of connection under the change of coordinates.

Consider the generator g of the cyclic group G_v given by 3.5. Since g acts linearly, the connection reacts to it like a $\binom{1}{2}$-tensor field. Thus by the considerations of 3.4 we get in the notation of 3.5, where $i, j, k = 1, \dots, n-1$:

$$\begin{gathered}
\Gamma^i_{jk} = \tilde\Gamma^i_{jk} \circ \sigma, \quad \Gamma^n_{jk} = \frac{1}{r} z^n \tilde\Gamma^n_{jk} \circ \sigma, \quad \Gamma^i_{jn} = r(z^n)^{r-1}\tilde\Gamma^i_{jn} \circ \sigma, \\
\Gamma^i_{nk} = r(z^n)^{r-1}\tilde\Gamma^i_{nk} \circ \sigma, \quad \Gamma^n_{jn} = \tilde\Gamma^n_{jn} \circ \sigma, \quad \Gamma^n_{nk} = \tilde\Gamma^n_{nk} \circ \sigma, \\
\Gamma^i_{nn} = r^2(z^n)^{r-2}\tilde\Gamma^i_{nn} \circ \sigma, \quad \Gamma^n_{nn} = r(z^n)^{r-1}\tilde\Gamma^n_{nn} \circ \sigma,
\end{gathered}$$

where the $\tilde\Gamma^a_{bc}$ are holomorphic functions of the coordinates y^a $(a = 1, \dots, n)$ introduced in 3.5.

Using the transformation formula for connections, we get the following formulas for the components γ^a_{bc} of the meromorphic connection $\sigma_*\Gamma$ with respect to the coordinates y^a

$$\begin{gathered}
\gamma^i_{jk} = \tilde\Gamma^i_{jk}, \quad \gamma^n_{jk} = y^n \tilde\Gamma^n_{jk}, \quad \gamma^i_{jn} = \tilde\Gamma^i_{jn}, \quad \gamma^i_{nk} = \tilde\Gamma^i_{nk}, \qquad (4.1.1) \\
\gamma^n_{jn} = \tilde\Gamma^n_{jn}, \quad \gamma^n_{nk} = \tilde\Gamma^n_{nk}, \quad \gamma^i_{nn} = \frac{1}{y^n}\tilde\Gamma^i_{nn}, \quad \gamma^n_{nn} = \tilde\Gamma^n_{nn} - \frac{r-1}{r y^n}.
\end{gathered}$$

Let $\tilde y^a$ for $a=1,\dots,n$ be other local coordinates centered at z and adapted to the stratification of Z_1. Then in a neighborhood of z we have

$$y^n = f\tilde y^n, \quad \tilde y^n = \tilde f y^n,$$

where f and $\tilde f$ are holomorphic functions in a neighborhood of z and $\tilde f f = 1$. Then we have

$$\frac{\partial y^n}{\partial \tilde y^i} = \frac{\partial f}{\partial \tilde y^i}\tilde y^n, \quad \frac{\partial \tilde y^n}{\partial y^i} = \frac{\partial f}{\partial y^i} y^n \quad (i=1,\dots,n-1)$$

and on $S=\{y^n=0\}$

$$\frac{\partial y^n}{\partial \tilde y^n} = f, \quad \frac{\tilde\partial y^n}{\partial y^n} = \tilde f.$$

Using these formulas one can check that in the coordinates $\tilde y^a$ the formulas 4.1.1 have the same form as in the coordinates y^a. For example, for the new component $\tilde\gamma^n_{nn}$ we have

$$\tilde\gamma^n_{nn} + \frac{r-1}{r\tilde y^n} = \frac{(r-1)\left(1-\tilde f\frac{\partial\tilde y^n}{\partial y^n}\left(\frac{\partial y^n}{\partial \tilde y^n}\right)^2\right)}{r y^n \tilde f} + h,$$

where h is a holomorphic function near z. Since on $S=\{y^n=0\}$ we have

$$1-\tilde f\frac{\partial\tilde y^n}{\partial y^n}\left(\frac{\partial y^n}{\partial \tilde y^n}\right)^2 = 1-\tilde f^2 f^2 = 0,$$

y^n divides in $\mathcal{O}_z$ the function

$$1-\tilde f\frac{\partial\tilde y^n}{\partial y^n}\left(\frac{\partial y^n}{\partial \tilde y^n}\right)^2.$$

Thus

$$\tilde\gamma^n_{nn} + \frac{r-1}{r\tilde y^n}$$

is holomorphic in a neighborhood of z.

