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On a Special Class of Frobenius Groups admitting Planar Partitions

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Abstract. Among all Frobenius Lie groups having a complement isomorphic either to $\mathbb{C}^\times$ or to $\mathbb{H}^\times$ and a kernel which is a vector group those are determined that admit a planar partition into closed subgroups. Moreover, it is shown that for each of these groups the exponential function induces a bijection between the set of planar partitions of the group and the set of planar partitions of the associated Lie algebra.

1. Introduction

One of the reasons for studying groups with partitions is in doing linear incidence geometry, in particular, topological incidence geometry. If one takes a group G with a partition $\mathcal{P}$, and sets $\mathcal{G} := \{gH \mid g \in G, H \in \mathcal{P}\}$, then the pair $(G, \mathcal{G})$ is a linear space on which the group G acts as a group of automorphisms via left translation (the notion "linear space" simply means that for any two distinct "points" $g, h \in G$ there exists exactly one "line" $L \in \mathcal{G}$ which contains both g and h). The structures obtained by this construction were first investigated by André in [1] and they are examples of point-regular geometries as considered in [11]. In dealing with stable planes (a special kind of topological linear spaces; see [6] for a definition) one is led to take G to be a Lie group and $\mathcal{P}$ a partition of G into closed subgroups of half dimension. Such partitions are called planar partitions. In [7] the author shows that for a Lie group G with a planar partition $\mathcal{P}$ the incidence structure $(G, \mathcal{G})$ is a stable plane exactly if the induced partition $\mathrm{L}\mathcal{P} := \{\mathrm{L}P \mid P \in \mathcal{P}\}$ of the Lie algebra $\mathrm{L}G$ is compact in the respective Grassmann topology.

So one has a construction method for point-regular stable planes which starts from a Lie group with planar partition, and now it remains to find Lie groups which admit such partitions. Thanks to the work of Plaumann and Strambach (cf. [8] and [9]) one already knows that such a group is either exponential (that is, the exponential function is a diffeomorphism) or is a Frobenius group whose kernel is a vector group and whose complements are isomorphic to $\mathbb{C}^\times$ or to $\mathbb{H}^\times$.

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In the present paper the author classifies those of the aforementioned Frobenius groups which admit planar partitions.

Throughout this note, all Lie groups are assumed to be real Lie groups of finite dimension and all vector spaces over skewfields are assumed to be left vector spaces.

2. Partitions of Vector Spaces, Algebras, and Groups

Definition 2.1. By a *partition of a vector space* we mean a set of non-trivial subspaces which cover the whole space and which pairwise intersect trivially. *Partitions of algebras and groups* are defined analogously. By a *partition of a Lie group* we mean a partition into closed subgroups. A partition is called *trivial* if it consists of one set only.

Examples 2.2. (a) For any vector space the set of all 1-dimensional subspaces is a partition, and the same holds true for Lie algebras. So if one has a collection of subalgebras of a Lie algebra which pairwise intersect trivially one can extend this to a partition of the algebra by adding a suitable set of 1-dimensional subalgebras.

(b) Let F be a skewfield and V a left vector space over F. On the F-vector space $F \times V$ we define a bracket multiplication by

$$[(a,v),(b,w)] := a(b,w) - b(a,v).$$

Endowed with this multiplication $F \times V$ becomes a Lie algebra over the center $Z(F)$ of F. We denote it by $\mathrm{dil}(V)$, or $\mathrm{dil}_n F$ if $V = F^n$. Algebras of this type are called *dilatation algebras.* As is easy to see from the definition, any F-subspace of $\mathrm{dil}(V)$ is a Lie subalgebra and hence, any partition of $\mathrm{dil}(V)$ into vector subspaces is a Lie algebra partition.

(c) Let F and V be as in (b). On the set $F^\times \times V$ we define a multiplication by

$$(a,v)(b,w) := (ab, aw + v).$$

Endowed with this multiplication $F^\times \times V$ becomes a group which we denote by $\mathrm{Dil}(V)$, or $\mathrm{Dil}_n F$ if $V = F^n$. Groups of this type are called *dilatation groups.* It is easy to see that for an F-subspace U of $F \times V$ the set $\tilde{U} := ((1,0) + U) \cap \mathrm{Dil}(V)$ is a subgroup of $\mathrm{Dil}(V)$ (cf. proof of Proposition 4.7). That means that for any partition $\mathfrak{P}$ of the Lie algebra $\mathrm{dil}(V)$ into F-subspaces the set $\mathcal{P} := \{\tilde{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ is a partition of the group $\mathrm{Dil}(V)$.

