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A Short Note on the Average Maximal Number of Balls in a Bin

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Abstract

We analyze the asymptotic behavior of the average maximal number of balls in a bin obtained by throwing uniformly at random r balls without replacement into n bins, T times. Writing the expected maximum as

$$\frac{r}{n}T + C_{n,r}\sqrt{T} + o(\sqrt{T}),$$

a recent preprint of Behrouzi-Far and Zeilberger asks for an explicit expression for $C_{n,r}$ in terms of n, r and π. In this short note, we find an expression for $C_{n,r}$ in terms of n, r and the expected maximum of n independent standard Gaussians. This provides asymptotics for large n as well as closed forms for small n—e.g., $C_{4,2} = \frac{3}{2\pi^{3/2}}\arccos(-1/3)$—and shows that computing a closed form for $C_{n,r}$ is precisely as hard as the difficult question of finding the expected maximum of n independent standard Gaussians.

1 Introduction

Suppose that you have n bins, and in each round, you throw r balls such that each ball lands in a different bin, with each of the $\binom{n}{r}$ possibilities equally likely. After T rounds, set $U(n,r;T)$ to be the maximum occupancy among the n bins. Set $A(n,r;T) = \mathbb{E}U(n,r;T) - \frac{r}{n}T$; utilizing a central limit theorem, one can show that $A(n,r;T) = C_{n,r}\sqrt{T} + o(\sqrt{T})$. A recent preprint [2] of Behrouzi-Far and Zeilberger asks for an explicit expression for $C_{n,r}$ in terms of n, r and π; they also calculate estimates for $C_{2,1}, C_{3,1}, C_{4,1}, C_{4,2}$ using recurrence relations derived with computer aid. As a motivation, Behrouzi-Far and Zeilberger [2] note that this problem arises in computer systems since load distribution across servers can be modeled with balls and bins.

Rather than utilizing exact computation in the vein of Behrouzi-Far and Zeilberger [2], we use a multivariate central limit theorem to prove the following:

Theorem 1. *Let $C_{n,r} = \lim_{T\to\infty} \frac{A(n,r;T)}{\sqrt{T}}$. Then*

$$C_{n,r} = \sqrt{\frac{r(n-r)}{n(n-1)}}\mathbb{E}\left(\max_{1\le j\le n} Z_j\right)$$

where Z_j are i.i.d. standard Gaussians.

The multivariate central limit theorem together with the continuous mapping theorem will give that $C_{n,r}$ is a maximum over some Gaussian vector (Lemma 5, Corollary 6, Lemma 7). Manipulating this Gaussian vector then relates this maximum to maximums of i.i.d. standard Gaussians (Lemma 8).

The expected maximum of n i.i.d. standard Gaussians appears to have no known closed form for general n and in fact the known forms for small n can be quite nasty; for instance, when $n = 5$, the expected value is $\frac{5}{2\pi^{3/2}} \arccos(-23/27)$. A short table of computed values is included in Section 3.

From here, we extract asymptotics for $n \to \infty$, uniformly in r:

Corollary 2. *As $n \to \infty$, we have*

$$C_{n,r} \sim \sqrt{\frac{2r(n-r)\log(n)}{n^2}}$$

uniformly in r.

Proof. Using the standard fact [4, Exercise 3.2.3] that $\mathbb{E}(\max_{1\le j\le n} Z_j) \sim \sqrt{2\log(n)}$ completes the proof (see [3, Section 10.5] for higher order information about $\max_{1\le j\le n} Z_j$). □

The exact form in Theorem 1 also picks up a nice combinatorial property:

Corollary 3. *For each n, the sequence $(C_{n,r})_{r=1}^{n-1}$ is log-concave.*

Proof. Log-concavity follows from the inequality

$$(r-1)(n-r+1)(r+1)(n-r-1) = (r^2-1)((n-r)^2-1) \le r^2(n-r)^2.$$

□

To prove Theorem 1, we use a multivariate central limit theorem to prove a limit theorem for $\frac{U(n,r;T)-\frac{r}{n}T}{\sqrt{T}}$ (Corollary 6), show that we can exchange the limit and expectation (Lemma 7), and then relate this expectation to the expected maximum of i.i.d. standard normals (Lemma 8).

2 Proving Theorem 1

We first must define the quantities of interest.