4.2. Theorem. *Let γ be a holomorphic complex linear connection on Z_0 such that for each $z\in Z_1\setminus Z_0$ it has an extension to a neighborhood of z whose components in the coordinates adapted to the stratification of Z_1 are defined by the formulas 4.1.1 where $\tilde\Gamma^a_{bc}$ are holomorphic. Then there is a unique G-invariant holomorphic complex linear connection Γ on V such that $\sigma_*\Gamma$ coincides with γ on Z_0. This remains true if we replace V by a G-open subset of G.*

Proof. Since σ is tale on the principal stratum, there is a unique G-invariant complex linear connection Γ_0 on $\sigma^{-1}(Z_0)$ such that $\sigma_*\Gamma_0=\gamma$. The condition of the theorem implies that the connection Γ_0 has a holomorphic extension to $\sigma^{-1}(Z_1)$. Then by Hartogs' extension theorem the connection Γ_0 has a unique holomorphic extension Γ to the whole of V. ∎

5. Lifts of diffeomorphisms of orbit spaces

5.1. Let G and G' be finite subgroups of $GL(V)$ and $GL(V')$ and let F be a holomorphic diffeomorphism $V \to V'$ which maps G-orbits to G'-orbits bijectively. Then the map F induces an isomorphism f of the sheaves $\mathfrak{F}_{V/G} \to \mathfrak{F}_{V'/G'}$, i.e. a holomorphic diffeomorphism of orbit spaces V/G and V'/G'.

Lemma. *There is a unique isomorphism $a : G \to G'$ such that $F \circ g = a(g) \circ F$ for every $g \in G$.*

Note that a and its inverse a^{-1} map complex reflections to complex reflections.

Proof. The cardinalities of the two groups are the same since F maps a generic regular orbit to a regular orbit. Consequently, it maps regular points to regular points and we have $\sigma' \circ F = f \circ \sigma : V \to V'/G'$ for a holomorphic diffeomorphism $f : V/G \to V'/G'$, where $\sigma : V \to V/G$ and $\sigma' : V' \to V'/G'$ are the quotient projections.

Fix some G-regular $v \in V$. Then $F(v)$ and $F(gv)$ for $g \in G$ are regular points of V' of the same orbit. Therefore, there is a unique $a(g) \in G$ such that $F(gv) = a(g)(F(v))$. We have $\sigma' \circ F \circ g = f \circ \sigma \circ g = f \circ \sigma = \sigma' \circ F = \sigma' \circ a(g) \circ F$. Since σ' is tale on V'_{reg} we see that $F \circ g = a(g) \circ F$ locally near v and thus globally. By uniqueness, the map $g \to a(g)$ is an isomorphism of G onto G'. ∎

In this section we study when a diffeomorphism f of the orbit spaces $Z \to Z'$ has a holomorphic lift F.

5.2. Corollary. *Let $F : V \to V$ be a holomorphic diffeomorphism which maps G-orbits onto G'-orbits, and $f : Z \to Z'$ the corresponding holomorphic diffeomorphism of the orbit spaces. Then f maps the isotropy type stratification of Z onto that of Z' and, moreover, it maps D_Z to $D_{Z'}$.*

Proof. This follows from Lemma 5.1 and the definition 3.3 of the reflection divisor. ∎

5.3. Theorem. *Let G and G' be two finite subgroups of $GL(V)$ and let $f : Z \to Z'$ be a holomorphic diffeomorphism of the corresponding orbit spaces such that $f(Z_0) = Z'_0$ and $f_*(D_Z) = D_{Z'}$. If Q is a holomorphic tensor field of type $\binom{p}{q}$ on Z_0 which satisfies the conditions of Theorem 3.7, then $f_*(Q)$ also satisfies these conditions on Z'_0 and thus there exists a unique G'-invariant holomorphic tensor field Q' of type $\binom{p}{q}$ such that $\sigma'_* Q'$ coincides with $f_* Q$ on Z'_0.*

This is also true for holomorphic connections if we replace Theorem 3.7 by Theorem 4.2. The theorem remains true if we replace V by invariant open subsets of V.