(d) If we assume V in example (c) to be finite dimensional over F and if we take F to be a locally compact connected skewfield (that is, F is isomorphic to one of the skewfields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), then the group $\mathrm{Dil}(V)$ is a Lie group and for any partition $\mathfrak{P}$ of the Lie algebra $\mathrm{dil}(V)$ into F-subspaces the set $\mathcal{P} := \{\tilde{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ is a partition of the Lie group $\mathrm{Dil}(V)$.

Since we are not interested in partitions of vector spaces, groups, or algebras in general but in those which give rise to geometric objects such as translation planes (see [6] for a definition) and stable planes, we now turn to so-called planar partitions. Before giving the definition we just note that any partition $\mathcal{P}$ of a Lie group G induces a partition $\mathrm{L}\mathcal{P} := \{\mathrm{L}P \mid P \in \mathcal{P}\}$ of the associated Lie algebra $\mathrm{L}G$

(as is easy to deduce from the fact that the exponential function $\exp : \mathrm{L}G \to G$ is a local homeomorphism).

Definition 2.3. A partition of a finite-dimensional vector space or a Lie algebra is called *planar* if it consists of subspaces of half dimension. A partition of a Lie group is called *planar* if the induced partition of the Lie algebra is planar.

Obviously, planar partitions of vector spaces or Lie groups can only exist in even dimension, but at the present point, it is not at all clear whether in any even dimension there exists a Lie group or a vector space which admits a planar partition. Indeed, there is a result which states that any even-dimensional vector space admits a planar partition (cf. [2]), and clearly, this result covers the Lie group case, since any finite-dimensional real vector space is an abelian Lie group. So the question arises to classify all Lie groups which admit planar partitions. In dealing with this problem one is led to the following result due to Plaumann and Strambach (cf. [9]).

Theorem 2.4. (Plaumann–Strambach) *Let G be a connected Lie group that admits a partition into subgroups of a fixed dimension $d > 1$. Then exactly one of the following holds:*

(i) *The exponential function is a diffeomorphism of $\mathrm{L}G$ onto G.*

(ii) *The group G is a Frobenius group whose kernel is a vector group and whose complement is isomorphic to $\mathbb{C}^\times$ or to $\mathbb{H}^\times$.*

In the remaining part of this note we are concerned with Frobenius groups mentioned in part (ii) of Theorem 2.4. Among these groups we classify exactly those which admit planar partitions. It turns out that, up to one exceptional class, all of these groups are dilatation groups over the quaternions or the complex numbers. Moreover, it is shown that for any such group the planar partitions are exactly the exponential images of the planar partitions of the corresponding Lie algebra.

3. A Result on Invariant Partitions

In the sequel we answer the following question: Consider a finite dimensional vector space V and an endomorphism φ of V. Under which conditions on φ the vector space V admits a partition into φ-invariant subspaces? The problem can also be formulated in terms of modules. This will be done in the proof of the following theorem.

Theorem 3.1. *Let F be a field, V a finite dimensional F-vector space, and let φ be a non-zero endomorphism of V. Then the vector space V admits a non-trivial partition into φ-invariant subspaces if, and only if, the subring $F[\varphi]$ of $\mathrm{End}_F V$ is a field, and $\dim_F F[\varphi] < \dim_F V$.*

Proof. We set $R := F[x]$ and define a F-algebra morphism $\mathrm{ev}_\varphi : R \to \mathrm{End}_F V$ by sending x to φ. With respect to this morphism the vector space V becomes an R-module and the φ-invariant subspaces are exactly the R-submodules of V.

Suppose first that $F[\varphi]$ is a field and $\dim_F F[\varphi] < \dim_F V$. Then V is a $F[\varphi]$-vector space and because of $\dim_F F[\varphi] < \dim_F V$ the set of all 1-dimensional $F[\varphi]$-subspaces is a non-trivial partition of V into R-submodules.

Now suppose that V admits a non-trivial partition $\mathcal{P}$ into R-submodules. As R is a principal ideal domain and V is a finitely generated R-module it is a direct sum of non-trivial cyclic submodules $Rv_1, \ldots, Rv_n$ (cf. [5]). We pick such a decomposition

$$V = Rv_1 \oplus \cdots \oplus Rv_n$$

which is maximal with respect to the number of summands, and claim that all these summands are isomorphic simple R-modules. If this is true, then $\ker \mathrm{ev}_\varphi$ is a maximal ideal in R and $F[\varphi]$ is a field, since it is isomorphic to $R/\ker \mathrm{ev}_\varphi$. So it remains to prove the claim. We do this by induction on n, starting with $n = 2$, since a cyclic R-module only admits the trivial partition.

