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Combinatorial Identities for Generalized Stirling Numbers Expanding f-Factorial Functions and the f-Harmonic Numbers

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Abstract

We introduce a class of $f(t)$-factorials, or $f(t)$-Pochhammer symbols, that includes many, if not most, well-known factorial and multiple factorial function variants as special cases. We consider the combinatorial properties of the corresponding generalized classes of Stirling numbers of the first kind that arise as the coefficients of the symbolic polynomial expansions of these f-factorial functions. The combinatorial properties of these more general parameterized Stirling number triangles include analogs of known expansions of the ordinary Stirling numbers by p-order harmonic number sequences, through the definition of a corresponding class of p-order f-harmonic numbers.

1 Introduction

1.1 Generalized f-factorial functions

1.1.1 Definitions

For any function, $f : \mathbb{N} \to \mathbb{C}$, and fixed non-zero indeterminates $x, t \in \mathbb{C}$, we introduce and define the *generalized $f(t)$-factorial function*, or alternately the *$f(t)$-Pochhammer symbol*, denoted by $(x)_{f(t),n}$, as the following products:

$$(x)_{f(t),n} = \prod_{k=1}^{n-1} \left(x + \frac{f(k)}{t^k} \right). \tag{1}$$

Within this article, we are interested in the combinatorial properties of the coefficients of the powers of x in the last product expansions which we consider to be generalized forms of the *Stirling numbers of the first kind* in this setting. Section 1.2 defines generalized Stirling numbers of both the first and second kinds and motivates the definitions of auxiliary triangles by special classes of formal power series generating function transformations and their corresponding negative-order variants considered in the references [18, 17].

1.1.2 Special cases

Key to the formulation of applications and interpreting the generalized results in this article is the observation that the definition of (1) provides an effective generalization of almost all other related factorial function variants considered in the references when $t \equiv 1$. The special cases of $f(n) := \alpha n + \beta$ for some integer-valued $\alpha \geq 1$ and $0 \leq \beta < \alpha$ lead to the motivations for studying these more general factorial functions in [17], and form the expansions of multiple α-factorial functions, $n!_{(\alpha)}$, studied in the triangular coefficient expansions defined by [15, 14]. The *factorial powers*, or *generalized factorials of t of order n and increment h*, denoted by $t^{(n,h)}$, or the *Pochhammer k-symbol* denoted by $(t)_{n,h} \equiv p_n(h, t) = t(t + h)(t + 2h) \cdots (t + (n - 1)h)$, studied in [3, 15, 2] form particular special cases, as do the the forms of the generalized *Roman factorials* and *Knuth factorials* for $n \geq 1$ defined in [8], and the *q-shifted factorial functions* considered in [10, 3]. When $(f(n), t) \equiv (q^{n+1}, 1)$ these products are related to the expansions of the finite cases of the *q-Pochhammer symbol* products, $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$, and the corresponding definitions of the generalized Stirling number triangles defined in (2) of the next subsection are precisely the *Gaussian polynomials*, or *q-binomial coefficients*, studied in relation to the q-series expansions and q-hypergeometric functions in [11, §17].

1.1.3 New results proved in the article

The results proved within this article, for example, provide new expansions of these special factorial functions in terms of their corresponding *p-order f-harmonic number sequences*,

$$F_n^{(p)}(t) := \sum_{1 \leq k \leq n} \frac{t^k}{f(k)^p},$$

which generalize known expansions of Stirling numbers by the ordinary *p-order harmonic numbers*, $H_n^{(p)} \equiv \sum_{1 \leq k \leq n} k^{-r}$, in [1, 14, 18, 17]. Still other combinatorial sums and properties satisfied by the symbolic polynomial expansions of these special case factorial functions follow as corollaries of the new results we prove in the next sections. The next subsection precisely expands the generalized factorial expansions of (1) through the generalized class of Stirling numbers of the first kind defined recursively by (2) below.