Proof. Since $f(Z_0) = Z'_0$ the tensor field $f_* Q$ is also holomorphic on Z'_0. Let $z \in Z_1 \setminus Z_0$. Then there is a complex space $S \in R_Z$ of codimension 1 such that $z \in S$. By assumption $f(z) \in Z'_1 \setminus Z'_0$ and $f(z) \in f(S) \in R_{Z'}$ and $r_z = e_S = e_{f(S)} = r_{f(z)}$. Now, obviously $f_* Q$ satisfies the conditions of Theorem 3.7 at $f(x)$. Thus there exists a G'-invariant holomorphic tensor field Q' on V with $\sigma'_* Q' = f_* Q$.

A similar argument applies to connections. ∎

5.4 Theorem. *Let G and G' be two finite subgroups of $GL(V)$. Let $f : Z \to Z'$ be a holomorphic diffeomorphism of the orbit spaces such that $f(Z_0) = Z'_0$ and $f_*(D_Z) = D_{Z'}$.*

Then f lifts to a holomorphic diffeomorphism $F : V \to V$, i.e. $\sigma' \circ F = f \circ \sigma$.

The local version is also true. Namely, if B is a ball in the vector space V centered at 0 (for an invariant Hermitian metric), $U = \sigma(B)$, and $f : U \to Z'$ is a local holomorphic diffeomorphism of U onto a neighborhood U' of $\sigma'(0)$ such that $f(U \cap Z_0) = U' \cap Z'_0$ and f maps $D_Z \cap U$ to $D_{Z'} \cap U'$, then there is a holomorphic lift $F : B \to V$.

Proof. Let Γ be the natural flat connection on V. Then Γ is uniquely defined by the holomorphic connection $\sigma_*\Gamma$ on Z_0 which satisfies the conditions of Theorem 4.2. By Theorem 5.3 there is a unique G-invariant holomorphic complex linear connection Γ' on V such that $\sigma'_*\Gamma'$ coincides with $f_*(\sigma_*\Gamma)$ on Z'_0. It is evident that Γ' is a torsion free flat connection, since Γ is it and Γ' is locally isomorphic to Γ on an open dense subset.

Let $v \in V$ be G-regular and let $v' \in V$ be G'-regular, such that $(f\circ\sigma)(v) = \sigma'(v')$. Then there is a biholomorphic map F of a neighborhood W of v onto a neighborhood of v' such that $\sigma' \circ F = f \circ \sigma$ on W and $F(v) = v'$. Moreover by construction F is a locally affine map of the affine space (V, Γ) into (V, Γ') equipped with the above structures of locally affine spaces, thus we have

$$F = \exp^{\Gamma'}_{v'} \circ T_vF \circ (\exp^{\Gamma}_v)^{-1} \tag{1}$$

where $\exp^{\Gamma}_v : T_vV \to V$ is the holomorphic geodesic exponential mapping centered at v given by the connection Γ and its induced spray. It is globally defined, thus complete and a holomorphic diffeomorphism since Γ is the standard flat connection. Likewise $\exp^{\Gamma'}_{v'}$ is the holomorphic exponential mapping of the flat connection Γ'. The formula above extends F to a globally defined holomorphic mapping if $\exp^{\Gamma'}_{v'} : T_vV \to V$ is also globally defined (complete). Assume for contradiction that this is not the case. Let F be maximally extended by equation (1); it still projects to $f : Z \to Z'$. We consider $\exp^{\Gamma'}_{v'}$ as a real exponential mapping, and then there is a real geodesic which reaches infinity in finite time and this is the image under F of a finite part $\exp^{\Gamma}_v([0, t_0)w)$ of a real geodesic of Γ emanating at v. The sequence $\exp^{\Gamma}_v((t_0 - 1/n)w)$ converges to $\exp^{\Gamma}_v(t_0w)$ in V, but its image under F diverges to infinity by assumption. On the other hand, the image under F is contained in the set $(\sigma')^{-1}(f\sigma(\exp^{\Gamma}_v([0, t_0]w)))$ which is compact since σ' is a proper mapping. Contradiction.