As $\mathcal{P}$ is a partition its elements cover V. Therefore we find $P_1, P_2 \in \mathcal{P}$ containing v_1 and v_2, respectively, and thus obtain $Rv_1 \subseteq P_1$ and $Rv_2 \subseteq P_2$, because P_1 and P_2 are R-submodules of V. Since we have $V = Rv_1 \oplus Rv_2$ and since $\mathcal{P}$ was supposed to be non-trivial this implies $P_1 = Rv_1$ and $P_2 = Rv_2$. Denoting the annihilator of an element $v \in V$ by $\mathrm{Ann}(v)$ the indecomposability of Rv_1 and Rv_2 implies that $\mathrm{Ann}(v_k) = \left(p_k^{n_k}\right)$ holds for suitable prime polynomials $p_k \in R$ and natural numbers n_k. Now, we have to show $n_1 = n_2 = 1$ and $(p_1) = (p_2)$. Without loss we assume $n_1 \leq n_2$. We distinguish three cases.

First case: $(p_1) \neq (p_2)$. In this case we have $\mathrm{Ann}(v_1) + \mathrm{Ann}(v_2) = R$. Taking into account that we also have $\mathrm{Ann}(v_1 + v_2) = \mathrm{Ann}(v_1) \cap \mathrm{Ann}(v_2)$ (since the sum $Rv_1 + Rv_2$ is direct) the chinese remainder theorem yields $R(v_1 + v_2) = Rv_1 \oplus Rv_2 = V$. So V is cyclic and therefore $\mathcal{P}$ has to be trivial, which was forbidden.

Second case: $(p_1) = (p_2)$ and $n_2 > 1$. Let $w_1 := p_1^{n_1 - 1} v_1$. Since $R(w_1 + v_2)$ is cyclic it is contained in some $P \in \mathcal{P}$. Because of $\{0\} \neq Rpv_2 \subseteq R(w_1 + v_2) \cap Rv_2$ we obtain $P = P_2$, which is impossible, since $w_1 + v_2 \notin P_2$.

The remaining third case is the desired one. So we have $Rv_1 \cong R/(p_1) = R/(p_2) \cong Rv_2$ and the claim is proved for $n = 2$.

Now let $n > 2$ and assume the claim holds true for $n - 1$. We write $Rv_k \cong_s Rv_l$ if Rv_k and Rv_l are isomorphic simple R-modules. Suppose there exist k, l such that $Rv_k \not\cong_s Rv_l$. Without loss let $(k, l) = (1, 2)$. Then we have $Rv_1 \not\cong_s Rv_3$ or $Rv_2 \not\cong_s Rv_3$ and we can assume that $Rv_2 \not\cong_s Rv_3$ holds. Setting $U := Rv_1 \oplus \ldots Rv_{n-1}$ and $W := Rv_2 \oplus \ldots Rv_n$ the induction hypothesis yields that U and W only admit trivial partitions into R-modules and therefore we find $P, Q \in \mathcal{P}$ containing U and W, respectively. Since we have $Rv_2 \subseteq U \cap W$ this implies $P = Q$ and thus $\mathcal{P} = \{P\}$, contradicting the fact that $\mathcal{P}$ is non-trivial. ∎

Because an algebraically closed field admits no proper finite extension one immediately gets the following consequence of Theorem 3.1.

Corollary 3.2. *Let F an algebraically closed field, V a finite dimensional F-vector space, and let φ be a non-zero endomorphism of V. Then the vector space V admits a non-trivial partition into φ-invariant subspaces if, and only if, the endomorphism φ is contained in the center of* $\mathrm{End}_F V$.

4. Application to Frobenius Lie Groups

Definition 4.1. By a *Frobenius group* we mean a group G that admits a semidirect decomposition $G = KN$ where N is a normal subgroup of G, called the *kernel* of G, and K is a complement of N such that for any $g \neq h \in N$ we have $gKg^{-1} \neq hKh^{-1}$ and the set $\{gKg^{-1} \,|\, g \in N\}$ together with N forms a partition of G. This partition is also called the *natural partition* of the group G. If G is supposed to be a Lie group we require K and N to be closed subgroups.

Remark 4.2. The condition in Definition 4.1 that $gKg^{-1} \neq hKh^{-1}$ holds for any two distinct elements $g, h \in N$ implies that the action of K on N by conjugation is free, i.e., for any $n \in N \setminus \{1\}$ we have $\{k \in K \,|\, knk^{-1} = n\} = \{1\}$. As we shall see in Example 4.3(b) this is sometimes sufficient for KN to be a Frobenius group. (Quite general, if a group G acts as a group of automorphisms on a group H, then we call this action *free* if for any element $h \in H \setminus \{1\}$ the stabilizer is trivial.)

Examples 4.3. (a) Any dilatation group $\mathrm{Dil}(V)$ is a Frobenius group with kernel $\{1\} \times V$.

