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Digit Sums Generalizing Binomial Coefficients

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Abstract

We define four different generalizations of binomial coefficients, using the digit sum of a string with digits in $\{0, 1, 2, \ldots, g\}$ for any positive integer g. We identify one generalization with the extended binomial coefficients, and we express every other generalization in terms of the extended binomial coefficients. We also express every generalization in terms of the binomial coefficients and find the explicit formula for each generalization.

1 Introduction

For any nonnegative integers n and k, the (n, k)-th binomial coefficient, denoted by $\binom{n}{k}$, is the coefficient of x^k in the expansion of $(x+1)^n$, i.e.,

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

and it satisfies the following recurrence relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1}$$

Using the recurrence relation, we construct Pascal's triangle (A007318) as follows:

$\binom{n}{k}$	$k=0$	1	2	3	4	5
$n=0$	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

To generalize the binomial coefficients, we let b be an integer greater than 1 and $g = b-1$ throughout this paper. Then we can express the *extended binomial coefficients* or *polynomial coefficients* as follows.

Definition 1. [5] For any nonnegative integer n and any integer k, the (n, k)-th b-nomial coefficient (or b-nomial number of type 1) is denoted by $\binom{n}{k}_b$ and satisfies

$$(x^g + x^{g-1} + \cdots + x + 1)^n = \sum_k \binom{n}{k}_b x^k. \tag{2}$$

Then the b-nomial coefficients satisfy the following recurrence relation [3]:

$$\binom{n}{k}_b = \sum_{i=0}^{g} \binom{n-1}{k-i}_b. \tag{3}$$

Using this recurrence relation, we construct a triangle and call it the *b-nomial triangle*. Then the 2-nomial triangle is Pascal's triangle, and b-nomial triangles for $b = 3$ and 4 (A027907 and A008287, respectively) are as follows [4]:

$\binom{n}{k}_3$	$k=0$	1	2	3	4	5	6	7	8	9	10
$n=0$	1										
1	1	1	1								
2	1	2	3	2	1						
3	1	3	6	7	6	3	1				
4	1	4	10	16	19	16	10	4	1		
5	1	5	15	30	45	51	45	30	15	5	1

$\binom{n}{k}_4$	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=0$	1															
1	1	1	1	1												
2	1	2	3	4	3	2	1									
3	1	3	6	10	12	12	10	6	3	1						
4	1	4	10	20	31	40	44	40	31	20	10	4	1			
5	1	5	15	35	65	101	135	155	155	135	101	65	35	15	5	1

Notice that Pascal's triangle is a lower triangular matrix but the b-nomial triangle for $b > 2$ is not. In this paper, we find two new generalizations of binomial coefficients, whose corresponding matrix is a lower triangular matrix. Out of these two matrices, one is symmetric just like Pascal's triangle, and the other is not.

To simplify the discussion below, we denote $l_b(m)$ and $s_b(m)$ as the length and the digit sum of the base-b representation of a nonnegative integer m, respectively. Since $s_b(m) \equiv m \pmod g$ for any integer m [11], there exists an integer k satisfying

$$s_b(g \cdot m) = g \cdot k.$$

Hence, we can ask the following questions for any positive integers n and k:

Question 1: How many nonnegative integers m satisfy $l_b(m) \leq n$ and $s_b(m) = k$?

Question 2: How many nonnegative integers m satisfy $l_b(m) = n$ and $s_b(m) = k$?

Question 3: How many nonnegative integers m satisfy $l_b(m) \leq n$ and $s_b(g \cdot m) = g \cdot k$?

Question 4: How many nonnegative integers m satisfy $l_b(m) = n$ and $s_b(g \cdot m) = g \cdot k$?

When $b = 2$, the answer to Questions 1 and 3 is the (n, k)-th binomial coefficient $\binom{n}{k}$ and the answer to Questions 2 and 4 is the $(n-1, k-1)$-th binomial coefficient $\binom{n-1}{k-1}$. Hence, by answering all these questions for any integer $b \geq 2$ and modifying the indices n and k, we can find four different generalizations of the binomial coefficients. Two of the generalizations are completely new, and both construct a lower triangular matrix for all b.

In Section 2, we clarify notation for this paper. Section 3 answers Questions 1 and 2, and Section 4 answers Questions 3 and 4 by defining new generalizations of binomial coefficients. In Section 5, we express each generalization in terms of the extended binomial coefficients. In Section 6, we express each generalization in terms of the binomial coefficients, and find the explicit formulas. In Section 7, we discuss the symmetry in each generalization and the sequences closely related to the generalizations.

2 Notation

We let Σ_b be the set $\{0, 1, 2, \ldots, g\}$ and Σ_b^* be the set of all finite strings consisting of digits in Σ_b. Every base-b representation of an integer is a finite string in Σ_b^*, and the set Σ_b^* also includes the *empty string*, which contains no digits, denoted by ϵ [9].

The notation for the number of digits in a string is as follows.

Notation 2. [9] For any finite string x and a digit a, let $|x|$ denote the number of digits in x, and $|x|_a$ denote the number of occurrences of digit a in x.

Lemma 3. *For any string x in Σ_b^*,*

$$|x| = \sum_{i=0}^{g} |x|_i.$$

For example, $|01011| = 5$, $|01011|_0 = 2$, $|01011|_1 = 3$, and $5 = 2+3$. We consider $|\epsilon| = 0$.

The following operation shows how to create a new string from given ones [9].

Definition 4. For any strings x and y and any positive integer n, the *concatenation* of x and y, denoted by xy, is the string obtained by joining x and y end-to-end, and x^n denotes the concatenation of n copies of x. That is, if $x = a_1a_2\cdots a_{|x|}$ and $y = b_1b_2\cdots b_{|y|}$ for some $a_i, b_i \in \Sigma_b$,

$$xy = a_1a_2\cdots a_{|x|}b_1b_2\cdots b_{|y|}, \text{ and } x^n = xx\cdots x \ (n \text{ times}).$$

By convention, x^0 is defined to be ϵ.

Lemma 5. *For any strings x and y and a nonnegative integer n, we have $|xy| = |x| + |y|$ and $|x^n| = n|x|$.*

For example, $101\ 00 = 10100$, $(10)^3 = 101010$, and $1 = 1\ (10)^0$. Then $|101\ 00| = |101| + |00| = 3 + 2 = 5$, $|(10)^3| = 3|10| = 3 \cdot 2 = 6$, and $|\epsilon| = |(10)^0| = 0$.

Since the base-b representation of an integer is a string in $\{0, 1, 2, \dots, g\}^*$, we call the base-b representation of an integer as a *b-ary string* throughout this paper. When we have to distinguish an integer and its b-ary string, we use the following notation.

Notation 6. For any integer m with its base-b representation x, we write $m = [x]_b$ or $(m)_b = x$.

For example, $5 = [12]_3$ and $(5)_3 = 12$. Then $([x]_b)_b = x$ for any b-ary string x and $[(m)_b]_b = m$ for any integer m. Throughout this paper, we use the convention that m is an integer and x is its b-ary string.

Notation 7. For any nonempty string x with $x = a_na_{n-1}\cdots a_2a_1$ for some digits a_i and any nonnegative integer m, we let $s(x)$ and $s_b(m)$ denote the digit sum of x and the digit sum of the b-ary string $(m)_b$, respectively, i.e.,

$$s(x) = \sum_{i=1}^{n} a_i \qquad \text{and} \qquad s_b(m) = s\left((m)_b\right).$$

By convention, we define $s(\epsilon) = 0$.

For example, $s(16) = 1 + 6 = 7$ and $s_3(16) = s(121) = 4$.

Definition 8. [2] For any string x with $x = a_na_{n-1}\cdots a_1$ for some digits a_i, the digit a_i is called *indispensable* in x, if $a_i = a_{i-1} = a_{i-2} = \cdots = a_{i-k+1} > a_{i-k}$ for some positive integer $k \le i + 1$, considering $a_0 = 0$, and *dispensable*, otherwise. We will follow this convention of dotting indispensable digits.