Any holomorphic lift F of a holomorphic diffeomorphism f is a holomorphic diffeomorphism of V which maps G-orbits onto G' orbits, by the following argument: Let F' be a holomorphic lift of f^{-1}. Evidently the map $F' \circ F$ preserves each G-orbit. Then, for a G-regular point $v \in V$, there is a $g \in G$ such that $F' \circ F = g$ in a neighborhood of v and, then, on the whole of V. Similarly $F \circ F' = g' \in G'$. This implies that F is a holomorphic diffeomorphism of V. By definition the lift F respects the partitions of V into orbits. ∎

We give a second proof of Theorem 5.4 based on the known results about the fundamental groups of V_{reg} and Z_0 for finite complex reflection groups. It is an extension of the proof of [8], using results of [2].

5.5. Lemma. *Let G and G' be two finite subgroups of $GL(V)$ and let $f : Z \to Z'$ be a holomorphic diffeomorphism of the corresponding orbit spaces. Suppose $v_0 \in V_{reg}$, $v_0' \in V'_{reg}$, and $f \circ \sigma(v_0) = \sigma'(v_0')$. If the image of the fundamental group $\pi_1(V_{reg}, v_0)$ under $f \circ \sigma$ is contained in the subgroup $\sigma'_*(\pi_1(V_{reg}), v_0')$ of $\pi_1(Z_0', \sigma'(v_0'))$, the holomorphic lift of $f \circ \sigma$ mapping v_0 to v_0' exists.*

Proof. Consider the restriction φ of the map $f \circ \sigma$ to V_{reg}. Since the restriction of σ to V_{reg} is a covering map onto Z_0, the condition of the lemma implies that there is a holomorphic lift F_0 of the map φ to V_{reg}. The map F_0 is bounded on $B \cap V_{\text{reg}}$ for each compact ball B in V since its image is contained in the compact set $(\sigma')^{-1}(f(\sigma(B)))$. Then by the Riemann extension theorem F_0 has a holomorphic extension F to V which is the required holomorphic lift of f. ∎

5.6. Next we prove Theorem 5.4 in the case when the group G is generated by complex reflections. Put

$$B := \pi_1(Z_0) \quad \text{and} \quad P := \pi_1(V_{\text{reg}}).$$

The groups B and P are called the *braid group* and the *pure braid group* associated to G, respectively. It is clear that the map σ induces an isomorphism of P onto a subgroup of B.

The following results about the groups B and P are well known (see, for example, [2]). The braid group B is generated by those elements which are represented by loops around the hypersurfaces $\sigma(H)$ for $H \in \mathfrak{H}$. The pure braid group P is generated by the elements of B of the type s^{e_H}, where s is any of the above generators of B represented by a loop around the hypersurface $\sigma(H)$. This implies the following

Proposition. *Suppose the group G is generated by complex reflections. Let f be a holomorphic diffeomorphism of the orbit space $Z = \mathbb{C}^n$ with $f(Z_0) = Z_0$ which also preserves D_Z. Then $f|_{Z_0}$ preserves the subgroup P of B.* ∎

The following proposition is an immediate consequence of Lemma 5.5 and Proposition 5.6.