Definition 4. Set $b(n, r; T)$ to be the random vector in $\{0, 1, \ldots, T\}^n$ denoting the occupancies of the bins at time T. Define X to be the random variable in $\{0, 1\}^n$ chosen uniformly among vectors $v \in \{0, 1\}^n$ with $\|v\|_{L^1} = r$. Let Γ be the covariance matrix of X.

This immediately gives a representation for $b(n, r; T)$, since $b(n, r; T) \overset{d}{=} \sum_{j=1}^{T} X_j$ where X_j are i.i.d. copies of X. Aiming to use a multivariate central limit theorem, we must calculate the covariance matrix Γ of X.

Lemma 5. *The matrix Γ is given by*

$$\Gamma_{i,j} = \begin{cases} \frac{r(n-r)}{n^2}, & \text{for } i = j; \\ -\frac{r(n-r)}{n^2(n-1)}, & \text{otherwise.} \end{cases}$$

Proof. The coordinates of X are Bernoulli random variables with success parameter r/n and covariance $\mathbb{E}[X^{(i)}X^{(j)}] = \frac{\binom{n-2}{r-2}}{\binom{n}{r}}$ for $i \neq j$. The covariance matrix Γ may then be calculated easily:

$$\Gamma_{j,j} = \frac{r}{n}\left(1 - \frac{r}{n}\right) = \frac{r(n-r)}{n^2}.$$

For $\Gamma_{i,j}$ with $i \neq j$, we compute

$$\Gamma_{i,j} = \frac{\binom{n-2}{r-2}}{\binom{n}{r}} - \frac{r^2}{n^2} = -\frac{r(n-r)}{n^2(n-1)}.$$

□

From here, the multivariate central limit theorem shows convergence in distribution.

Corollary 6. *We have*

$$\frac{U(n, r; T) - \frac{r}{n}T}{\sqrt{T}} \xrightarrow{d} \max\{Y_1, \ldots, Y_n\}$$

where $(Y_1, \ldots, Y_n)$ is a mean-zero multivariate Gaussian with covariance matrix Γ, and the convergence is in distribution.

Proof. The multivariate central limit theorem [4, Thm. 3.9.6] implies that

$$\frac{b(n, r; T) - \mathbb{E}b(n, r; T)}{\sqrt{T}} \to \mathcal{N}(0, \Gamma).$$

The identity $\mathbb{E}b(n, r; T) = (\frac{r}{n}, \ldots, \frac{r}{n})$ together with an application of the continuous mapping theorem to the maximum function implies the Corollary. □

To gain information about $A(n,r;T)$, we need to show that not only do we have convergence in distribution, but that we can switch the order of taking limits and expectation.

Lemma 7.

$$C_{n,r} := \lim_{T\to\infty} \frac{A(n,r;T)}{\sqrt{T}} = \mathbb{E}\max\{Y_1,\dots,Y_n\}$$

where $(Y_1,\dots,Y_n)$ are jointly Gaussian with mean 0 and covariance matrix given by Γ as defined in Lemma 5.

Proof. Our strategy is to show uniform integrability of $\widehat{U}(T) := (U(n,r;T) - \frac{r}{n}T)/\sqrt{T}$; for $j \in \{1,2,\dots,n\}$, let $b^{(j)}$ denote the number of balls in bin j. Then by a union bound, we have

$$\mathbb{P}\left(\left|U(n,r;T) - \frac{r}{n}T\right| \geq \lambda\sqrt{T}\right) \leq n\mathbb{P}\left(\left|b^{(1)} - \frac{r}{n}T\right| \geq \lambda\sqrt{T}\right). \tag{1}$$

Since $b^{(1)}$ is a sum of independent Bernoulli random variables, we may apply Hoeffding's inequality (e.g., [1, Thm. 7.2.1]) to bound

$$\mathbb{P}\left(\left|b^{(1)} - \frac{r}{n}T\right| \geq \lambda\sqrt{T}\right) \leq 2\exp\left(-2\lambda^2\right).$$

Thus, for each T and $K > 0$ we have

$$\mathbb{E}\left(|\widehat{U}(T)| \cdot \mathbf{1}_{|\widehat{U}(T)|\geq K}\right) \leq 2n\int_K^\infty e^{-2\lambda^2}\,d\lambda.$$

This goes to zero uniformly in T as $K \to \infty$, thereby showing that the family $(\widehat{U}(T))_{T\geq 0}$ is uniformly integrable. Since uniform integrability together with convergence in distribution implies convergence of means, Corollary 6 completes the proof. □

All that remains now is to relate $\mathbb{E}\max\{Y_1,\dots,Y_n\}$ to the right-hand-side of Theorem 1.