(b) Example (a) is a special case of the following construction: Let V be a finite dimensional vector space over some skewfield F and let $G \subseteq \mathrm{GL}(V)$ be a non-trivial group that acts freely on V. Then the semidirect product $G \ltimes V$ is a Frobenius group with kernel $\{1\} \times V$. In order to see this, we have to show that for any element $(g, v) \in (G \setminus \{1\}) \times V$ there exists exactly one element $w \in V$ such that

$$\big(g, v + (1 - g)w\big) = (1, w)(g, v)(1, w)^{-1} \in G \times \{0\}.$$

By freeness of the action of G on V we obtain that $1 - g \in \mathrm{End}_F V$ is injective and thus invertible. So $w := -(1 - g)^{-1}v \in V$ is the unique element with the desired property.

(c) There is a generalisation of dilatation groups which one obtains by admitting $F = (F, +, \cdot)$ in the definition of $\mathrm{Dil}_n F$ to be a nearfield (that is, $(F, +)$ is an abelian group, $(F^\times, \cdot)$ is a group, $a(x + y) = ax + ay$ and $0x = 0$ hold for all $a, x, y \in F$). For any nearfield the set $K(F) := \{k \in F \,|\, (\forall x, y \in F)\, (x + y)k = xk + yk\}$ is a skewfield and F is a left vector space over $K(F)$ (cf. [4]). If F is finite-dimensional over $K(F)$, then $\mathrm{Dil}_n F$ is a Frobenius group with kernel $\{1\} \times F^n$, which immediately follows from the discussion in Example (b).

(d) One can for each positive real number b define proper nearfield $\mathbb{H}_b := (\mathbb{H}, \circ_b, +)$ by letting $x \circ_b y := x|x|^{ib}y|x|^{-ib}$ for $x, y \in \mathbb{H}$. (A theorem due to Kalscheuer states that any locally compact connected nearfield which is not a skewfield is isomorphic to exactly one of the nearfields $\mathbb{H}_b$ (cf. [10] 64.20).) Now, for any such nearfield $\mathbb{H}_b$ the group $\mathrm{Dil}_n\mathbb{H}_b$ is a Frobenius Lie group (cf. [6] 3.1).

We call a linear representation $\lambda : G \to \mathrm{GL}(V)$ *free* if the associated action of G on V is free in the sense of Remark 4.2. Throughout the remaining part of this note $\mathbb{F}$ stands for the field of the complex numbers or the quaternions, furthermore, we write $S_{\mathbb{F}}$ for the subgroup consisting of all elements of modulus 1 in the multiplicative group $\mathbb{F}^\times$.

Proposition 4.4. *Let G be a Frobenius Lie group whose kernel is a vector group V and whose complements are isomorphic to $S_{\mathbb{F}}$. Then G is isomorphic to one of the groups $S_{\mathbb{F}} \ltimes \mathbb{F}^n$ where the action of $S_{\mathbb{F}}$ on $\mathbb{F}^n$ is given by scalar multiplication. In particular, the kernel V is an $\mathbb{F}$-vector space and for any subgroup H of G that contains a complement of G the subgroup $H \cap V$ is an $\mathbb{F}$-subspace of V.*

Proof. We have $G \cong S_{\mathbb{F}} \ltimes_\lambda V$ with some representation $\lambda : S_{\mathbb{F}} \to \mathrm{GL}(V)$. As $S_{\mathbb{F}}$ is a compact group this representation is completely reducible whence we have a decomposition $V = V_1 \oplus \cdots \oplus V_n$ into irreducible $S_{\mathbb{F}}$-modules. Now the assumption on G to be a Frobenius group implies that each of the respective subrepresentations is free. By looking at the irreducible representations of $S_{\mathbb{C}} \cong \mathrm{SO}_2\mathbb{R}$ one gets that the only ones that are free are those which are equivalent to the natural representation on $\mathbb{C}$ given by left multiplication.