5.7. Proposition. *Suppose the groups G and G' are generated by complex reflections. Let $f : Z \to Z'$ be a holomorphic diffeomorphism between the corresponding orbit spaces, such that $f(Z_0) = Z_0'$ and $f_*(D_Z) = D_{Z'}$.*

Then f has a holomorphic lift F to V. ∎

Second proof of 5.4. Now let $G \subset GL(V)$ be a finite group and let G_1 be the subgroup generated by all complex reflections in G. Clearly G_1 is a normal subgroup of G. Let $G_2 = G/G_1$. Let $\sigma_1^1, \dots, \sigma_1^n$ be a system of homogeneous generators of $\mathbb{C}[V]^{G_1}$ and $\sigma_1 : V \to \mathbb{C}^n$ the corresponding orbit map. Then the action of G on V induces the action of the group G_2 on $V_1 := \mathbb{C}^n = \sigma_1(V)$. Since each representation of the group G_2 is completely reducible, by standard arguments of invariant theory, we may assume that the generators σ_1^i's are chosen in such a way that the above action of G_2 on $V_1 = \mathbb{C}^n$ is linear. Then the representation of G_2 on V_1 contains no complex reflections. Let $\sigma_2^1, \dots, \sigma_2^m$ be a system of homogeneous generators of $\mathbb{C}[V_1]^{G_2}$ and $\sigma_2 : V_1 \to \mathbb{C}^m$ the corresponding orbit map. Then $\sigma^i = \sigma_2^i \circ \sigma_1$ $(i = 1, \dots, m)$ is a system of generators of $\mathbb{C}[V]^G$ with orbit map $\sigma = \sigma_2 \circ \sigma_1$. Similarly for G'.

Let $f : Z \to Z'$ be a holomorphic diffeomorphism, such that $f(Z_0) = Z_0'$ and $f_*(D_Z) = D_{Z'}$. Since the group G_2 contains no complex reflections the set $V_{1,\mathrm{reg}}$ of regular points of the action of G_2 on V_1 is obtained from V_1 by removing some subsets of codimension ≥ 2. And similarly for G'. Then the fundamental group $\pi_1(V_{1,\mathrm{reg}}) = \pi_1(V_1) = 0$ is trivial and by lemma 5.5 the diffeomorphism f has a holomorphic lift $F_1 : V_1 \to V_1'$ which is a holomorphic diffeomorphism mapping the principal stratum to the principal stratum, and the reflection divisor to the reflection divisor, since G_2 contains no complex reflections on V_1. Thus the diffeomorphism F_1 has a holomorphic lift to V by Proposition 5.7, which is a holomorphic lift of f. ∎

6. An intrinsic characterization of a complex orbifold

We recall the definition of orbifold.

6.1. Definition. *[11] Let X be a Hausdorff space. An atlas of a smooth n-dimensional orbifold on X is a family $\{U_i\}_{i\in I}$ of open sets that satisfy:*

1. *$\{U_i\}_{i\in I}$ is an open cover of X.*

2. *For each $i \in I$ we have a local uniformizing system consisting of a triple $(\tilde U_i, G_i, \varphi_i)$, where $\tilde U_i$ is a connected open subset of $\mathbb{R}^n$ containing the origin, G_i is a finite group of diffeomorphisms acting effectively and properly on $\tilde U_i$, and $\varphi_i : \tilde U_i \to U_i$ is a continuous map of $\tilde U_i$ onto U_i such that $\varphi_i \circ g = \varphi_i$ for all $g \in G_i$ and the induced map of $\tilde U_i/G_i$ onto U_i is a homeomorphism. The finite group G_i is called a local uniformizing group.*

3. *Given $\tilde x_i \in \tilde U_i$ and $\tilde x_j \in \tilde U_j$ such that $\varphi_i(\tilde x_i) = \varphi_j(\tilde x_j)$, there is a diffeomorphism $g_{ij} : \tilde V_j \to \tilde V_i$ from a neighborhood $\tilde V_j \subseteq \tilde U_j$ of $\tilde x_j$ onto a neighborhood $\tilde V_i \subseteq \tilde U_i$ of $\tilde x_i$ such that $\varphi_j = \varphi_i \circ g_{ij}$.*

Two atlases are equivalent if their union is again an atlas of a smooth orbifold on X. An orbifold is the space X with an equivalence class of atlases of smooth orbifolds on X.

