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New prime gaps between 10^{15} and 5×10^{16}

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Abstract

The interval from 10^{15} to 5×10^{16} was searched for first occurrence prime gaps and maximal prime gaps. One hundred and twenty-two new first occurrences were found, including four new maximal gaps, leaving 1048 as the smallest gap whose first occurrence remains uncertain. The first occurrence of any prime gap of 1000 or greater was found to be the maximal gap of 1132 following the prime 1693182318746371. A maximal gap of 1184 follows the prime 43841547845541059. More extensive tables of prime gaps are maintained at http://www.trnicely.net.

1. Introduction

We restrict our discussion to the positive integers. Let Q denote the sequence of prime numbers, $Q = \{2, 3, 5, 7, 11, \ldots, q_k, q_{k+1}, \ldots\}$, and D the sequence of differences of consecutive prime numbers, $D = \{1, 2, 2, 4, \ldots, q_{k+1} - q_k, \ldots\}$.

A *prime gap* G is the interval bounded by two consecutive prime numbers q_k and q_{k+1}. The *measure* (size, magnitude) g of a prime gap G is the difference $g = q_{k+1} - q_k$ of its bounding primes. A prime gap is often specified by its measure g and its initial prime $p_1 = q_k$, and less often by the measure g and the terminal prime $p_2 = q_{k+1}$. A prime gap of measure g contains $g - 1$ consecutive composite integers. The measures of the prime gaps are the

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successive elements of the sequence D. Since 2 is the only even prime, every prime gap is of even measure, with the sole exception of the prime gap of measure 1 following the prime 2.

In illustration, a gap of measure $g = 6$ (or simply a gap of 6) follows the prime $p_1 = 23$, while a gap of 10 follows the prime 139.

It is elementary that gaps of arbitrarily large measure exist, since, as observed by Lucas [11], for $n > 0$ the integer $(n+1)! + 1$ must be followed by at least n consecutive composites, divisible successively by $2, 3, \ldots, n+1$; however, $n+1$ represents only a lower bound on the measure of such gaps.

The *merit* M of a prime gap of measure g following the prime p_1 is defined as $M = g/\ln(p_1)$. It is the ratio of the measure of the gap to the "average" measure of gaps near that point; as a consequence of the Prime Number Theorem, the average difference between consecutive primes near x is approximately $\ln(x)$.

A prime gap of measure g is considered a *first occurrence prime gap* when no smaller consecutive primes differ by exactly g, i.e., when this is the first appearance of the positive integer g in the sequence D. Thus, the gap of 4 following 7 is a first occurrence, while the gap of 4 following 13 is not. Note that this usage of the compound adjective *first occurrence* carries no implication whatsoever regarding historical precedence of discovery. Multiple instances of gaps of 1048 are known, but none is yet known to be a first occurrence, even though one of them bears an earliest historical date of discovery. This terminology follows that of Young and Potler [20], and produces more concise phrasing than some past and present alternative nomenclature.

A prime gap of measure g is titled *maximal* if it strictly exceeds all preceding gaps, i.e., the difference between any two consecutive smaller primes is $< g$, so that g exceeds all preceding elements of D. Thus the gap of 6 following the prime 23 is a maximal prime gap, since each and every smaller prime is followed by a gap less than 6 in measure; but the gap of 10 following the prime 139, while a first occurrence, is not maximal, since a larger gap (the gap of 14 following the prime 113) precedes it in the sequence of integers. Maximal prime gaps are *ipso facto* first occurrence prime gaps as well.

Furthermore, the term *first known occurrence prime gap* is used to denote a prime gap of measure g which has not yet been proven to be (and may or may not be) the true first occurrence of a gap of measure g; this situation arises from an incomplete knowledge of the gaps (and primes) below the first known occurrence. Thus, Nyman discovered a gap of 1048 following the prime 88089672331629091, and no smaller instance is known; but since his exhaustive scan extended only to 5×10^{16}, this gap remains for the moment merely a first known occurrence, not a first occurrence. First known occurrences serve as upper bounds for first occurrences not yet established.

The search for first occurrence and maximal prime gaps was previously extended to 10^{15} by the works of Glaisher [7], Western [18], Lehmer [10], Appel and Rosser [1], Lander and Parkin [9], Brent [2, 3], Young and Potler [20], and Nicely [12]. The present work extends this upper bound to 5×10^{16}. The calculations are currently being continued beyond 5×10^{16} by Tomás Oliveira e Silva [17], as part of a project generating numerical evidence for the Goldbach conjecture.