The analogous result holds true for $S_{\mathbb{H}}$ but it requires a little more work in order to prove it. We focus on V_1. By complexification we get a representation $S_{\mathbb{H}} \to \mathrm{GL}_{\mathbb{C}}(V_1 \otimes \mathbb{C})$, for which we have two possibilities: $V_1 \otimes \mathbb{C}$ is an irreducible complex $S_{\mathbb{H}}$-module or it decomposes into two isomorphic irreducible complex submodules. In both cases the associated action of $S_{\mathbb{H}}$ on $V_1 \otimes \mathbb{C}$ is free if the action of $S_{\mathbb{H}}$ on V_1 is free. In the latter case even V_1 has a complex structure and the real representation $S_{\mathbb{H}} \to \mathrm{GL}(V_1)$ can be viewed as a complex representation with respect to this structure. So we have to deal with irreducible complex representations of $S_{\mathbb{H}}$. Since $S_{\mathbb{H}} \cong \mathrm{SU}_2\mathbb{C}$ we are concerned with irreducible representations of $\mathrm{SU}_2\mathbb{C}$, but these are well-known. Up to isomorphism they all can be obtained by the following construction (see [3] II.5): let $\mathbb{C}[z_1, z_2]$ be the vector space of complex polynomials in two commuting variables and let P_n, $n \in \mathbb{N}_0$, be the subspace of homogeneous polynomials of degree n. Viewing the elements of $\mathbb{C}[z_1, z_2]$ as functions on $\mathbb{C}^2$ we can define an action of $\mathrm{SU}_2\mathbb{C}$ on $\mathbb{C}[z_1, z_2]$ by letting $gp(z) := p(g^{-1}z)$ where $g \in \mathrm{SU}_2\mathbb{C}$, $p \in \mathbb{C}[z_1, z_2]$, and $z \in \mathbb{C}^2$. Then P_n is an $(n+1)$-dimensional irreducible $\mathrm{SU}_2\mathbb{C}$-submodule. In order to see that only one of these irreducible submodules, namely P_1, leads to a free action, we argue as follows: clearly, the action of $\mathrm{SU}_2\mathbb{C}$ on P_0 is not free. In order to see that this is likewise the case for P_n if $n > 1$, take $p := (z \mapsto z_1^n) \in P_n$ and $g = \mathrm{diag}(a, \bar{a}) \in \mathrm{SU}_2\mathbb{C}$ where $a \in \mathbb{C} \setminus \{1\}$ is an n-th root of unity. Then we have $gp = p$ but $g \neq 1$.

Since there is no 2-dimensional real representation of $\mathrm{SU}_2\mathbb{C}$ whose complexification could yield the aforementioned representation on P_1, we get that the action of $\mathrm{SU}_2\mathbb{C}$ on P_1 is the only free action of $\mathrm{SU}_2\mathbb{C}$ on a real vector space. By uniqueness, this action of $\mathrm{SU}_2\mathbb{C}$ on P_1 is equivalent to the action of $S_{\mathbb{H}}$ on $\mathbb{H}$ by left multiplication. So an irreducible $\mathbb{H}^\times$-module decomposes into irreducible $S_{\mathbb{H}}$-submodules which we can identify with $\mathbb{H}$ and on which $S_{\mathbb{H}}$ acts via multiplication from the left. ∎

The next result is a consequence of Result 6.3 in [8]. Nevertheless, we give here a direct proof.

Lemma 4.5. *The Lie groups $\mathbb{C}^\times$ and $\mathbb{H}^\times$ only admit the trivial partition.*

Proof. Let $\mathcal{P}$ be a partition of $\mathbb{C}^\times$. If $\mathcal{P}$ were non-trivial then it would consist of the images of all one-parameter subgroups of $\mathbb{C}^\times$. Because a one-parameter subgroup of $\mathbb{C}^\times$ is given by a homomorphism $\alpha_c : \mathbb{R} \to \mathbb{C}^\times : t \mapsto e^{ct}$, where $c = a + bi \in \mathbb{C}$, this would imply $\{e^d \mid d \in \mathbb{N}\frac{2\pi a}{b}\} \subseteq \alpha_c(\mathbb{R}) \cap \alpha_1(\mathbb{R})$ for $c \in \mathbb{C} \setminus \mathbb{R}$, contradicting that fact that the elements of $\mathcal{P}$ intersect trivially.

Now let $\mathcal{P}$ be a partition of $\mathbb{H}^\times$. Using what we have already shown, we obtain that for any $h \in \mathbb{H}^\times$ the subgroup $C_h := h\mathbb{C}^\times h^{-1}$ of $\mathbb{H}^\times$ is contained in some $P_h \in \mathcal{P}$, and this implies that $\mathcal{P}$ is trivial, since we have $\bigcap\{C_h \mid h \in \mathbb{H}^\times\} = \mathbb{R}^\times$ and $\bigcup\{C_h \mid h \in \mathbb{H}^\times\} = \mathbb{H}^\times$. ∎

Proposition 4.6. *Let G be a Frobenius Lie group whose kernel is a vector group V and whose complements are isomorphic to $\mathbb{F}^\times$. Then any planar partition of G induces a partition of V into $\mathbb{F}^\times$-invariant subspaces.*