We let $\iota(x)$ denote the number of indispensable digits in x, and we let $\iota_b(m) = \iota((m)_b)$ for any nonnegative integer m. By convention, we define $\iota(\epsilon) = 0$.

For example, if $x = \dot{2}13345\dot{7}\dot{7}\dot{4}$, the digits 2, 7, 7, and 4 are indispensable in x and the digits 1, 3, 3, 4. and 5 are dispensable in x. Hence, $\iota(x) = 4$. If $m = 16$, the ternary string $(m)_3 = 121$ so $\iota_3(16) = \iota(1\dot{2}\dot{1}) = 2$.

To simplify arguments, we use the following notation as well.

Notation 9. For any positive integer n, we let Σ_b^n and σ_b^n denote as follows:

$$\Sigma_b^n = \{x \in \Sigma_b^* : |x| = n\} \text{ and } \sigma_b^n = \{x \in \Sigma_b^n : x \neq 0y \text{ for any string } y\}.$$

By convention, we let $\Sigma_b^0 = \sigma_b^0 = \{\epsilon\}$.

For example, $\Sigma_3^2 = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}$ and $\sigma_3^2 = \{10, 11, 12, 20, 21, 22\}$.

Notation 10. For any set X of strings, and any nonnegative integer k, we have

$$X[k] = \{x \in X : s(x) = k\} \text{ and } X\langle k\rangle = \{x \in X : \iota(x) = k\}.$$

For example, $\Sigma_3^2[2] = \{02, 11, 20\}$ and $\Sigma_3^2\langle 1\rangle = \{0\dot{1}, 0\dot{2}, \dot{1}0, 1\dot{2}, \dot{2}0\}$.

3 Addressing Questions 1 and 2

First, we find a combinatorial interpretations for the extended binomial coefficients as follows.

Theorem 11. *For any nonnegative integer n and any integer k, the number of b-ary strings of length n with digit sum k is the (n, k)-th b-nomial coefficient $\binom{n}{k}_b$, i.e.,*

$$\binom{n}{k}_b = |\Sigma_b^n[k]| = |\{x \in \Sigma_b^n : s(x) = k\}|. \tag{4}$$

Proof. Since the empty string ϵ is the only string of length 0 and $s(\epsilon) = 0$,

$$|\Sigma_b^0[0]| = |\{\epsilon\}| = 1 \text{ and } |\Sigma_b^0[k]| = |\phi| = 0 \text{ for nonzero } k.$$

Since the string 0^n is the only string of length n with digit sum 0,

$$|\Sigma_b^n[0]| = |\{0^n\}| = 1 \text{ for any nonzero } n.$$

Hence, the sequence $(|\Sigma_b^n[k]|)_{n,k\geq 0}$ has the same initial conditions as the b-nomial coefficients. We just need to show the sequences $(|\Sigma_b^n[k]|)_{n,k\geq 0}$ has the same recurrence relation as (3). For any digit $a \in \{0, 1, 2, \ldots, g\}$, we let

$$B_a = \{x \in \Sigma_b^n[k] : x = ya \text{ for any string } y\}.$$

Then the set $\Sigma_b^n[k]$ is partitioned by the B_a, i.e.,

$$\Sigma_b^n[k] = \bigcup_{a=0}^{g} B_a \text{ and } B_a \cap B_{a'} = \phi \text{ for } a \neq a'. \tag{5}$$

Consider a function $f_a : B_a \to \Sigma_b^{n-1}[k-a]$ with $f_a(ya) = y$ for any $y \in \Sigma_b^{n-1}$. Since $s(ya) = k$ iff $s(y) = k-a$, and $y_1 a = y_2 a$ iff $y_1 = y_2$, the function f_a is bijective. Hence, $|B_a| = |\Sigma_b^{n-1}[k-a]|$ so by (5),

$$|\Sigma_b^n[k]| = \sum_{a=0}^{g} |B_a| = \sum_{a=0}^{g} |\Sigma_b^{n-1}[k-a]|.$$

Therefore, the sequence $(|\Sigma_b^n[k]|)_{n,k\geq 0}$ has the same recurrence relation as (3). □

Hence, the (n,k)-th b-nomial coefficient is the answer to Question 1.

Lemma 12. *For any positive integers n and k,*

$$\binom{n}{k}_b = |\{m \in \mathbb{N} : l_b(m) \leq n \text{ and } s_b(m) = k\}|.$$

Using the combinatorial interpretation, we can also generalize the hockey stick identity for the binomial coefficients[6]:

$$\binom{n}{k} = \sum_{i=1}^{n} \binom{n-i}{k-1}. \tag{6}$$

Theorem 13. *For any positive integers n and k,*

$$\binom{n}{k}_b = \sum_{j=1}^{g} \sum_{i=1}^{n} \binom{n-i}{k-j}_b.$$

Proof. For any nonzero digit $a \in \Sigma_b$ and nonnegative integer $i < n$, we let

$$B_{a0^i} = \{x \in \Sigma_b^n[k] : x = ya0^i \text{ for any string } y\} = \{ya0^i : y \in \Sigma_b^{n-i-1}[k-a]\}.$$

Since $k > 0$, the string $0^n \notin \Sigma_b^n[k]$. Thus, the set $\Sigma_b^n[k]$ is partitioned by the set B_{a0^i}. That is,

$$\Sigma_b^n[k] = \bigcup_{a=1}^{g} \bigcup_{i=0}^{n-1} B_{a0^i}, \text{ and } B_{a0^i} \cap B_{a'0^{i'}} = \phi \text{ for } a \neq a' \text{ or } i \neq i'.$$

Hence,

$$\binom{n}{k}_b = |\Sigma_b^n[k]| = \sum_{a=1}^{g} \sum_{i=0}^{n-1} |B_{a0^i}|.$$

Since $|\{ya0^i : y \in \Sigma_b^{n-i-1}[k-a]\}| = |\Sigma_b^{n-i-1}[k-a]|$, the size $|B_{a0^i}| = |\Sigma_b^{n-i-1}[k-a]| = \binom{n-i-1}{k-a}_b$. By adjusting indices i and a, we have the relation. □

Now we consider strings with a nonzero leading digit. For any positive integer n and any integer k, we consider

$$|\sigma_b^n[k]| = |\{x \in \sigma_b^n : s(x) = k\}|.$$

Since $\sigma_b^0 = \{\epsilon\}$ and $s(\epsilon) = 0$, we have $|\sigma_b^o[k]| = 1$ if $k = 0$ and $|\sigma_b^o[k]| = 0$ otherwise. Then it is obvious that the size $|\sigma_b^n[k]|$ is the answer to Question 2.

Lemma 14. *For any positive integers n and k,*

$$|\sigma_b^n[k]| = |\{m \in \mathbb{N} : l_b(m) = n \text{ and } s_b(m) = k\}|.$$

When $b = 2$, since digit 1 is the only nonzero digit, the leading digit for every string in σ_2^n is 1, and the digit sum is the same as the number of 1 digits. Thus,

$$\begin{aligned}|\sigma_2^n[k]| = |\{1x \in \sigma_2^n : s(1x) = k\}| &= |\{x \in \Sigma_2^{n-1} : s(x) = k-1\}| \\ &= |\{x \in \Sigma_2^{n-1} : |x|_1 = k-1\}| = \binom{n-1}{k-1}.\end{aligned}$$

When $b > 2$, we sort the strings in σ_b^n according to digit sums to calculate the number $|\sigma_b^n[k]|$. For example, since $\sigma_3^2 = \{10\} \cup \{11, 20\} \cup \{12, 21\} \cup \{22\}$,

$$\begin{aligned}&|\sigma_3^2[1]| = |\{10\}| = 1; && |\sigma_3^2[2]| = |\{11, 20\}| = 2;\\ &|\sigma_3^2[3]| = |\{12, 21\}| = 2; && |\sigma_3^2[4]| = |\{22\}| = 1.\end{aligned}$$

Then the first few numbers for $|\sigma_b^n[k]|$ when $b = 2$, 3, and 4 are as follows:

$\lvert\sigma_2^n[k]\rvert$	$k=0$	1	2	3	4	5	6
$n=0$	1						
1		1					
2		1	1				
3		1	2	1			
4		1	3	3	1		
5		1	4	6	4	1	

$\lvert\sigma_3^n[k]\rvert$	$k=0$	1	2	3	4	5	6	7	8	9	10
$n=0$	1										
1		1	1								
2		1	2	2	1						
3		1	3	5	5	3	1				
4		1	4	9	13	13	9	4	1		
5		1	5	14	26	35	35	26	14	5	1

$\lvert\sigma_4^n[k]\rvert$	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=0$	1															
1		1	1	1												
2		1	2	3	3	2	1									
3		1	3	6	9	10	9	6	3	1						
4		1	4	10	19	28	34	34	28	19	10	4	1			
5		1	5	15	34	61	91	115	124	115	91	61	34	15	5	1

The sequences $(|\sigma_b^n[k]|)_{n,k\geq 0}$ for $b=3$ and 10 are identified with A005773 and A071976, respectively, but in general, the numbers are not studied much. In this paper, we adjust the indices n and k to study the numbers as a new generalization of the binomial coefficients as follows.