If we take in the definition of orbifold $\mathbb{C}^n$ instead of $\mathbb{R}^n$ and require that G_i is a finite group of holomorphic diffeomorphisms acting effectively and properly on $\tilde U_i$ and the maps g_{ij} are biholomorphic, we get the definition of complex analytic n-dimensional orbifold.

6.2. Theorem. *[11] Let M be a smooth manifold and G a proper discontinuous group of diffeomorphisms of M. Then the orbit space M/G has a natural structure of smooth n-dimensional orbifold. If M is a complex n-dimensional manifold and G is a group of holomorphic diffeomorphisms of M, the orbit space M/G is a complex n-dimensional orbifold.*

6.3 Definitions. In the definition of atlas of a complex orbifold on X we can always take $\tilde U_i$ to be balls of the space $\mathbb{C}^n$ (with respect to some Hermitian metric) centered at the origin and the finite subgroups G_i to be subgroups of the $GL(n)$

acting naturally on $\mathbb{C}^n$. In the sequel we consider atlases of complex orbifolds satisfying these conditions.

Let X be a complex orbifold with an atlas $(\tilde{U}_i, G_i, \varphi_i)$. A function $f : U_i \to \mathbb{C}$ is called holomorphic if $f \circ \varphi_i$ is a holomorphic function on $\tilde{U}_i$. The germs of holomorphic functions on X define a *sheaf* $\mathfrak{F}_X$ on X. It is evident that the sheaf $\mathfrak{F}_X$ depends only on the structure of complex orbifold on X.

Consider a uniformizing system $(\tilde{U}_i, G_i, \varphi_i)$ of the above atlas and the corresponding action of G_i on $\mathbb{C}^n$. Then we have the isotropy type stratification of the orbit space $\mathbb{C}^n/G_i$, the induced stratification of U_i, and the divisor D_{U_i}.

By corollary 5.2 we get the *stratification on* X by gluing the strata on the U_i's. Denote by X_0 the principal stratum of this stratification. By definition, for each $x \in X_0$, for each uniformizing system $(\tilde{U}_i, G_i, \varphi_i)$, and for each $y \in \tilde{U}_i$ such that $\varphi_i(y) = x$, the isotropy group G_y of y is trivial. Note that X_0 is a complex manifold. Note that X_1 is also a complex manifold since this holds locally as noted in 3.5.

Denote by R_X the *set of all strata of codimension* 1 of X. Since the pullbacks of the reflection divisors D_{U_i} to $U_i \cap U_j$ agree by 5.2 we may glue them into the reflection divisor D_X on X_1.

6.4. Definition. *Let X and $\tilde{X}$ be two smooth orbifolds. The orbifold $\tilde{X}$ is called a covering orbifold for X with a projection $p : \tilde{X} \to X$ if p is a continuous map of underlying topological spaces and each point $x \in X$ has a neighborhood $U = \tilde{U}/G$ (where $\tilde{U}$ is an open subset of $\mathbb{R}^n$) for which each component V_i of $p^{-1}(U)$ is isomorphic to $\tilde{U}/G_i$, where $G_i \subseteq G$ is some subgroup. The above isomorphisms $U = \tilde{U}/G$ and $V_i = \tilde{U}/G_i$ must respect the projections.*

Note that the projection p in the above definition is not necessarily a covering of the underlying topological spaces. It is clear that a covering orbifold for a complex orbifold is a complex orbifold. Hereafter we suppose that all orbifolds and their covering orbifolds are connected.

6.5. Theorem. *[11] An orbifold X has a universal covering orbifold $p : \tilde{X} \to X$. More precisely, if $x \in X_0$, $\tilde{x} \in \tilde{X}_0$ and $p(\tilde{x}) = x$, for any other covering orbifold $p' : \tilde{X}' \to X$ and $\tilde{x}' \in \tilde{X}'$ such that $p'(\tilde{x}') = x$ there is a cover $q : \tilde{X} \to \tilde{X}'$ such that $p = p' \circ q$ and $q(\tilde{x}) = \tilde{x}'$. For any points $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is a deck transformation of $\tilde{X}$ taking $\tilde{x}$ to $\tilde{x}'$.*

Now we prove the main theorem of this section.