Proof. Let $G = \mathbb{F}^\times \ltimes_\lambda V$ with some representation $\lambda : \mathbb{F}^\times \to \mathrm{GL}(V)$ and assume that G admits a planar partition $\mathcal{P}$. Since $\mathbb{C}^\times$ and $\mathbb{H}^\times$ only admit the trivial partition any element of $\mathcal{P}$ which is not contained in V contains a complement of G. As all complements of G are conjugate to $\mathbb{F}^\times$ by an element of V, and as V is abelian, this implies that for any $Q \in \mathcal{Q} := \{P \in \mathcal{P} \mid P \not\subseteq V\}$ the subgroup $Q \cap V$ is an $\mathbb{F}^\times$-invariant subspace of V. So it remains to show that any $P \in \mathcal{P} \setminus \mathcal{Q}$ is also an $\mathbb{F}^\times$-invariant subspace of V. Since $\dim P + \dim P' = \dim G > \dim V$ holds for all $P, P' \in \mathcal{P}$ the set $\mathcal{P} \setminus \mathcal{Q}$ contains at most one element and hence, the assertion now follows from the fact that $\bigcup\{Q \cap V \mid Q \in \mathcal{Q}\}$ is an $\mathbb{F}^\times$-invariant subset of V. ∎

If one views the group $\mathrm{Dil}_n\mathbb{F}$ as a group of matrices by identifying the element (a, v) with the matrix $\begin{pmatrix} a\mathbf{1}_n & v \\ & 1 \end{pmatrix}$, and doing the analogue for the Lie algebra $\mathrm{dil}_n\mathbb{F}$, one sees that $\mathrm{dil}_n\mathbb{F}$ is the Lie algebra of $\mathrm{Dil}_n\mathbb{F}$ and that the exponential function is given by

$$\exp(a, v) = \begin{cases} \left(e^a, (e^a - 1)a^{-1}v\right) & \text{if } a \neq 0 \\ (1, v) & \text{otherwise} \end{cases}.$$

Using this, easy calculations show that $\exp : \mathrm{dil}_n\mathbb{F} \to \mathrm{Dil}_n\mathbb{F}$ is surjective and that

$$\exp \mathbb{F}x \subseteq (1, 0) + \mathbb{F}x$$

holds for each $x \in \mathrm{dil}_n\mathbb{F}$ (see [6] 3.1 for the details). Both facts together imply that

$$\exp \mathbb{F}x = ((1, 0) + \mathbb{F}x) \cap \mathrm{Dil}_n\mathbb{F} \tag{1}$$

holds for each $x \in \mathrm{dil}_n\mathbb{F}$, since the 1-dimensional $\mathbb{F}$-subspaces of $\mathrm{dil}_n\mathbb{F}$ form a partition of $\mathrm{dil}_n\mathbb{F}$.

Proposition 4.7. *If U is an $\mathbb{F}$-subspace of the Lie algebra $\mathrm{dil}_n\mathbb{F}$, then we have $\exp U = ((1, 0) + U) \cap \mathrm{Dil}_n\mathbb{F}$ and $\exp U$ is a closed subgroup of $\mathrm{Dil}_n\mathbb{F}$. In particular, if $\mathfrak{P}$ is any partition of the Lie algebra $\mathrm{dil}_n\mathbb{F}$ into $\mathbb{F}$-subspaces, then the set $\exp \mathfrak{P} := \{\exp \mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}$ is a partition of the Lie group $\mathrm{Dil}_n\mathbb{F}$.*

Proof. The fact that $\exp U = ((1,0)+U)\cap \mathrm{Dil}_n\mathbb{F}$ holds for any $\mathbb{F}$-subspace U of $\mathrm{dil}_n\mathbb{F}$ is an immediate consequence of equation (1), since any $\mathbb{F}$-subspace is cover by its 1-dimensional $\mathbb{F}$-subspaces. In order to see that $\exp U$ is a group, we pick $(1+a,v),(1+b,w)\in \exp U$ and compute

$$\begin{aligned}(1+a,v)(1+b,w) &= \big((1+a)(1+b),\,(1+a)v+u\big)\\ &= (1,0)+(a,u)+(1+a)(b,v),\end{aligned}$$

and

$$(1+a,v)^{-1} = \big((1+a)^{-1},-(1+a)^{-1}v\big) = (1,0)-(1+a)^{-1}(a,v).$$

Since U is an $\mathbb{F}$-subspace and $a,b\neq -1$ both results are contained in $\exp U$. ■

Theorem 4.8. *Let $\mathbb{F}\in\{\mathbb{C},\mathbb{H}\}$ and let G be a Frobenius Lie group whose kernel is a vector group and whose complements are isomorphic to $\mathbb{F}^\times$. If G admits a planar partition, then exactly one of the following holds:*

(i) *The group G is isomorphic to one of the groups $\mathrm{Dil}_n\mathbb{F}$.*

(ii) *The group G is isomorphic to one of the groups $\mathrm{Dil}_1\mathbb{H}_b$.*

Moreover, the planar partitions of $\mathrm{Dil}_n\mathbb{F}$ are exactly the exponential images of the planar partitions of $\mathrm{dil}_n\mathbb{F}$, which all consist of $\mathbb{F}$-subspaces. The only planar partitions of the groups $\mathrm{Dil}_1\mathbb{H}_b$ are the natural ones.