Definition 15. For any nonnegative integer n and any integer k, the (n,k)-th b-nomial number of type 2, denoted by $\binom{n}{k}_{b2}$, is defined as

$$\binom{n}{k}_{b2} = |\sigma_b^{n+1}[k+1]| = |\{x \in \sigma_b^{n+1} : s(x) = k+1\}|.$$

By convention, we define $\binom{-1}{k} = 1$ if $k=-1$ and $\binom{-1}{k} = 0$ otherwise.

Then the number $\binom{n}{k}_{22} = \binom{n}{k}$, and the first few numbers for $\binom{n}{k}_{b2}$ for nonnegative integers n and k when $b=3$ and 4 are as follows:

$\binom{n}{k}_{32}$	$k=0$	1	2	3	4	5	6	7	8	9	10	
$n=0$	1	1										
1	1	2	2	1								
2	1	3	5	5	3	1						
3	1	4	9	13	13	9	4	1				
4	1	5	14	26	35	35	26	14	5	1		
5	1	6	20	45	75	96	96	75	45	20	6	1

$\binom{n}{k}_{42}$	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=0$	1	1	1													
1	1	2	3	3	2	1										
2	1	3	6	9	10	9	6	3	1							
3	1	4	10	19	28	34	34	28	19	10	4	1				
4	1	5	15	34	61	91	115	124	115	91	61	34	15	5	1	
5	1	6	21	55	115	201	301	391	445	445	391	301	201	115	55	21

By the definitions, we find the following basic properties for the b-nomial numbers of type 1 and 2.

Lemma 16. *For any nonnegative integers n and k:*

(i) $\binom{n}{k}_b = \sum_{i=0}^{n} \binom{i-1}{k-1}_{b2}$*;*

(ii) $\binom{n}{k}_{b2} = \binom{n+1}{k+1}_b - \binom{n}{k+1}_b$*;*

(iii) $\binom{n}{k}_{b2} \neq 0$ *iff* $0 \leq k < g(n+1)$*;*

(iv) $\sum_{k\geq 0} \binom{n}{k}_b = b^n$ *and* $\sum_{k\geq 0} \binom{n}{k}_{b2} = g \cdot b^n$*;*

(v) $\binom{n}{k}_b = \binom{n-1}{k}_{b2}$ *if* $k = 0, 1, 2, \ldots, g-1$*;*

(vi) $\binom{0}{k}_{b2} = 1$ *if* $k = 0, 1, 2, \ldots, g-1$ *and* $\binom{0}{k}_{b2} = 0$ *otherwise.*

Proof. Since $\sigma_b^0 = \{\epsilon\}$, the set $\Sigma_b^n[k]$ can be partitioned as follows:

$$\Sigma_b^n[k] = \bigcup_{i=0}^{n} \{0^{n-i}x : x \in \sigma_b^i[k]\}.$$

Since $\binom{i-1}{k-1}_{b2} = |\sigma_b^i[k]| = |\{0^{n-i}x : x \in \sigma_b^i[k]\}|$, we have (i) by Theorem 11 and Definition 15. Then (i) provides (ii). For any string x in σ_b^{n+1}, $1 \le s(x) \le g(n+1)$, so we have (iii). Since $\Sigma_b^n = \bigcup_{k\ge 0} \Sigma_b^n[k]$ and $\sigma_b^n = \bigcup_{k\ge 0} \sigma_b^n[k]$, we have (iv). For (v), we consider $f : \Sigma_b^n[k] \to \sigma_b^n[k+1]$ by $f(a_1a_2\ldots a_n) = (a_1+1)a_2\ldots a_n$ for any digit a_i in Σ_b. Since $k \le g-1$, the digit $a_1 \le g-1$ so a_1+1 is a nonzero leading digit. Then f is a well-defined one-to-one mapping, so $|\Sigma_b^n[k]| = |\sigma_b^n[k+1]|$. Hence, (v) holds. Since $\sigma_b^1 = \{1, 2, \ldots, g\}$ and $s(a) = a$ for any $a = 1, 2, \ldots, g$, (vi) holds. □

Even if the b-nomial numbers of type 1 and 2 have different initial conditions, they have the same recurrence relations. We can also find a generalization of the hockey stick identity for the b-nomial numbers of type 2.

Corollary 17. *For any positive integer n and any integer k,*

$$\binom{n}{k}_{b2} = \sum_{i=0}^{g} \binom{n-1}{k-i}_{b2} = \sum_{j=1}^{g}\sum_{i=1}^{n+1} \binom{n-i}{k-j}_{b2}.$$

Proof. By Lemma 16 (ii) and the recurrence relation (3), we have the first recurrence relation for the b-nomial numbers of type 2:

$$\begin{aligned}\binom{n}{k}_{b2} &= \binom{n+1}{k+1}_b - \binom{n}{k+1}_b = \sum_{i=0}^{g}\binom{n}{k+1-i}_b - \sum_{i=0}^{g}\binom{n-1}{k+1-i}_b \\ &= \sum_{i=0}^{g}\left(\binom{n}{k+1-i}_b - \binom{n-1}{k+1-i}_b\right) = \sum_{i=0}^{g}\binom{n-1}{k-i}_{b2}.\end{aligned}$$

By Lemma 16 (ii) and Theorem 13, we have the second relation for the b-nomial numbers of type 2.

$$\begin{aligned}\binom{n}{k}_{b2} &= \binom{n+1}{k+1}_b - \binom{n}{k+1}_b \\ &= \sum_{j=1}^{g}\sum_{i=1}^{n+1}\binom{n+1-i}{k+1-j}_b - \sum_{j=1}^{g}\sum_{i=1}^{n}\binom{n-i}{k+1-j}_b \\ &= \sum_{i=1}^{g}\binom{0}{k+1-j}_b + \sum_{j=1}^{g}\sum_{i=1}^{n}\left(\binom{n+1-i}{k+1-j}_b - \binom{n-i}{k+1-j}_b\right) \\ &= \sum_{i=1}^{g}\binom{-1}{k-j}_{b2} + \sum_{j=1}^{g}\sum_{i=1}^{n}\binom{n-i}{k-j}_{b2} = \sum_{j=1}^{g}\sum_{i=1}^{n+1}\binom{n-i}{k-j}_{b2},\end{aligned}$$

since

$$\binom{0}{k}_b = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \binom{-1}{k}_{b2} = \begin{cases} 1, & \text{if } k = -1; \\ 0, & \text{otherwise.} \end{cases}$$

□

4 Addressing Questions 3 and 4

First, we define a new sequence to answer Question 3.

Definition 18. For any positive integer n and any integer k, the (n, k)-th b-nomial number of type 3, denoted by $\binom{n}{k}_{b3}$, is the number of b-ary strings of length n with k indispensable digits, i.e.,

$$\binom{n}{k}_{b3} = |\Sigma_b^n\langle k\rangle| = |\{x \in \Sigma_b^n : \iota(x) = k\}|.$$

By convention, we define $\binom{0}{k}_{b3} = 1$ if $k = 0$ and $\binom{0}{k}_{b3} = 0$ otherwise.