1.2 Definitions of generalized f-factorial Stirling numbers

We first employ the next recurrence relation to define the generalized triangle of Stirling numbers of the first kind, which we denote by $\left[{n \atop k}\right]_{f(t)} := [x^{k-1}](x)_{f(t),n}$, or just by $\left[{n \atop k}\right]_{f}$ when the context is clear, for natural numbers $n, k \geq 0$ [14, *cf.* §3.1] [1].

$$\begin{bmatrix} n \\ k \end{bmatrix}_{f(t)} = f(n-1) \cdot t^{1-n} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{f(t)} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{f(t)} + [n = k = 0]_{\delta} \tag{2}$$

Notice that the triangular coefficients defined by (2) are equivalent to the $b_i = 0$ case of the more general recurrence studied in [9, 19]. We also define the corresponding generalized forms of the *Stirling numbers of the second kind*, denoted by $\left\{{n \atop k}\right\}_{f(t)}$, so that we can consider inversion relations and combinatorial analogs of known identities for the ordinary triangles by the sum

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{f(t)} = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j} f(j)^n}{t^{jn} \cdot j!},$$

from which we can prove the following form of a particularly useful generating function transformation motivated in the references when $f(n)$ has a Taylor series expansion in integral powers of n about zero [14, *cf.* §3.3] [5, *cf.* §7.4] [16, 17]:

$$\sum_{0 \leq j \leq n} \frac{f(j)^k}{t^{jk}} z^j = \sum_{0 \leq j \leq k} \begin{Bmatrix} k \\ j \end{Bmatrix}_{f(t)} z^j \times D_z^{(j)} \left[\frac{1 - z^{n+1}}{1 - z} \right]. \tag{3}$$

The negative-order cases of the infinite series transformation in (3) are motivated in [17] where we define modified forms of the Stirling numbers of the second kind by

$$\begin{Bmatrix} k \\ j \end{Bmatrix}_{f^*} = \sum_{1 \leq m \leq j} \binom{j}{m} \frac{(-1)^{j-m}}{j! \cdot f(m)^k},$$

which then implies that the transformed ordinary and exponential zeta-like power series enumerating generalized polylogarithm functions and the f-harmonic numbers, $F_n^{(p)}(t)$, are expanded by the following two series variants [17]:

$$\begin{aligned} \sum_{n \geq 1} \frac{z^n}{f(n)^k} &= \sum_{j \geq 0} \begin{Bmatrix} k \\ j \end{Bmatrix}_{f^*} \frac{z^j \cdot j!}{(1-z)^{j+1}} \\ \sum_{n \geq 1} \frac{F_n^{(r)}(1) z^n}{n!} &= \sum_{j \geq 0} \begin{Bmatrix} k \\ j \end{Bmatrix}_{f^*} \frac{z^j \cdot e^z (j+1+z)}{(j+1)}. \end{aligned}$$

[1] The bracket symbol $[\texttt{cond}]_\delta$ denotes *Iverson's convention* which evaluates to exactly one of the values in $\{0, 1\}$ and where $[\texttt{cond}]_\delta = 1$ if and only if the condition `cond` is true.

We focus on the combinatorial relations and sums involving the generalized positive-order Stirling numbers in the next few sections.

2 Generating functions and expansions by f-harmonic numbers

2.1 Motivation from a technique of Euler

We are motivated by Euler's original technique for solving the *Basel problem* of summing the series, $\zeta(2) = \sum_n n^{-2}$, and later more generally all even-indexed integer zeta constants, $\zeta(2k)$, in closed-form by considering partial products of the sine function [6, pp. 38-42]. In particular, we observe that we have both an infinite product and a corresponding Taylor series expansion in z for $\sin(z)$ given by

$$\sin(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z \prod_{j \geq 1} \left(1 - \frac{z^2}{j^2 \pi^2}\right).$$

Then if we combine the form of the coefficients of z^3 in the partial product expansions at each finite $n \in \mathbb{Z}^+$ with the known trigonometric series terms defined such that $[z^3]\sin(z) = -\frac{1}{3!}$ given on each respective side of the last equation, we see inductively that

$$H_n^{(2)} = -\pi^2 \cdot [z^2] \prod_{1 \leq j \leq n} \left(1 - \frac{z^2}{j^2 \pi^2}\right) \qquad \longrightarrow \qquad \zeta(2) = \frac{\pi^2}{6}.$$

In our case, we wish to similarly enumerate the p-order f-harmonic numbers, $F_n^{(p)}(t)$, through the generalized product expansions defined in (1).