Proof. Let $G=\mathbb{F}^\times \ltimes_\lambda V$ and assume that G admits a planar partition $\mathcal{P}$. Then, in view of Proposition 4.4 and Proposition 4.6, the kernel V is an $\mathbb{F}$-vector space and the set $\mathcal{P}_V := \{P\cap V \mid P\in\mathcal{P}\}$ is a partition of V into $\mathbb{F}^\times$-invariant vector subspaces. Identifying $\mathbb{F}^\times$ with $\mathbb{R}\times S_\mathbb{F}$ via the map $(t,s)\mapsto e^ts$ we obtain that $\alpha := \lambda|_\mathbb{R} : \mathbb{R}\to \mathrm{GL}(V)$ is a one-parameter group with $\operatorname{im}\alpha\subseteq \mathrm{GL}_\mathbb{F}V$ and $\alpha(t)P\subseteq P$ for any $P\in\mathcal{P}_V$ and any $t\in\mathbb{R}$. Hence, there exists some $X\in \mathrm{End}_\mathbb{F}V$ such that $\alpha(t)=\exp(tX)$ for any $t\in\mathbb{R}$. We claim that $\mathcal{P}_V$ is an X-invariant partition, and in order to prove this, we pick a non-zero vector $v\in V$. As $\mathcal{P}_V$ is a partition of V there exists exactly one element $P_v\in\mathcal{P}_V$ that contains v. Since P_v is invariant under $\alpha(\mathbb{R})$ it contains the image of the curve $\alpha_v:\mathbb{R}\to V : t\mapsto \exp(tX)v$ and since it is closed in V it also contains the tangent vector $Xv=\dot\alpha_v(0)$, as was to be shown. Now we set $n:=\dim_\mathbb{F} V$ and distinguish several cases.

Let first $\mathbb{F}=\mathbb{C}$. If $n=1$ then we have $\mathrm{End}_\mathbb{C}V=\mathbb{C}\mathrm{id}_V$ and thus $X\in\mathbb{C}\mathrm{id}_V$. If $n>1$ then Corollary 3.2 applies and yields that X is contained in $\mathbb{C}\mathrm{id}_V$. Consequently, α is of the form $t\mapsto e^{ct}\mathrm{id}_V$ for some $c\in\mathbb{C}$. In fact, we even have $c\in\mathbb{C}\setminus\mathbb{R}i$, because of the injectivity of α. Now the map $(t,s)\mapsto e^{ct}s$ induces an automorphism of $\mathbb{C}^\times$, and this automorphism again induces an isomorphism of $G=\mathbb{C}^\times\ltimes_\lambda V$ onto $\mathrm{Dil}_n\mathbb{C}$.

Now let $\mathbb{F}=\mathbb{H}$ and $n>1$. As V is a $\mathbb{H}$-vector space it is a vector space over any subfield K of $\mathbb{H}$. We regard $\mathbb{C}$ as a subfield of $\mathbb{H}$ and set $\mathcal{K}:=\{h\mathbb{C}h^{-1}\mid h\in\mathbb{H}^\times\}$. Obviously, we have inclusions $\mathrm{GL}_\mathbb{H}V\subseteq \mathrm{GL}_K V\subseteq \mathrm{GL}(V)$ for any $K\in\mathcal{K}$. As in case $\mathbb{F}=\mathbb{C}$ we obtain $\alpha(\mathbb{R})\subseteq K^\times\mathrm{id}_V$ for any $K\in\mathcal{K}$, and hence

$$\alpha(\mathbb{R})\subseteq \bigcap_{K\in\mathcal{K}} K^\times\mathrm{id}_V = \mathbb{R}^\times\mathrm{id}_V$$

since $\bigcap \mathcal{K} = \mathbb{R}$. Consequently, α is of the form $t \mapsto e^{rt}\mathrm{id}_V$ for some $r \in \mathbb{R}$. Now, in analogy with the preceding case, we obtain that the map $(t,s) \mapsto e^{rt}s : \mathbb{R} \times S_{\mathbb{H}} \to \mathbb{H}^\times$ induces an isomorphism of $G = \mathbb{H}^\times \ltimes_\lambda V$ onto $\mathrm{Dil}_n\mathbb{H}$.