2. Computational Technique

The calculations were carried out over a period of years, distributed asynchronously among numerous personal computers, taking advantage of otherwise idle CPU time. Nyman accomplished the bulk of the computations; employing as many as eighty systems from 1998 to 2002, he accounted for the survey of the region from $1.598508912 \times 10^{15}$ through 5×10^{16}. Nicely's enumerations of prime gaps began in the summer of 1995, but the portion reported here was carried out from 1997 to 1999, over the interval from 10^{15} to $1.598508912 \times 10^{15}$, the number of systems in use varying from about five to twenty-five. The algorithms employed the classic sieve of Eratosthenes, with the addition of a few speed enhancing optimizations, to carry out an exhaustive generation and analysis of the differences between consecutive primes. More sophisticated techniques for locating large prime gaps, such as scanning through arithmetic progressions, were rendered impractical by the fact that the search for first occurrences was being carried out concurrently with other tasks; Nicely was enumerating prime constellations, while Nyman was gathering comprehensive statistics on the frequency distribution of prime gaps.

Among the measures taken to guard against errors (whether originating in logic, software, or hardware), the count $\pi(x)$ of primes was maintained and checked periodically against known values, such as those published by Riesel [14], and especially the extensive values computed recently by Silva [17]. In addition, Nicely has since duplicated Nyman's results through 4.5×10^{15}.

3. Computational Results

Table 3 lists the newly discovered first occurrence prime gaps resulting from the present study; maximal gaps are indicated by a double dagger (‡). Each table entry shows the measure g of the gap and the initial prime p_1. The fifteen gaps between 10^{15} and $1.598508912 \times 10^{15}$ are due to Nicely; all the rest were discovered by Nyman.

4. Observations

As a collateral result of his calculations, Nyman has computed for the count of twin primes the value $\pi_2(5 \times 10^{16}) = 47177404870103$, the maximum argument for which this function has been evaluated. Nyman also obtained $\pi(5 \times 10^{16}) = 1336094767763971$ for the corresponding count of primes; this is the largest value of x for which $\pi(x)$ has been determined by direct enumeration, and confirms the value previously obtained by Deléglise and Rivat [5], using indirect sieving methods. Nyman has also generated frequency tables for the distribution of all prime gaps below 5×10^{16}.

Listings of the 423 previously known first occurrence prime gaps (including 61 maximal gaps), those below 10^{15}, have been published collectively by Young and Potler [20] and Nicely [12], and are herein omitted for brevity.

A comprehensive listing of first occurrence and maximal prime gaps, annotated with additional information, is available at Nicely's URL. Nicely also maintains at his URL extensive lists of first known occurrence prime gaps, lying beyond the present upper bound of exhaustive computation, and discovered mostly by third parties, notably Harvey Dubner [6]. These lists exhibit specific gaps for every even positive integer up to 10884, as well as for other scattered even integers up to 233822; for some of the gaps exceeding 8000 in magnitude, the