2.2 Generating the integer order f-harmonic numbers

We first define a shorthand notation for another form of generalized "*f-factorials*" that we will need in expanding the next products as follows:

$$n!_f := \prod_{j=1}^{n} f(j) \qquad \text{and} \qquad n!_{f(t)} := \prod_{j=1}^{n} \frac{f(j)}{t^j} = \frac{n!_f}{t^{n(n+1)/2}}.$$

If we let $\zeta_p \equiv \exp(2\pi\imath/p)$ denote the *primitive p^{th} root of unity* for integers $p \geq 1$, and define the coefficient generating function, $\widetilde{f}_n(w) \equiv \widetilde{f}_n(t; w)$, by

$$\widetilde{f}_n(w) := \sum_{k \geq 2} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{f(t)} w^k = \left(\prod_{j=1}^{n} \left(w + f(j) t^{-j}\right) - \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{f(t)} \right) w,$$

we can factor the partial products in (1) to generate the p-order f-harmonic numbers in the following forms:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{t^{kp}}{f(k)^p} &= \frac{t^{pn(n+1)/2}}{(n!_f)^p}[w^{2p}]\left((-1)^{p+1}\prod_{m=0}^{p-1}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{f(t)} \zeta_p^{m(k-1)} w^k\right) && (4)\\
&= \frac{t^{pn(n+1)/2}}{(n!_f)^p}[w^{2p}]\left(\sum_{j=0}^{p-1} \frac{(-1)^j w^j\ p}{p-j}\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{f(t)}^{j} \widetilde{f}_n(w)^{p-j}\right)\\
\sum_{k=1}^{n} \frac{t^{k}}{f(k)^p} &= \frac{t^{n(n+1)/2}}{(n!_f)^p}[w^{2p}]\left((-1)^{p+1}\prod_{m=0}^{p-1}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{f\left(t^{1/p}\right)} \zeta_p^{m(k-1)} w^k\right). && (5)
\end{aligned}$$

Example 1 (Special Cases)**.** For a fixed f and any indeterminate $t \neq 0$, let the shorthand notation $\bar{F}_n(k) := \left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]_{f(t)}$. Then the following expansions illustrate several characteristic forms of these prescribed partial sums for the first several special cases of (4) when $2 \leq p \leq 5$:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{t^{2k}}{f(k)^2} &= \frac{t^{n(n+1)}}{(n!_f)^2}\left(\bar{F}_n(2)^2 - 2\bar{F}_n(1)\bar{F}_n(3)\right) && (6)\\
\sum_{k=1}^{n} \frac{t^{3k}}{f(k)^3} &= \frac{t^{3n(n+1)/2}}{(n!_f)^3}\left(\bar{F}_n(2)^3 - 3\bar{F}_n(1)\bar{F}_n(2)\bar{F}_n(3) + 3\bar{F}_n(1)^2\bar{F}_n(4)\right)\\
\sum_{k=1}^{n} \frac{t^{4k}}{f(k)^4} &= \frac{t^{4n(n+1)}}{(n!_f)^4}\left(\bar{F}_n(2)^4 - 4\bar{F}_n(1)\bar{F}_n(2)^2\bar{F}_n(3) + 2\bar{F}_n(1)^2\bar{F}_n(3)^2 + 4\bar{F}_n(1)^2\bar{F}_n(2)\bar{F}_n(4)\right.\\
&\qquad\left. - 4\bar{F}_n(1)^3\bar{F}_n(5)\right)\\
\sum_{k=1}^{n} \frac{t^{5k}}{f(k)^5} &= \frac{t^{5n(n+1)/2}}{(n!_f)^5}\left(\bar{F}_n(2)^5 - 5\bar{F}_n(1)\bar{F}_n(2)^3\bar{F}_n(3) + 5\bar{F}_n(1)^2\bar{F}_n(2)\bar{F}_n(3)^2\right.\\
&\qquad + 5\bar{F}_n(1)^2\bar{F}_n(2)^2\bar{F}_n(4) - 5\bar{F}_n(1)^3\bar{F}_n(3)\bar{F}_n(4)\\
&\qquad\left. - 5\bar{F}_n(1)^3\bar{F}_n(2)\bar{F}_n(5) + 5\bar{F}_n(1)^4\bar{F}_n(6)\right).
\end{aligned}$$