6.6. Theorem. *An n-dimensional complex orbifold X is uniquely determined by the sheaf of holomorphic functions $\mathfrak{F}_X$, the principal stratum X_0, and the reflection divisor D_X.*

Proof. For each $x \in X$, there exists $V = \mathbb{C}^m$, a finite group $G \subset GL(m)$, a ball B in V centered at 0, an open subset U of X containing x, and an isomorphism $\psi : \pi(B) \to U$ between the sheaves $\mathfrak{F}_Z|_{\pi(B)}$ and $\mathfrak{F}_X|_U$. Consider the map $\pi : V \to Z = V/G$, the stratum Z_0 and the reflection divisor D_Z. We suppose also that $\psi(Z_0 \cap B/G) \subseteq X_0$ and $\psi_*(D_{\pi(B)}) = D_U$. It suffices to prove that the germ of the uniformizing system $\{B, G, \psi \circ \pi|B\}$ at x is the germ of some uniformizing system of the orbifold X.

Let $y \in V_{\text{reg}} \cap B$. Then the ring $\mathfrak{F}_Z(\pi(y))$ of germs of $\mathfrak{F}_Z$ at $\pi(y)$ is isomorphic to the ring of germs of holomorphic functions on $\mathbb{C}^n$ at 0 and thus we have $m = n$.

Consider the uniformizing system $(\tilde{U}_i, G_i, \varphi_i)$ of the orbifold X, where $\tilde{U}_i$ is a ball in $\mathbb{C}^n$ centered at the origin, G_i is a finite subgroup of the group $GL(n)$ acting naturally on $V = \mathbb{C}^n$, and where $\varphi_i(0) = x$. Consider the map $\pi_i : V \to V/G_i$ given by some system of generators of $\mathbb{C}[V]^{G_i}$. We may assume that $\varphi_i = \psi_i \circ \pi_i|_{\tilde{U}_i}$, where $\psi_i : \mathfrak{F}_{\tilde{U}_i/G_i} \to \mathfrak{F}_{U_i}$ is an isomorphism of sheaves.

$$\begin{tikzcd}
\mathbb{C}^n & B \arrow[l, hook'] \arrow[rr, dotted, "F"'] \arrow[d, "\pi"'] & & \tilde{U}_i \arrow[r, hook] \arrow[d, "\pi_i"] & \mathbb{C}^n \\
& B/G \arrow[rr, dotted, "f"'] \arrow[dr, "\psi"'] & & \tilde{U}_i/G_i \arrow[dl, "\psi_i"] & \\
& & U & &
\end{tikzcd}$$

Then the maps ψ and ψ_i define a map (germ) f of a holomorphic diffeomorphism B/G to U_i/G_i at $0 := \pi(0)$ such that $f(0) = 0 := \pi_i(0)$. Then f induces an isomorphism $\mathfrak{F}_{V/G}(0) \to \mathfrak{F}_{V/G_i}(0)$, it maps $(B/G)_0$ to $(\tilde{U}_i/G_i)_0$ and $f_*(D_{B/G}) = D_{\tilde{U}_i/G_i}$. Thus by theorem 5.4 there is a germ of a holomorphic diffeomorphism $F : B \to \tilde{U}_i$ which is equivariant for a suitable isomorphism $G \to G_i$. ■

6.7. Corollary. *Let M be a complex simply connected manifold, G a proper discontinuous group of holomorphic diffeomorphisms of M, and $\mathfrak{F}_X$ the corresponding sheaf on the orbifold $X = M/G$. The G-manifold M is a universal covering orbifold for the orbifold X and it is defined uniquely up to a natural isomorphism of universal coverings by the sheaf $\mathfrak{F}_X$, the principal stratum X_0, and by the reflection divisor D_X.*

Proof. Evidently the manifold M is a covering orbifold for X. If $\tilde{X}$ is a universal covering orbifold for X, by definition 6.4 there is a cover $q : \tilde{X} \to M$. By definition $\tilde{X}$ should be a manifold and q a cover of manifolds. Therefore, q is a diffeomorphism. Then the statement of the corollary follows from theorem 6.6. ■

An automorphism of the sheaf $\mathfrak{F}_X$ is called a holomorphic diffeomorphism of the orbit space X. Theorem 6.5 and corollary 6.7 imply the following analogue of Theorem 5.4.