When $b = 2$, digit 1 is the only nonzero digit, so every digit 1 in a binary string is indispensable. Thus,

$$|\Sigma_2^n\langle k\rangle| = |\{x \in \Sigma_2^n : |x|_1 = k\}| = \binom{n}{k}.$$

Hence, the number $\binom{n}{k}_{23} = \binom{n}{k}$. When $b > 2$, we can sort the strings in Σ_b^n to calculate $\binom{n}{k}_{b3}$. For example, since

$$\Sigma_3^2 = \{00\} \bigcup \{0\dot{1}, 0\dot{2}, \dot{1}0, 1\dot{2}, \dot{2}0\} \bigcup \{\dot{1}\dot{1}, \dot{2}\dot{1}, \dot{2}\dot{2}\},$$

we have $\binom{2}{k}_{33} = 0$ for any $k \neq 0, 1, 2$, and

$$\binom{2}{0}_{33} = |\{00\}| = 1; \binom{2}{1}_{33} = |\{0\dot{1}, 0\dot{2}, \dot{1}0, \dot{2}0, 1\dot{2}\}| = 5; \binom{2}{2}_{33} = |\{\dot{1}\dot{1}, \dot{2}\dot{1}, \dot{2}\dot{2}\}| = 3.$$

Then the first few numbers for $\binom{n}{k}_{b3}$ when $b = 3$ and 4 are as follows:

$\binom{n}{k}_{33}$	$k = 0$	1	2	3	4	5
$n = 0$	1	0	0	0	0	0
1	1	2	0	0	0	0
2	1	5	3	0	0	0
3	1	9	13	4	0	0
4	1	14	35	26	5	0
5	1	20	75	96	45	6

$\binom{n}{k}_{43}$	$k = 0$	1	2	3	4	5
$n = 0$	1	0	0	0	0	0
1	1	3	0	0	0	0
2	1	9	6	0	0	0
3	1	19	34	10	0	0
4	1	34	115	91	15	0
5	1	55	301	445	201	21

Since $s_b(g \cdot m) = g \cdot \iota_b(m)$ [2], the number $\binom{n}{k}_{b3}$ is the answer to Question 3.

Lemma 19. *For any positive integers n and k,*

$$\binom{n}{k}_{b3} = |\{m \in \mathbb{N} : l_b(m) \leq n \text{ and } s_b(g \cdot m) = g \cdot k\}|.$$

Now, we consider strings with a nonzero leading digit to answer Question 4. For any positive integer n and any integer k, we consider the number of strings in the set σ_b^n with k indispensable digits, i.e.,

$$|\sigma_b^n\langle k\rangle| = |\{x \in \sigma_b^n : \iota(x) = k\}|.$$

Since $\sigma_b^0 = \{\epsilon\}$ and $\iota(\epsilon) = 0$, we have

$$|\sigma_b^0\langle k\rangle| = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $s_b(g \cdot m) = g \cdot \iota_b(m)$ [2], the number $|\sigma_b^n\langle k\rangle|$ is the answer to Question 3.

Lemma 20. *For any positive integers n and k,*

$$|\sigma_b^n\langle k\rangle| = |\{m \in \mathbb{N} : l_b(m) = n \text{ and } s_b(g \cdot m) = g \cdot k\}|.$$

When $b = 2$, since digit 1 is the only nonzero digit, the leading digit for every string in σ_2^n is 1 and every digit 1 is indispensable. Thus, for any positive integers n and k,

$$|\sigma_2^n\langle k\rangle| = |\{1x \in \sigma_2^n : \iota(1x) = k\}| = |\{x \in \Sigma_2^{n-1} : \iota(x) = k - 1\} = \binom{n-1}{k-1}.$$

When $b > 2$, we can sort the strings in σ_b^n to calculate the number $|\sigma_b^n\langle k\rangle|$. For example, since

$$\sigma_3^2 = \{\dot{1}0, \dot{2}0, 1\dot{2}\} \bigcup \{\dot{1}\dot{1}, \dot{2}\dot{1}, \dot{2}\dot{2}\},$$

we have $|\sigma_3^2\langle k\rangle| = 0$ for $k \neq 1, 2$, and

$$|\sigma_3^2\langle 1\rangle| = |\{\dot{1}0, \dot{2}0, 1\dot{2}\}| = 3; \ |\sigma_3^2\langle 2\rangle| = |\{\dot{1}\dot{1}, \dot{2}\dot{1}, \dot{2}\dot{2}\}| = 3.$$

Hence, the first few numbers for $|\sigma_b^n\langle k\rangle|$ when $b = 2$ and 3 are as follows:

$\lvert\sigma_2^n\langle k\rangle\rvert$	$k = 0$	1	2	3	4	5
$n = 0$	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	2	1	0	0
4	0	1	3	3	1	0
5	0	1	4	6	4	1

$\lvert\sigma_3^n\langle k\rangle\rvert$	$k = 0$	1	2	3	4	5
$n = 0$	1	0	0	0	0	0
1	0	2	0	0	0	0
2	0	3	3	0	0	0
3	0	4	10	4	0	0
4	0	5	22	22	5	0
5	0	6	40	70	40	6

Since $|\sigma_b^{n+1}\langle k+1\rangle| = \binom{n}{k}$ for any nonnegative integers n and k, we modify the indices n and k to find another generalization of the binomial coefficients.

Definition 21. For any nonnegative integer n and any integer k, the (n,k)-th b-nomial number of type 4, denoted by $\binom{n}{k}_{b4}$, satisfies

$$\binom{n}{k}_{b4} = |\sigma_b^{n+1}\langle k+1\rangle| = |\{x \in \sigma_b^{n+1} : \iota(x) = k+1\}|.$$

By convention, we define $\binom{-1}{k}_{4b} = 1$ if $k = -1$ and $\binom{-1}{k}_{4b} = 0$ otherwise.

Obviously, the number $\binom{n}{k}_{24} = \binom{n}{k}$, and the first few b-nomial numbers of type 4 for any nonnegative integers n and k when $b = 3$ and 4 are as follows:

$\binom{n}{k}_{34}$	$k=0$	1	2	3	4	5
$n=0$	2	0	0	0	0	0
1	3	3	0	0	0	0
2	4	10	4	0	0	0
3	5	22	22	5	0	0
4	6	40	70	40	6	0
5	7	65	171	171	65	7

$\binom{n}{k}_{44}$	$k=0$	1	2	3	4	5
$n=0$	3	0	0	0	0	0
1	6	6	0	0	0	0
2	10	28	10	0	0	0
3	15	81	81	15	0	0
4	21	186	354	186	21	0
5	28	371	1137	1137	371	28

By the definitions, we find the following basic properties for the b-nomial numbers of type 3 and type 4.

Lemma 22. *For any nonnegative integers n and k:*

(i) $\binom{n}{k}_{b3} = \sum_{i=0}^{n} \binom{i-1}{k-1}_{b4}$;

(ii) $\binom{n}{k}_{b4} = \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3}$;

(iii) $\binom{n}{k}_{b3} = 0 = \binom{n}{k}_{b4}$ *if* $k > n$;

(iv) $\sum_{k\geq 0} \binom{n}{k}_{b3} = b^n$ *and* $\sum_{k\geq 0} \binom{n}{k}_{b4} = g \cdot b^n$;

(v) $\binom{n}{n}_{b3} = \binom{n-1}{n-1}_{b4}$;

(vi) $\binom{1}{1}_{b3} = g = \binom{0}{0}_{b4}$.