For each fixed integer $p > 1$, the particular partial sums defined by the ordinary generating function, $\widetilde{f}_n(w)$, correspond to a function in n that is fixed with respect to the lower indices for the triangular coefficients defined by (2). Moreover, the resulting coefficient expansions enumerating the f-harmonic numbers at each $p \geq 2$ are isobaric in the sense that the sum of the indices over the lower index k is $2p$ in each individual term in these finite sums.

2.3 Expansions of the generalized coefficients by f-harmonic numbers

The *elementary symmetric polynomials* depending on the function f implicit to the product-based definitions of the generalized Stirling numbers of the first kind expanded through (1) provide new forms of the known p-order harmonic number, or *exponential Bell polynomial*, expansions of the ordinary Stirling numbers of the first kind enumerated in the references [1, 12, 4, 13]. Thus, if we first define the weighted sums of the f-harmonic numbers, denoted $w_f(n,m)$, recursively according to an identity for the Bell polynomials, $\ell \cdot Y_{n,\ell}(x_1, x_2, \ldots)$, for $x_k \equiv (-1)^k F_n^{(k)}(t^k)(k-1)!$ as [13, §4.1.8]

$$w_f(n+1,m) := \sum_{0 \leq k < m} (-1)^k F_n^{(k+1)}(t^{k+1})(1-m)_k w_f(n+1, m-1-k) + [m=1]_{\delta},$$

we can expand the generalized coefficient triangles through these weighted sums as

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{f(t)} &= \frac{n!_f}{(k-1)!}\, w_f(n+1,k) \\ &= \sum_{j=0}^{k-2} \begin{bmatrix} n+1 \\ k-1-j \end{bmatrix}_{f(t)} \frac{(-1)^j F_n^{(j+1)}(t^{j+1})}{(k-1)} + n!_{f(t)} \cdot [k=1]_{\delta}. \end{aligned} \tag{7}$$

The weighted terms of

$$(1-m)_k \equiv (-1)^k (m-1)_k = (-1)^k (m-1)m(m+1)\cdots(m-2+k)$$

are expressed in terms of the *Pochhammer symbol* defined by $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_n := 1$ when $n = 0$. This definition of the weighted f-harmonic sums for the generalized triangles in (2) implies the special case expansions given in the next corollary.

Corollary 2 (Weighted f-Harmonic Sums for the Generalized Stirling Numbers)**.** *The first few special case expansions of the coefficient identities in* (7) *are stated for fixed* f, $t \neq 0$, *and integers* $n \geq 0$ *in the following forms:*