Lemma 8. *Let $(Y_1,\dots,Y_n)$ be jointly Gaussian with mean 0 and covariance matrix Γ. Then*

$$\mathbb{E}(\max\{Y_1,\dots,Y_n\}0 = \sqrt{\frac{r(n-r)}{n(n-1)}}\,\mathbb{E}\left(\max_{1\leq j\leq n} Z_j\right)$$

where the variables Z_j are i.i.d. standard Gaussians.

Proof. Consider a multivariate Gaussian $(W_1,\dots,W_n)$ with mean 0 and covariance matrix given by

$$\widetilde{\Gamma}_{i,j} = \begin{cases} \frac{n}{n-1}, & \text{for } i = j; \\ -\frac{n}{(n-1)^2}, & \text{for } i \neq j. \end{cases}$$

Since $\Gamma = \frac{r(n-r)(n-1)}{n^3}\widetilde{\Gamma}$, we have

$$(Y_1,\dots,Y_n) \overset{d}{=} \sqrt{\frac{r(n-r)(n-1)}{n^3}}(W_1,\dots,W_n). \tag{2}$$

The vector $(W_1, \ldots, W_n)$ can in fact be realized by setting $W_j = Z_j - \frac{\sum_{i \neq j} Z_i}{n-1}$ with Z_i i.i.d. standard Gaussians. This is because the two vectors are both mean-zero multivariate Gaussians and have the same covariance matrix. Setting $S_n = \sum_{i=1}^n Z_i$, we note

$$W_j = -\frac{S_n}{n-1} + \frac{n}{n-1} Z_j$$

thereby implying

$$\max_{1 \leq j \leq n} \{W_j\} = -\frac{S_n}{n-1} + \left(\frac{n}{n-1}\right) \max_{1 \leq j \leq n} \{Z_j\}.$$

Taking expectations and utilizing (2) completes the proof. □

Remark 9. The final piece of the proof of Lemma 8—relating the expected maximum of the process $(Z_j - \frac{\sum_{i \neq j} Z_i}{n-1})_{j=1}^n$ to that of $(Z_j)_{j=1}^n$—is due to a Math Overflow answer of Iosef Pinelis [5]. Further, the vector $(W_1, \ldots, W_n)$ is in fact equal in distribution to the vector $(Z_1, \ldots, Z_n)$ conditioned to sum to 0.

Theorem 1 now follows from combining Lemmas 7 and 8.

3 Comparison with numerical values

Theorem 1 proves an equality for $C_{n,r}$, although for large n, the expectation on the right-hand-side of Theorem 1 appears to have no known closed form. Calculating these values for small n is tricky and tedious; we reproduce a few values of $\mathbb{E}(\max_{1 \leq j \leq n} Z_j)$ which can be computed precisely, as calculated by Selby [6]:

n	$\mathbb{E}(\max_{1 \leq j \leq n} Z_j)$
2	$\pi^{-1/2}$
3	$(3/2)\pi^{-1/2}$
4	$3\pi^{-3/2} \arccos(-1/3)$
5	$(5/2)\pi^{-3/2} \arccos(-23/27)$

Table 1: List of expected values of maximum of Gaussians.

We can then use these to obtain exact values for the values of $C_{n,r}$ predicted in Behrouzi-Far and Zeilberger [2], and note that their predictions are quite close:

	Exact value	Numerical approximation	Predicted value [2]
$C_{2,1}$	$\frac{1}{\sqrt{2\pi}}$	$0.39894\ldots$	$0.3989\ldots$
$C_{3,1}$	$\frac{\sqrt{3}}{2\sqrt{\pi}}$	$0.48860\ldots$	$0.489\ldots$
$C_{4,1}$	$\frac{3}{2\pi^{3/2}}\arccos(-1/3)$	$0.51469\ldots$	$0.516\ldots$
$C_{4,2}$	$\frac{\sqrt{3}}{\pi^{3/2}}\arccos(-1/3)$	$0.59431\ldots$	$0.59430\ldots$

Table 2: Comparison of exact values of $C_{n,r}$ with predictions from Behrouzi-Far and Zeilberger [2].

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