Proof. Equations (i) and (ii) are obtained by Definitions 18 and 21. (iii) holds, because there cannot be more indispensable digits than the total number of digits. (iv) holds, because the sum of $\binom{n}{k}_{b3}$ for fixed n is the total number of b-ary strings of length n, and the sum of $\binom{n}{k}_{b4}$ for fixed n is the total number of b-ary strings of length $n+1$ with a nonzero leading digit. (v) holds, because $\binom{n-1}{n-1}_{b4} = \binom{n}{n}_{b3} - \binom{n-1}{n}_{b3} = \binom{n}{n}_{b3}$ by (ii) and $\binom{n-1}{n}_{b3} = 0$ by (iii). (vi) is obtained by $|\Sigma_b^1\langle 1\rangle| = |\{1, 2, \ldots, g\}| = |\sigma_b^1\langle 1\rangle|$. □

5 In terms of extended binomial coefficients

Section 3 shows that the b-nomial numbers of type 2 are expressed in terms of the extended binomial coefficients. In this section, we express the b-nomial numbers of type 3 and type 4 in terms of the extended binomial coefficients as well.

First, we define the following sequence to help the further calculations.

Definition 23. For any digit $a \in \{0, 1, 2, \ldots, g-1\}$, any positive integer n, and any integer k, we let Z_{ba}^n denote the set of b-ary strings of length $n+1$ with a dispensable leading digit a. That is,

$$Z_{ba}^n = \begin{cases} \{x \in \Sigma_b^{n+1} : [0^{n+1}]_b \leq [x]_b \leq [0g^n]_b\}, & \text{if } a = 0; \\ \{x \in \Sigma_b^{n+1} : [a^n(a+1)]_b \leq [x]_b \leq [ag^n]_b\}, & \text{if } a = 1, 2, \ldots, g-1. \end{cases}$$

We let $C_{ba}(n,k)$ denote the number of strings in Z_{ba}^n with k indispensable digits, i.e.,

$$C_{ba}(n,k) = |Z_{ba}^n\langle k\rangle|.$$

For example, since

$$Z_{41}^2 = \{11\dot{2}, 11\dot{3}, 1\dot{2}0, 12\dot{3}, 1\dot{3}0\} \cup \{1\dot{2}\dot{1}, 1\dot{2}\dot{2}, 1\dot{3}\dot{1}, 1\dot{3}\dot{2}, 1\dot{3}\dot{3}\};\ Z_{42}^2 = \{22\dot{3}, 2\dot{3}0\} \cup \{2\dot{3}\dot{1}, 2\dot{3}\dot{2}, 2\dot{3}\dot{3}\},$$

we have

$$C_{41}(2,1) = 5 = C_{41}(2,2); \quad C_{42}(2,1) = 2; \quad C_{42}(2,2) = 3.$$

The following shows the first few numbers for $C_{41}(n,k)$ and $C_{42}(n,k)$:

$C_{41}(n,k)$	1	2	3	4	5
$n=1$	2	0	0	0	0
2	5	5	0	0	0
3	9	24	9	0	0
4	14	71	71	14	0
5	20	166	310	166	20

$C_{42}(n,k)$	1	2	3	4	5
$n=1$	1	0	0	0	0
2	2	3	0	0	0
3	3	12	6	0	0
4	4	31	40	10	0
5	5	65	155	101	15

Since $Z_{b0}^n = \{0x : x \in \Sigma_b^n\}$ and digit 0 is always dispensable,

$$|Z_{b0}^n\langle k\rangle| = |\{0x : x \in \Sigma_b^n \text{ and } \iota(0x) = k\}| = |\{x \in \Sigma_b^n : \iota(x) = k\}| = |\Sigma_b^n\langle k\rangle|.$$

Hence, we identify the number $C_{b0}(n,k)$ with the (n,k)-th b-nomial number of type 3.

Lemma 24. *For any positive n and k,*

$$C_{b0}(n,k) = \binom{n}{k}_{b3}.$$

To calculate $C_{ba}(n,k)$ for any digit $a \in \{0,1,2,\dots,g-1\}$, we define a set Y_a as follows:

$$Y_a = \begin{cases} \{x \in \Sigma_b^m : [0^n]_b \le [x]_b \le [1^n]_b\}, & \text{if } a = 0; \\ \{x \in \Sigma_b^n : [a^{n-1}(a+1)]_b \le [x]_b \le [(a+1)^n]_b\}, & \text{if } a = 1,2,\dots,g-1. \end{cases} \tag{7}$$

Then the set Σ_b^n is partitioned by the Y_a, i.e.,

$$\Sigma_b^n = \bigcup_{a=0}^{g-1} Y_a \quad \text{and} \quad Y_a \cap Y_{a'} = \phi. \tag{8}$$

We identify the size $|Y_a\langle k\rangle|$ as the $(n, gk-a)$-th b-nomial coefficient as follows.

Lemma 25. *For any digit $a \in \{0,1,2,\dots,g-1\}$ and any positive integers n and k,*

$$|Y_a\langle k\rangle| = \binom{n}{gk-a}_b.$$

Proof. We define a function

$$f : Y_a\langle k\rangle \to \Sigma_b^n[gk-a] \text{ with } f(x) = (g \cdot [x]_b - [a0^n]_b)_b\,,$$

for any $x \in Y_a\langle k\rangle$. If $a = 1,2,\dots,g-1$, we have

$$g \cdot a = [a-1, b-a]_b \text{ and } g \cdot (a+1) = [a, g-a]_b. \tag{9}$$

Hence, we can calculate the bounds of the products of g by the numbers represented by the strings in Y_a as follows:

$$\begin{array}{ll} \text{if } x \in Y_0, & [00^n]_b = [0^n]_b \le g \cdot [x]_b \le [g^n]_b = [0g^n]_b; \\ \text{if } x \in Y_a (a \ne 0), & [a0^{n-1}(g-a)]_b \le g \cdot [x]_b \le [ag^{n-1}(g-a)]_b. \end{array}$$

Thus, there exists $y \in \Sigma_b^n$ such that $g \cdot [x]_b = [ay]_b$ for every $a = 0,1,2,\dots,g-1$, so

$$g \cdot [x]_b - [a0^n]_b = [0y]_b = [y]_b \text{ for some string } y \in \Sigma_b^n.$$

Since $\iota(x) = k$, $s_b(g \cdot [x]_b) = g \cdot \iota(x) = g \cdot k$, so

$$s_b(g \cdot [x]_b - [a0^n]_b) = s_b([y]_b) = s_b([ay]_b) - a = s_b(g \cdot [x]_b) - a = g \cdot k - a.$$

Thus, f is well-defined. We just need to show f is bijective so that

$$|Y_a\langle k\rangle| = |\Sigma_b^n[gk-a]| = \binom{n}{gk-a}_b.$$

It is obvious that f is injective, since if $f(x_1) = f(x_2)$, $g \cdot [x_1]_b = g \cdot [x_2]_b$ so $x_1 = x_2$. To prove f is surjective, we consider a string $y \in \Sigma_b^n[gk - a]$. Then $s(y) = g \cdot k - a$ so $s(ay) = s(y) + a = g \cdot k$. Thus, the number $[ay]_b$ is divisible by g, so

$$g \cdot \iota_b \left(\frac{[ay]_b}{g} \right) = s_b \left(g \cdot \frac{[ay]_b}{g} \right) = s_b([ay]_b) = g \cdot k, \text{ so } \iota_b \left(\frac{[ay]_b}{g} \right) = k.$$

Since $[0^n]_b \leq [y]_b \leq [g^n]_b$,

$$[a0^n]_b \leq [ay]_b \leq [ag^n]_b. \tag{10}$$

If $a \neq 0$, the number $[a0^{n-1}(g-1)]_b$ is the least multiple of g, and the number $[ag^{n-1}(g-a)]_b$ is the greatest multiple of g in the interval (10). Since the number $[ay]_b$ is a multiple of g,

$$\begin{aligned} &\text{if } a = 0, \quad [0^n]_b = [00^n]_b \leq [ay]_b \leq [0g^n]_b = [g^n]_b; \\ &\text{if } a \neq 0, \quad [a0^{n-1}(g-a)]_b \leq [ay]_b \leq [ag^{n-1}(g-a)]_b. \end{aligned}$$

Hence, by (9), we have

$$\begin{aligned} &\text{if } a = 0, \quad [0^n]_b \leq \frac{[ay]_b}{g} \leq [1^n]_b; \\ &\text{if } a \neq 0, \quad [a^{n-1}(a+1)] \leq \frac{[ay]_b}{g} \leq [(a+1)^n]_b. \end{aligned}$$

Therefore, $\left(\frac{[ay]_b}{g}\right)_b \in Y_a\langle k \rangle$ and

$$f\left(\left(\frac{[ay]_b}{g} \right)_b \right) = \left(g \cdot \frac{[ay]_b}{g} - [a0^n]_b \right)_b = ([0y]_b)_b = y.$$

∎

Now we express $C_{ba}(n, k)$ in terms of extended binomial coefficients.