$$\begin{aligned} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{f(t)} &= \frac{n!_f}{t^{n(n+1)/2}}\, F_n^{(1)}(t) \\ \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{f(t)} &= \frac{n!_f}{2\, t^{n(n+1)/2}} \left(F_n^{(1)}(t)^2 - F_n^{(2)}(t^2)\right) \\ \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_{f(t)} &= \frac{n!_f}{6\, t^{n(n+1)/2}} \left(F_n^{(1)}(t)^3 - 3F_n^{(1)}(t)F_n^{(2)}(t^2) + 2F_n^{(3)}(t^3)\right) \\ \begin{bmatrix} n+1 \\ 5 \end{bmatrix}_{f(t)} &= \frac{n!_f}{24\, t^{n(n+1)/2}} \Big(F_n^{(1)}(t)^4 - 6F_n^{(1)}(t)^2 F_n^{(2)}(t^2) + 3F_n^{(2)}(t^2)^2 + 8F_n^{(1)}(t)F_n^{(3)}(t^3) \\ &\qquad - 6F_n^{(4)}(t^4)\Big). \end{aligned} \tag{8}$$

Proof. These expansions are computed explicitly using the recursive formula in (7) for the first few cases of the lower triangle index $2 \leq k \leq 5$. $\square$

We will return to the expansions of these coefficients in (7) to formulate new finite sum identities providing functional relations between the p-order f-harmonic number sequences in the next section.

2.4 Combinatorial sums and functional equations for the f-harmonic numbers

The next several properties give interesting expansions of the p-order f-harmonic numbers recursively over the parameter p that can then be employed to remove, or at least significantly obfuscate, the current direct cancellation problem with these forms phrased by the examples in (6) and in (8).

Proposition 3. *For any fixed $p \geq 1$ and $n \geq 0$, we have the following coefficient product identities generating the p-order f-harmonic numbers, $F_n^{(p)}(t)$:*

$$\begin{aligned} F_n^{(p+1)}(t) &= F_n^{(p)}(t) + \frac{(-1)^p t^{n(n+1)/2}}{t^{\frac{pn(n+1)}{2(p+1)}} n!_f} \begin{bmatrix} n+1 \\ p+2 \end{bmatrix}_{f(t^{1/(p+1)})} \qquad (9) \\ &\quad + \sum_{j=0}^{p-1} \frac{p\,(-1)^{j+1} t^{n(n+1)/2}}{t^{\frac{jn(n+1)}{2p}} (n!_f)^{p-j}(p-j)} \left(\sum_{\substack{0 \leq i_1,\ldots,i_{p-j} \leq j \\ i_1+\cdots+i_{p-j}=j}} \begin{bmatrix} n+1 \\ i_1+2 \end{bmatrix}_{f(t^{1/p})} \cdots \begin{bmatrix} n+1 \\ i_{p-j}+2 \end{bmatrix}_{f(t^{1/p})} \right) \\ &\quad + \sum_{j=0}^{p-1} \sum_{i=0}^{j} \frac{(p+1) t^{n(n+1)/2} (-1)^j}{t^{\frac{jn(n+1)}{2(p+1)}} (n!_f)^{p+1-j}(p+1-j)} \begin{bmatrix} n+1 \\ i+2 \end{bmatrix}_{f(t^{1/(p+1)})} \times \\ &\quad \times \left(\sum_{\substack{0 \leq i_1,\ldots,i_{p-j} \leq j-i \\ i_1+\cdots+i_{p-j}=j-i}} \prod_{m=1}^{p-j} \begin{bmatrix} n+1 \\ i_m+2 \end{bmatrix}_{f(t^{1/(p+1)})} \right). \end{aligned}$$

Proof. To begin with, observe the following rephrasing of the partial sums expansions from equations (4) and (5) as

$$\begin{aligned} F_n^{(p+1)}(t) &= \frac{t^{n(n+1)/2}}{(n!_f)^{p+1}} \sum_{j=0}^{p} \frac{(p+1)\ (-1)^j}{(p+1-j)} \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{f\left(t^{1/(p+1)}\right)}^j [w^{2p+2-j}] \widetilde{f}_n(w)^{p+1-j} \\ &= \frac{(p+1)(-1)^p t^{n(n+1)/2}}{t^{\frac{pn(n+1)}{2(p+1)}} n!_f} \begin{bmatrix} n+1 \\ p+2 \end{bmatrix}_{f(t^{1/(p+1)})} \\ &\quad + \sum_{j=0}^{p-1} \frac{(p+1)(-1)^j t^{n(n+1)/2}}{t^{\frac{jn(n+1)}{2(p+1)}} (n!_f)^{p+1-j}(p+1-j)} [w^j] \left(\frac{\widetilde{f}_n(w)}{w^2} \right)^{p+1-j}. \end{aligned}$$