Gap	Following the prime	Gap	Following the prime	Gap	Following the prime
796	1271309838631957	928	10244316228469423	1010	21743496643443551
812	1710270958551941	930	3877048405466683	1012	22972837749135871
824	1330854031506047	932	10676480515967939	1014	13206732046682519
838	1384201395984013	934	8775815387922523	1016	25488154987300883
842	1142191569235289	936	2053649128145117	1018	37967240836435909
846	1045130023589621	938	3945256745730569	1020	24873160697653789
848	2537070652896083	940	9438544090485889	1022	10501301105720969
850	2441387599467679	942	10369943471405191	1024	22790428875364879
852	1432204101894959	944	4698198022874969	1026	14337646064564951
854	1361832741886937	946	8445899254653313	1028	16608210365179331
856	1392892713537313	948	5806170698601659	1030	21028354658071549
858	1464551007952943	950	5000793739812263	1032	19449190302424919
864	2298355839009413	952	3441724070563411	1034	11453766801670289
866	2759317684446707	954	8909512917643439	1036	36077433695182153
868	1420178764273021	956	7664508840731297	1038	28269785077311409
870	1598729274799313	958	6074186033971933	1040	46246848392875127
874	1466977528790023	960	5146835719824811	1042	33215047653774409
876	1125406185245561	962	9492966874626647	1044	7123663452896833
878	2705074880971613	964	5241451254010087	1046	25702173876611591
882	3371055452381147	966	5158509484643071	1050	13893290219203981
884	1385684246418833	968	19124990244992669	1054	26014156620917407
886	4127074165753081	970	10048813989052669	1056	11765987635602143
888	2389167248757889	972	4452510040366189	1058	28642379760272723
890	3346735005760637	974	10773850897499933	1060	15114558265244791
892	2606748800671237	976	14954841632404033	1062	15500910867678727
894	2508853349189969	978	12040807275386881	1064	43614652195746623
896	3720181237979117	980	19403684901755939	1068	23900175352205171
898	4198168149492463	982	18730085806290949	1072	40433690575714297
900	2069461000669981	984	11666708491143997	1074	33288359939765017
902	1555616198548067	986	34847474118974633	1076	20931714475256591
904	3182353047511543	988	11678629605932719	1084	41762363147589283
908	2126985673135679	990	2764496039544377	1098	25016149672697549
910	1744027311944761	992	4941033906441539	1100	21475286713974413
912	2819939997576017	994	3614455901007619	1102	39793570504639117
914	3780822371661509	996	14693181579822451	1106	29835422457878441
‡916	1189459969825483	998	11813551133888459	1108	43986327184963729
918	2406868929767921	1000	22439962446379651	1120	19182559946240569
920	4020057623095403	1002	14595374896200821	1122	31068473876462989
922	4286129201882221	1004	7548471163197917	‡1132	1693182318746371
‡924	1686994940955803	1006	37343192296558573	‡1184	43841547845541059
926	6381944136489827	1008	5356763933625179		

Table 1. First occurrence prime gaps between 10^{15} and 5×10^{16}. ‡ denotes a maximal gap

bounding integers have only been proved strong probable primes (based on multiple Miller's tests).

The largest gap herein established as a first occurrence is the maximal gap of 1184 following the prime 43841547845541059, discovered 31 August 2002 by Nyman. The smallest gap whose first occurrence remains uncertain is the gap of 1048.

The maximal gap of 1132 following the prime 1693182318746371, discovered 24 January 1999 by Nyman, is the first occurrence of any "kilogap", i.e., any gap of measure 1000 or greater. Its maximality persists throughout an extraordinarily large interval; the succeeding maximal gap is the gap of 1184 following the prime 43841547845541059. The ratio of the initial primes of these two successive maximal gaps is ≈ 25.89, far exceeding the previous extreme ratio of ≈ 7.20 for the maximal gaps of 34 (following 1327) and 36 (following 9551), each discovered by Glaisher [7] in 1877. Furthermore, the gap of 1132 has the greatest merit (≈ 32.28) of any known gap; the maximal gap of 1184 is the only other one below 5×10^{16} having a merit of 30 or greater.

The gap of 1132 is also of significance to the related conjectures put forth by Cramér [4] and Shanks [16], concerning the ratio $g/\ln^2(p_1)$. Shanks reasoned that its limit, taken over all first occurrences, should be 1; Cramér argued that the limit superior, taken over all prime gaps, should be 1. Granville [8], however, provides evidence that the limit superior is $\geq 2e^{-\gamma} \approx 1.1229$. For the 1132 gap, the ratio is ≈ 0.9206, the largest value observed for any $p_1 > 7$, the previous best being ≈ 0.8311 for the maximal gap of 906 following the prime 218209405436543, discovered by Nicely [12] in February, 1996.

Several models have been proposed in an attempt to describe the distribution of first occurrence prime gaps, including efforts by Western [18], Cramér [4], Shanks [16], Riesel [14], Rodriguez [15], Silva [17], and Wolf [19]. We simply note here Nicely's empirical observation that all first occurrence and maximal prime gaps below 5×10^{16} obey the following relationship:

$$0.122985 \cdot \sqrt{g} \cdot \exp\sqrt{g} \;<\; p_1 \;<\; 2.096 \cdot g \cdot \exp\sqrt{g} \quad . \tag{1}$$

The validity of (1) for *all* first occurrence prime gaps remains a matter of speculation. Among its corollaries would be the conjecture that every positive even integer represents the difference of some pair of consecutive primes, as well as a fairly precise estimate for the answer to the question posed in 1964 by Paul A. Carlson to Daniel Shanks [16], to wit, the location of the first occurrence of one million consecutive composite numbers. The argument $g = 1000002$ entered into (1) yields the result $2.4 \times 10^{436} < p_1 < 4.2 \times 10^{440}$, which is near the middle of Shanks' own estimate of $10^{300} < p_1 < 10^{600}$.

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