Lemma 26. *For any digit $a \in \{0, 1, 2, \ldots, g-1\}$ and any positive integers n and k,*

$$C_{ba}(n, k) = \sum_{i=a}^{g-1} \binom{n}{gk - i}_b. \tag{11}$$

Proof. We define

$$f : Z_{ba}^n\langle k \rangle \to \bigcup_{i=a}^{g-1} Y_i\langle k \rangle \text{ with } f(ax) = x.$$

Since the leading digit a is always dispensable, $\iota(ax) = \iota(x)$. Hence, f is a well-defined one-to-one mapping, so $|Z_{ba}^n\langle k \rangle| = |\cup_{i=a}^{g-1} Y_i\langle k \rangle|$. By (8),

$$|Z_{ba}^n\langle k \rangle| = \sum_{i=a}^{g-1} |Y_i\langle k \rangle|.$$

By Definition 23 and Lemma 25, we obtain (11). ∎

Therefore, we can express the b-nomial numbers of type 3 and 4 in terms of the b-nomial numbers of type 1 and 2, respectively, as follows.

Theorem 27. *For any positive integers n and k:*

(i) $\displaystyle \binom{n}{k}_{b3} = \sum_{i=0}^{g-1} \binom{n}{gk-i}_{b} = \binom{n+1}{gk+1}_{b} - \binom{n}{gk+1}_{b} = \binom{n+1}{gk}_{b} - \binom{n}{gk-g}_{b};$

(ii) $\displaystyle \binom{n}{k}_{b4} = \sum_{i=1}^{g} \binom{n}{gk-i}_{b2} = \binom{n+1}{gk+g}_{b2} - \binom{n}{gk+g}_{b2} = \binom{n+1}{gk+g-1}_{b2} - \binom{n}{gk-1}_{b2}.$

Proof. By Lemma 24 and 26, we have the first expression for $\binom{n}{k}_{b3}$ in (i). By (3), we have the last two expressions in (i).

Since $\binom{n}{k}_{b4} = \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3}$ and $\binom{n}{k}_{b2} = \binom{n+1}{k+1}_{b} - \binom{n}{k+1}_{b}$ by Lemma 22 (ii) and Lemma 16 (ii), we apply the first expression for $\binom{n}{k}_{b3}$ in (i) to have

$$\begin{aligned} \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3} &= \sum_{i=0}^{g-1} \binom{n+1}{g(k+1)-i}_{b} - \sum_{i=0}^{g-1} \binom{n}{g(k+1)-i}_{b} \\ &= \sum_{i=1}^{g} \binom{n}{g(k+1)-i}_{b2}, \end{aligned} \tag{12}$$

which provides the first expression for $\binom{n}{k}_{b4}$ in (ii). Similarly, we apply the second expression for $\binom{n}{k}_{b3}$ in (i). Since $g(k+1)+1 = gk+b$, we have

$$\begin{aligned} \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3} &= \left(\binom{n+2}{gk+b}_{b} - \binom{n+1}{gk+b}_{b}\right) - \left(\binom{n+1}{gk+b}_{b} - \binom{n}{gk+b}_{b}\right) \\ &= \binom{n+1}{gk+b-1}_{b2} - \binom{n}{gk+b-1}_{b2}, \end{aligned} \tag{13}$$

which provides the second expression for $\binom{n}{k}_{b4}$ in (ii). Similarly, we apply the last expression for $\binom{n}{k}_{b3}$ in (i). Since $g(k+1)-g = gk$, we have

$$\begin{aligned} \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3} &= \left(\binom{n+2}{gk+g}_{b} - \binom{n+1}{gk}_{b}\right) - \left(\binom{n+1}{gk+g}_{b} - \binom{n}{gk}_{b}\right) \\ &= \left(\binom{n+2}{gk+g}_{b} - \binom{n+1}{gk+g}_{b}\right) - \left(\binom{n+1}{gk}_{b} - \binom{n}{gk}_{b}\right) \\ &= \binom{n+1}{gk+g-1}_{b2} - \binom{n}{gk-1}_{b2}, \end{aligned} \tag{14}$$

which provides the last expression in (ii). □

Applying Theorem 27 (i) to Lemma 22 (ii), we express the b-nomial numbers of type 4 in terms of the extended binomial coefficients as follows.

Corollary 28. *For any positive n and k,*

$$\begin{aligned}\binom{n}{k}_{b4} &= \binom{n+2}{gk+b}_b - 2\binom{n+1}{gk+b}_b + \binom{n}{gk+b}_b \\ &= \binom{n}{gk}_b + \sum_{i=1}^{g-1}\binom{n+1}{gk+i}_b \\ &= g\cdot\binom{n}{gk}_b + \sum_{i=1}^{g-1}(g-i)\cdot\left(\binom{n}{gk+i}_b + \binom{n}{gk-i}_b\right).\end{aligned}$$

Proof. Since $\binom{n}{k}_{b4} = \binom{n+1}{k+1}_{b3} - \binom{n}{k+1}_{b3}$, the first identity in (13) provides the first expression. By the first identity in (14), we have

$$\binom{n}{k}_{b4} = \left(\binom{n+2}{gk+g}_b - \binom{n+1}{gk}_b - \binom{n+1}{gk+g}_b\right) + \binom{n}{gk}_b.$$

Since $\binom{n+2}{gk+g}_b = \sum_{i=0}^{g}\binom{n+1}{gk+i}_b$ by (3), we have the second expression. By the first identity in (12) and (3), we have

$$\binom{n}{k}_{b4} = \sum_{i=1}^{g}\left(\binom{n+1}{gk+i}_b - \binom{n}{gk+i}_b\right) = \sum_{i=1}^{g}\sum_{j=1}^{g}\binom{n}{gk+i-j}_b.$$

The number $i-j$ for $1 \le i, j \le g$ is as follows:

$i-j$	$j=1$	2	3	$\cdots$	$g-2$	$g-1$	g
$i=1$	0	-1	-2	$\cdots$	$3-g$	$2-g$	$1-g$
2	1	0	-1	$\cdots$	$4-g$	$3-g$	$2-g$
3	2	1	0	$\cdots$	$5-g$	$4-g$	$3-g$
$\vdots$				$\ddots$			
$g-2$	$g-3$	$g-4$	$g-5$	$\cdots$	0	-1	-2
$g-1$	$g-2$	$g-3$	$g-4$	$\cdots$	1	0	-1
g	$g-1$	$g-2$	$g-3$	$\cdots$	2	1	0

Hence, we have

$$\begin{aligned}\binom{n}{k}_{b4} &= g\cdot\binom{n}{gk}_b \\ &\quad + (g-1)\cdot\binom{n}{gk+1}_b + (g-2)\cdot\binom{n}{gk+2}_b + \cdots + 1\cdot\binom{n}{gk+g-1}_b \\ &\quad + (g-1)\cdot\binom{n}{gk-1}_b + (g-2)\cdot\binom{n}{gk-2}_b + \cdots + 1\cdot\binom{n}{gk-(g-1)}_b.\end{aligned}$$

Therefore, we have the last expression for $\binom{n}{k}_{b4}$. □

6 Explicit formulas

Neuschel showed that the extended binomial coefficients can be expressed as a sum of Hermite polynomials and Bernoulli numbers [7]. In this section, we express the b-nomial numbers of each type, including extended binomial coefficients, in terms of the binomial coefficients.