The coefficients involved in the partial sum forms for each sequence of $F_n^{(p)}(t)$ are implicitly tied to the form of $t \mapsto t^{1/p}$ in the triangle definition of (2). Given this distinction, let the generating function $\widetilde{f}$ be defined equivalently in the more careful definition as $\widetilde{f}_n(w) :\equiv \widetilde{f}_n(t;\ w)$. The powers of the generating function $\widetilde{f}_n(w)$ from the previous equations satisfy the coefficient term expansions according to the next equation [5, *cf.* §7.5].

$$\begin{aligned}
[w^{2p-j}]\widetilde{f}_n(w)^{p-j} &:= [w^{2p-j}]\widetilde{f}_n(t;\ w)^{p-j} = [w^j]\left(\frac{\widetilde{f}_n(t;\ w)}{w^2}\right)^{p-j} \\
&= \sum_{\substack{0 \leq i_1, \ldots, i_{p-j} \leq j \\ i_1 + \cdots i_{p-j} = j}} \begin{bmatrix} n+1 \\ i_1+2 \end{bmatrix}_{f(t)} \cdots \begin{bmatrix} n+1 \\ i_{p-j}+2 \end{bmatrix}_{f(t)}
\end{aligned}$$

Then by taking the difference of the harmonic sequence terms over successive indices $p \geq 1$ and at a fixed index of $n \geq 1$, the stated recurrences for these p-order sequences result. □

The generating function series over n in the next proposition is related to the forms of the *Euler sums* considered in [1] and to the context of the generalized zeta function transformations considered in [17] briefly noted in the introduction. We suggest the infinite sums over these generalized identities for $n \geq 1$ as a topic for future research exploration in the concluding remarks of Section 4.

Proposition 4 (Functional Equations for the f-Harmonic Numbers)**.** *For any integers $n \geq 0$ and $p \geq 2$, we have the following functional relations between the p-order and $(p-1)$-order f-harmonic numbers over n and p:*

$$\begin{aligned}
F_{n+1}^{(p)}(t^p) &= F_n^{(p)}(t^p) + \sum_{1 \leq j < p} \begin{bmatrix} n+2 \\ p+1-j \end{bmatrix}_{f(t)} \frac{(-1)^{p+1-j} t^{j(n+1)}}{f(n+1)^j (n+1)!_{f(t)}} + \begin{bmatrix} n+1 \\ p \end{bmatrix}_{f(t)} \frac{(-1)^{p+1}}{(n+1)!_{f(t)}} \\
&= F_n^{(p)}(t^p) + \frac{t^{(p-1)(n+1)}}{f(n+1)^{p-1}} + \frac{(-1)^{p-1}}{(n+1)!_{f(t)}} \left(\begin{bmatrix} n+1 \\ p \end{bmatrix}_{f(t)} + \begin{bmatrix} n+1 \\ p-1 \end{bmatrix}_{f(t)} \right) \\
&\quad + \begin{bmatrix} n+2 \\ p \end{bmatrix}_{f(t)} \frac{(-1)^p t^{n+1}}{f(n+1)(n+1)!_{f(t)}} \\
&\quad + \sum_{j=0}^{p-3} \begin{bmatrix} n+2 \\ j+2 \end{bmatrix}_{f(t)} \frac{(-1)^{j+1} \left(f(n+1) t^{-(n+1)} - 1\right) t^{(p-1-j)(n+1)}}{f(n+1)^{p-1-j} (n+1)!_{f(t)}}.
\end{aligned}$$