It remains to treat the case where $\mathbb{F} = \mathbb{H}$ and $n = 1$. Identifying V with $\mathbb{H}$, we obtain $\mathrm{End}_{\mathbb{H}}V = \{\rho_h \mid h \in \mathbb{H}\}$, where ρ_h denotes multiplication from the right with h. Since the map $h \mapsto \rho_{\bar{h}} : \mathbb{H} \to \mathrm{End}_{\mathbb{H}}V$ is an isomorphism of $\mathbb{R}$-algebras the one-parameter group $\alpha : \mathbb{R} \to \mathrm{GL}_{\mathbb{H}}V$ is determined by a continuous injective homomorphism $\beta : \mathbb{R} \to \mathbb{H}^\times$. Such a homomorphism is of the form $t \mapsto e^{ht}$ for some $h \in \mathbb{H}$. Because of the injectivity of β, we even have $h \in \mathbb{H} \setminus \mathrm{Pu}\,\mathbb{H}$, where $\mathrm{Pu}\,\mathbb{H}$ denotes set of pure quaternions. As a commutative real subalgebra of $\mathbb{H}$ the algebra $\mathbb{R}[e^{ht}]$ is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$ and hence, after conjugation with a suitable quaternion, we can assume $\mathbb{R}[e^{ht}] \subseteq \mathbb{C}$. Thus we have $h = a + bi \in \mathbb{C} \setminus \mathbb{R}i$ and by rescaling we can achieve $a = 1$. Furthermore, we can assume $b \geq 0$, if necessary by replacing the representation α by the equivalent real representation $\bar{\alpha} : t \mapsto \kappa \circ \alpha(t) \circ \kappa$, where $\kappa : \mathbb{H} \to \mathbb{H} : x \mapsto \bar{x}$ denotes conjugation in $\mathbb{H}$. Now the action of $\mathbb{H}^\times$ on $\mathbb{H}$ is given by $(a,v) \mapsto a|a|^{-1}v\overline{|a|^h} = av|a|^{-bi}$. For $b = 0$ the group G is obviously isomorphic with $\mathrm{Dil}_1\mathbb{H}$. If $b > 0$ then the map $(a,v) \mapsto (a|a|^{-bi}, v)$ from $\mathbb{H}^\times \times \mathbb{H}$ onto itself gives an isomorphism of the group G onto $\mathrm{Dil}_1\mathbb{H}_b$.

Let $\mathfrak{P}$ be a planar partition of $\mathrm{dil}_n\mathbb{F}$. We show that $\mathfrak{P}$ consists of $\mathbb{F}$-subspaces. Because of the dimension formula there exists at most one element of $\mathfrak{P}$ which is contained in the subspace $V := \{0\} \times \mathbb{F}^n$, and for any element $\mathfrak{p} \in \mathfrak{P}$ which is not contained in V we obtain that the projection of $\mathfrak{p}$ onto $\mathbb{F} \times \{0\}$ along V is surjective. Noting this and that $[(a,0),v] = av$ holds for any $a \in \mathbb{F}$ and any $v \in V$ one sees as in the proof of Proposition 4.6 that the set $\mathfrak{P}_V := \{\mathfrak{p} \cap V \mid \mathfrak{p} \in \mathfrak{P}\}$ consists of $\mathbb{F}$-subspaces of V. Let $\mathfrak{p} \in \mathfrak{P}$ be an element which is not contained in V and let $(1,v) \in \mathfrak{p} \setminus V$. For any $a \in \mathbb{F}$ we find $w \in \mathbb{F}^n$ such that $(a,w) \in \mathfrak{p}$. Now we compute $(0, w - av) = [(1,v),(a,w)] \in \mathfrak{p}$ and obtain $a(1,v) = (a,w) - (0, w - av) \in \mathfrak{p}$, as desired. Since $\mathfrak{P}$ consists of $\mathbb{F}$-subspaces of $\mathrm{dil}_n\mathbb{F}$ its exponential image $\exp\mathfrak{P}$ is a planar partition of $\mathrm{Dil}_n\mathbb{F}$, according to Proposition 4.7.

If $\mathcal{P}$ is a planar partition of $\mathrm{Dil}_n\mathbb{F}$, then $\mathrm{L}\mathcal{P}$ is a planar partition of $\mathrm{dil}_n\mathbb{F}$, and thus consists of $\mathbb{F}$-subspaces. So we can apply Proposition 4.7 and obtain $\mathcal{P} = \exp \mathrm{L}\mathcal{P}$.

The rest follows from the fact that for any of the groups $\mathrm{Dil}_1\mathbb{H}_b$ the complements only admit the trivial partition, since they all are isomorphic with $\mathbb{H}^\times$. ∎

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