Theorem 29. *For any nonnegative integers n and k:*

$$
\begin{aligned}
&(i) && \binom{n}{k}_b = \sum_{i_1+i_2+\cdots+i_g=k} \binom{n}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{g-1}}{i_g};\\
&(ii) && \binom{n}{k}_{b2} = \sum_{i_1+i_2+\cdots+i_g=k+1} \binom{n}{i_1-1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{g-1}}{i_g};\\
&(iii) && \binom{n}{k}_{b3} = \sum_{i_1+i_2+\cdots+i_g=gk+1} \binom{n}{i_1-1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{g-1}}{i_g};\\
&(iv) && \binom{n}{k}_{b4} = \sum_{i_1+i_2+\cdots+i_g=gk+b} \binom{n}{i_1-2}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{g-1}}{i_g}.
\end{aligned}
$$

Proof. (i) For any string $x = a_1a_2\cdots a_n$ with a digit $a_i \in \Sigma_b$, we can find the unique string x_j for each $j = 0, 1, 2, \ldots, g$ as follows:

$$x_j = a'_1a'_2\cdots a'_n, \text{ where } a'_i = \begin{cases} a_i, & \text{if } a_i < j; \\ j, & \text{if } a_i \geq j. \end{cases}$$

Hence, every string in $\Sigma_b^n[k]$ can be uniquely constructed as follows: For any positive integer i_j satisfying $\sum_{j=1}^{g} i_j = k$:

- Let $x_0 = 0^n$;
- For $j = 1, 2, \ldots, g$, construct string x_j, by choosing i_j digits $(j-1)$ in the string x_{j-1} and replacing each chosen digit $j-1$ with digit j;
- Let $x = x_g$.

Then each string $x_j \in \{0, 1, 2, \ldots, j\}^n$ for all $j = 0, 1, 2, \ldots, g$, so the string $x \in \Sigma_b^n$. Since there is no change in the occurrences of digit j after constructing the string x_{j+1},

$$|x|_j = |x_{j+1}|_j = i_j - i_{j+1} \text{ for all } j = 1, 2, \ldots, g-1, \text{ and } |x|_g = |x_g|_g = i_g.$$

Hence, the digit sum of the string x is

$$s(x) = \sum_{j=1}^{g} j \cdot |x|_j = \sum_{j=1}^{g-1} j \cdot (i_j - i_{j+1}) + g \cdot i_g = \sum_{j=1}^{g} i_j = k.$$

Since the string $x \in \Sigma_b^n$, the string $x \in \Sigma_b^n[k]$.

Since there are $\binom{n}{i_1}$ ways to construct x_1 from x_0 and $\binom{i_{j-1}}{i_j}$ ways to construct x_j from x_{j-1} for each $j = 2, 3, \ldots, g$, there are $\binom{n}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{g-1}}{i_g}$ distinct strings in $\Sigma_b^n[k]$. Therefore, the right-hand side of the equation (i) counts the number of strings in $\Sigma_b^n[k]$. Since the size $|\Sigma_b^n[k]| = \binom{n}{k}_b$, the equation (i) holds.

(ii) To construct a string in $\sigma_b^{n+1}[k+1]$, we start with the string $x_0 = 10^n$ and we construct x_1 by choosing $i_1 - 1$ copies of the digit 0 in x_0 and replacing each chosen digit 0 with the digit 1. Then the string x_1 has i_1 digits 1 because of the leading digit. After that, we have the same procedures to construct x_j for all $j = 2, 3, \ldots, g$ as (i). Since we want the digit sum to be $k+1$, the sum $\sum i_j = k+1$. Therefore, we have the equation (ii).

(iii) By (i) and Theorem 27 (i), we have

$$\begin{aligned}\binom{n}{k}_{b3} &= \binom{n+1}{gk+1}_b - \binom{n}{gk+1}_b \\ &= \sum_{\sum i_j = gk+1} \binom{n+1}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{g-1}}{i_g} - \sum_{\sum i'_j = gk+1} \binom{n}{i'_1}\binom{i'_1}{i'_2}\cdots\binom{i'_{g-1}}{i'_g}.\end{aligned}$$

We assume $i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+1$ and $i'_g \le i'_{g-1} \le \cdots \le i'_2 \le i'_1 \le n$. However, since $\binom{n}{n+1} = 0$, we can let $i'_1 \le n+1$ so we set $i_j = i'_j$. Hence,

$$\binom{n}{k}_{b3} = \sum \binom{n+1}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{g-1}}{i_g} - \sum \binom{n}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{g-1}}{i_g},$$

where $\sum_{j=1}^{g} i_j = gk+1$. Since $\binom{n+1}{i_1} - \binom{n}{i_1} = \binom{n}{i_1-1}$, we have the equation (iii).

(iv) Similarly, by (ii) and Theorem 27 (ii), we have

$$\begin{aligned}\binom{n}{k}_{b4} &= \binom{n+1}{gk+g}_{b2} - \binom{n}{gk+g}_{b2} \\ &= \sum \binom{n+1}{i_1-1}\binom{i_1}{i_2}\cdots\binom{i_{g-1}}{i_g} - \sum \binom{n}{i_1-1}\binom{i_1}{i_2}\cdots\binom{i_{g-1}}{i_g}\end{aligned}$$

where $\sum_{j=1}^{g} i_j = gk+g+1 = gk+b$. Since $\binom{n+1}{i_1-1} - \binom{n}{i_1-1} = \binom{n}{i_1-2}$, we have (iv). □

Note that we can assume the indices i_j in Theorem 29 satisfy the following:

$$\begin{aligned}&(i) && i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n;\\ &(ii) && i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+1;\\ &(iii) && i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+1;\\ &(iv) && i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+2,\end{aligned}$$

because otherwise, we have $\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{g-1}}{i_g} = 0$, $\binom{n}{i_1} = 0$, $\binom{n}{i_1-1} = 0$, or $\binom{n}{i_1-2} = 0$.

Using the explicit formula for the binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

we find the following explicit formulas for the b-nomial numbers of each type.

Theorem 30. *For any nonnegative integers n and k:*

$$(i) \quad \binom{n}{k}_b = \sum_{i_1+i_2+\cdots+i_g=k} \frac{n!}{(n-i_1)!(i_1-i_2)!(i_2-i_3)!\cdots(i_{g-1}-i_g)!i_g!};$$

$$(ii) \quad \binom{n}{k}_{b2} = \sum_{i_1+i_2+\cdots+i_g=k+1} \frac{i_1 \cdot n!}{(1+n-i_1)!(i_1-i_2)!(i_2-i_3)!\cdots(i_{g-1}-i_g)!i_g!};$$

$$(iii) \quad \binom{n}{k}_{b3} = \sum_{i_1+i_2+\cdots+i_g=gk+1} \frac{i_1 \cdot n!}{(1+n-i_1)!(i_1-i_2)!(i_2-i_3)!\cdots(i_{g-1}-i_g)!i_g!};$$

$$(iv) \quad \binom{n}{k}_{b4} = \sum_{i_1+i_2+\cdots+i_g=gk+b} \frac{i_1(i_1-1) \cdot n!}{(2+n-i_1)!(i_1-i_2)!(i_2-i_3)!\cdots(i_{g-1}-i_g)!i_g!},$$

where the indices i_j satisfy

$$\begin{aligned}
&0 \le i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n \quad \text{for } (i);\\
&0 \le i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+1 \text{ for } (ii);\\
&0 \le i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+1 \text{ for } (iii);\\
&0 \le i_g \le i_{g-1} \le \cdots \le i_2 \le i_1 \le n+2 \text{ for } (iv).
\end{aligned}$$

7 Symmetry

In this section, we discuss symmetry of the sequences defined in Sections 3, 4, and 5. By the definitions, the binomial coefficients and extended binomial coefficients have symmetry: for any nonnegative integer n,

$$\binom{n}{k} = \binom{n}{n-k} \text{ and } \binom{n}{k}_b = \binom{n}{gn-k}_b. \tag{15}$$

The b-nomial numbers of type 2 and 4 and the sequence $(C_{b1}(n,k))_{n,k>0}$ also have symmetry as follows.