6.8. Theorem. *Let M be a complex simply connected manifold, G a proper discontinuous group of holomorphic diffeomorphisms of M, and $\mathfrak{F}_X$ the corresponding sheaf on the orbifold $X = M/G$. Each holomorphic diffeomorphism f of the orbit space X preserving X_0 and D_X has a holomorphic lift F to M, which is G-equivariant with respect to an automorphism of G. The lift F is unique up to composition by an element of G.*

Proof. By theorem 6.6 and corollary 6.7 the manifold M with the map $f \circ p : M \to X$, where $p : M \to X$ is the projection, is a universal covering orbifold for X. Then there is a holomorphic diffeomorphism $F : M \to M$ such that $p \circ F = f \circ p$. The equivariance property holds locally by 5.1, thus globally. The lift is uniquely given by choosing $F(x)$ for a regular point x in the orbit $f(p(x))$. ■

6.9. Let V be a complex vector space with a linear action of a finite group G. The group $\mathbb{C}^*$ acts on V by homotheties and induces an action on $Z = V/G$.

Corollary. *In this situation, the G-module V is uniquely defined up to a linear isomorphism by the sheaf $\mathfrak{F}_{V/G}$ with the action of $\mathbb{C}^*$, by Z_0, and the reflection divisor D_Z.* ∎

Proof. Consider the orbit space $Z = V/G$ of a G-module V with the sheaf $\mathfrak{F}_{V/G}$, regular stratum Z_0, reflection divisor D_Z, and the action of $\mathbb{C}^*$ induced by the action of $\mathbb{C}^*$ on V by homotheties. Suppose that we have another G'-module V' with the same data on $Z' = V'/G'$ such that there is a biholomorphic map $f : Z \to Z'$ preserving these data. By Theorem 4.5 there is a biholomorphic lift $F : V \to V'$, and by lemma 5.1 there is an isomorphism $a : G \to G'$ such that $F \circ g = a(g) \circ F$. Thus we may assume that $G = G'$, $V = V'$, $Z = Z'$, and a is the identity map. By definition the pullback A of the vector field on the orbit space V/G defined by the action of the group $\mathbb{C}^*$ on V/G coincides with the vector field on V defined by the above action of the group $\mathbb{C}^*$ on V. By construction $F^*A = A$ and then the map F commutes with the action of $\mathbb{C}^*$ on V, i.e. for each $t \in \mathbb{C}^*$ and $v \in V$ we have $F(tv) = tF(v)$. Since F is biholomorphic it is a linear automorphism of the vector space V. By definition F is then an automorphism of the G-module V. ∎

6.10. Tensor fields and connections on orbifolds. The local results in section 3 show that the correct definition of a $\binom{p}{q}$-tensor field Q on an orbifold X is as follows: Q is a meromorphic $\binom{p}{q}$-tensor field on X_1 such that $\operatorname{div}_{D_X}(Q) \geq 0$.

Likewise, we can define connections on orbifolds by requiring the local conditions of section 4.

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A. Kriegl
Institut für Mathematik,
Universität Wien,
Strudlhofgasse 4, A-1090 Wien,
Austria.
Andreas.Kriegl@univie.ac.at

M. Losik
Saratov State University,
ul. Astrakhanskaya, 83,
410026 Saratov, Russia
LosikMV@info.sgu.ru

P. W. Michor
Institut für Mathematik,
Universität Wien,
Strudlhofgasse 4, A-1090 Wien,
Austria; and
Erwin Schrödinger Institut
für Mathematische Physik,
Boltzmanngasse 9, A-1090 Wien,
Austria
Peter.Michor@esi.ac.at

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