Lemma 31. *For any nonnegative integers n and k:*

(i) $\binom{n}{k}_{b2} = \binom{n}{g(n+1)-(k+1)}_{b2};$

(ii) $\binom{n}{k}_{b4} = \binom{n}{n-k}_{b4};$

(iii) $C_{b1}(n,k) = C_{b1}(n, n-k+1).$

Proof. (i) Let x be a string in $\sigma_b^{n+1}[k+1]$. Then there exist digits $a_i \in \Sigma_b$ satisfying $x = a_1 a_2 \cdots a_n a_{n+1}$ with $a_1 \neq 0$ and $\sum a_i = k+1$. Then we define a string x' such that

$$x' = a_1' a_2' \cdots a_n' a_{n+1}', \text{ where } a_i' = \begin{cases} b - a_i, & \text{if } i = 1; \\ g - a_i, & \text{otherwise.} \end{cases}$$

Since $a_1 \neq 0$, the digit $a_1' \in \Sigma_b$ so every digit $a_i' \in \Sigma_b$ for all i. Since $a_1 \neq b$, the digit $a_1 \neq 0$. Since the sum $\sum a_i = k+1$,

$$s(x') = gn + b - \sum a_i = gn + g + 1 - (k+1) = g(n+1) - k.$$

Thus, the string $x' \in \sigma_b^{n+1}[g(n+1) - k]$. Hence, the function

$$f : \sigma_b^{n+1}[k+1] \to \sigma_b^{n+1}[g(n+1) - k] \text{ with } f(x) = f(x')$$

is well-defined. Since $a_i \neq c_i$ iff $a_i' \neq c_i'$, the function f is injective.

To prove f is surjective, we consider a string $y \in \sigma_b^{n+1}[g(n+1) - k]$ with digits c_i such that $y = c_1 c_2 \cdots c_n c_{n+1}$. Then the string

$$y' = c_1' c_2' \cdots c_n' c_{n+1}', \text{ where } c_{i'} = \begin{cases} b - c_i, & \text{if } i = 1; \\ g - c_i, & \text{otherwise,} \end{cases}$$

satisfies $f(y') = y$ and $y' \in \sigma_b^{n+1}[k+1]$, since $s(y') = gn + b - \sum c_i = gn + b - (gn + g - k) = k+1$. Therefore, f is bijective, so $|\sigma_b^{n+1}[k+1]| = |\sigma_b^{n+1}[g(n+1) - k]|$.

(ii) Applying (15) to the second identity in Corollary 28,

$$\begin{aligned} \binom{n}{k}_{b4} &= \binom{n}{gk}_b + \sum_{i=1}^{g-1} \binom{n+1}{gk+i}_b \\ &= \binom{n}{gn - gk}_b + \sum_{i=1}^{g-1} \binom{n+1}{g(n+1) - gk - i}_b \\ &= \binom{n}{g(n-k)}_b + \sum_{i=1}^{g-1} \binom{n+1}{g(n-k) + g - i}_b \\ &= \binom{n}{g(n-k)}_b + \sum_{i=1}^{g-1} \binom{n+1}{g(n-k) + i}_b = \binom{n}{n-k}_{b4}. \end{aligned}$$

(iii) Similarly, by Lemma 26 and (15),

$$\begin{aligned} C_{b1}(n,k) &= \sum_{i=1}^{g-1} \binom{n}{gk - i}_b = \sum_{i=1}^{g-1} \binom{n}{gn - gk + i}_b \\ &= \sum_{i=1}^{g-1} \binom{n}{gn - gk + g - i}_b = \sum_{i=1}^{g-1} \binom{n}{g(n-k+1) - i}_b = C_{b1}(n, n-k+1). \end{aligned}$$

□

Even if $\binom{n}{k}_{23} = \binom{n}{n-k}_{23}$ and $C_{b1}(n,k) = C_{b1}(n, n-k+1)$, the b-nomial numbers of type 3 and the sequence $[C_{ba}(n,k)]_{n,k\geq 0}$ are not symmetric in general. That is,

$$\binom{n}{k}_{b3} \neq \binom{n}{n-k}_{b3} \text{ for any } b > 2 \text{ and } C_{ba}(n,k) \neq C_{ba}(n, n-k+1) \text{ for any } a > 1.$$

However, we can find the following relations for symmetric entries.

Lemma 32. *For any positive integer n, any integer k, and any digit $a = 2, 3, \ldots, g-1$:*

(i) $\binom{n}{n-k+1}_{b3} - \binom{n}{k}_{b3} = \binom{n-1}{n-k+1}_{b3} - \binom{n-1}{k}_{b3}$;

(ii) $C_{ba}(n, n-k+1) - C_{ba}(n,k) = C_{b,b-a}(n, n-k+1) - C_{b,b-a}(n,k)$.

Proof. (i) By (15) and Theorem 27 (i),

$$\begin{aligned}
\binom{n}{n-k+1}_{b3} &= \sum_{i=0}^{g-1} \binom{n}{g(n-k+1)-i}_b = \sum_{i=0}^{g-1} \binom{n}{gn-(gk-g+i)}_b \\
&= \sum_{i=0}^{g-1} \binom{n}{gk-(g-i)}_b = \sum_{i=1}^{g} \binom{n}{gk-i}_b \\
&= \sum_{i=0}^{g-1} \binom{n}{gk-i}_b - \binom{n}{gk}_b + \binom{n}{g(k-1)}_b \\
&= \binom{n}{k}_{b3} - \binom{n}{gk}_b + \binom{n}{g(k-1)}_b.
\end{aligned} \tag{16}$$

By adjusting the indices n and k in (16), we have

$$\binom{n-1}{n-k+1}_{b3} = \binom{n-1}{n-1-(k-1)+1}_{b3} = \binom{n-1}{k-1}_{b3} - \binom{n-1}{g(k-1)}_b + \binom{n-1}{g(k-2)}_b. \tag{17}$$

By subtracting (17) from (16) and applying Theorem 27 (i), we have

$$\begin{aligned}
&\binom{n}{n-k+1}_{b3} - \binom{n-1}{n-k+1}_{b3} \\
&= \binom{n}{k}_{b3} - \binom{n-1}{k-1}_{b3} - \left(\binom{n}{gk}_b - \binom{n-1}{g(k-1)}_b\right) + \left(\binom{n}{g(k-1)}_b - \binom{n-1}{g(k-2)}_b\right) \\
&= \binom{n}{k}_{b3} - \binom{n-1}{k-1}_{b3} - \binom{n-1}{k}_{b3} + \binom{n-1}{k-1}_{b3} = \binom{n}{k}_{b3} - \binom{n-1}{k}_{b3},
\end{aligned}$$

which provides the relation in (i).

(ii) By (15) and Lemma 26, we have

$$\begin{aligned} C_{ba}(n, n-k+1) &= \sum_{i=a}^{g-1} \binom{n}{gn-gk+g-i}_b = \sum_{i=a}^{g-1} \binom{n}{gk-g+i}_b = \sum_{i=1}^{g-a} \binom{n}{gk-i}_b \\ &= \sum_{i=1}^{g-1} \binom{n}{gk-i}_b - \sum_{i=g-a+1}^{g-1} \binom{n}{gk-i}_b = C_{b1}(n,k) - C_{b,g-a+1}(n,k). \end{aligned} \tag{18}$$

By adjusting the index k, we have

$$C_{ba}(n,k) = C_{b1}(n, n-k+1) - C_{b,g-a+1}(n, n-k+1). \tag{19}$$

By subtracting (19) from (18) and applying Lemma 31 (iii), we have

$$C_{ba}(n, n-k+1) - C_{ba}(n,k) = C_{b,g-a+1}(n, n-k+1) - C_{b,g-a+1}(n,k).$$

Since $g-a+1 = b-a$, we have the relation in (ii). □

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