Proof. First, notice that (7) implies that we have the following weighted harmonic number sums for the p-order f-harmonic numbers:

$$F_n^{(p)}(t^p) = \sum_{1 \leq j < p} \begin{bmatrix} n+1 \\ p+1-j \end{bmatrix}_{f(t)} \frac{(-1)^{p+1-j} F_n^{(j)}(t^j)}{n!_{f(t)}} + \begin{bmatrix} n+1 \\ p+1 \end{bmatrix}_{f(t)} \frac{p(-1)^{p+1}}{n!_{f(t)}}.$$

Next, we use (2) twice to expand the differences of the left-hand-side of the previous equation as

$$\begin{aligned}
\frac{t^{p(n+1)}}{f(n+1)^p} &= F_{n+1}^{(p)}(t^p) - F_n^{(p)}(t^p) \\
&= \sum_{1\leq j<p} \begin{bmatrix} n+2 \\ p+1-j \end{bmatrix}_{f(t)} \frac{(-1)^{p+1-j} F_{n+1}^{(j)}(t^j)}{(n+1)!_{f(t)}} - \sum_{1\leq j<p} \begin{bmatrix} n+1 \\ p+1-j \end{bmatrix}_{f(t)} \frac{(-1)^{p+1-j} F_n^{(j)}(t^j)}{n!_{f(t)}} \\
&\qquad + \begin{bmatrix} n+2 \\ p+1 \end{bmatrix}_{f(t)} \frac{p(-1)^{p+1}}{(n+1)!_{f(t)}} - \frac{f(n+1)}{t^{n+1}} \begin{bmatrix} n+1 \\ p+1 \end{bmatrix}_{f(t)} \frac{p(-1)^{p+1}}{(n+1)!_{f(t)}} \\
&= \sum_{1\leq j<p} \begin{bmatrix} n+2 \\ p+1-j \end{bmatrix}_{f(t)} \frac{(-1)^{p+1-j} t^{j(n+1)}}{f(n+1)^j (n+1)!_{f(t)}} - \sum_{1\leq j<p} \begin{bmatrix} n+1 \\ p-j \end{bmatrix}_{f(t)} \frac{(-1)^{p-j} F_n^{(j)}(t^j)}{(n+1)!_{f(t)}} \\
&\qquad + \begin{bmatrix} n+1 \\ p \end{bmatrix}_{f(t)} \frac{p(-1)^{p+1}}{(n+1)!_{f(t)}} \\
&= \sum_{1\leq j<p} \begin{bmatrix} n+2 \\ p+1-j \end{bmatrix}_{f(t)} \frac{(-1)^{p+1-j} t^{j(n+1)}}{f(n+1)^j (n+1)!_{f(t)}} - \begin{bmatrix} n+1 \\ p \end{bmatrix}_{f(t)} \frac{(p-1)(-1)^{p+1}}{(n+1)!_{f(t)}} \\
&\qquad + \begin{bmatrix} n+1 \\ p \end{bmatrix}_{f(t)} \frac{p(-1)^{p+1}}{(n+1)!_{f(t)}}.
\end{aligned}$$

The second identity is verified similarly by combining the coefficient terms as in the last equations and adding the right-hand-side differences of the $(p-1)$-order f-harmonic numbers to the first identity. $\square$

One immediate corollary that is demonstrated in the next example provides new expansions of the p-order harmonic numbers in terms of the ordinary triangle of Stirling numbers of the first kind corresponding to the case where $(f(n), t) \equiv (n, 1)$ in the previous proposition. Similar expansions of identities related to the generalized generating function transformations in [17] result for the special cases of the proposition where $(f(n), t) \equiv (\alpha n + \beta, t)$ for some application-dependent prescribed $\alpha, \beta \in \mathbb{C}$ defined such that $-\frac{\beta}{\alpha} \notin \mathbb{Z}$. Another special case worth noting and independently expanding provides analogous relations between the q-binomial coefficients implicit to the forms of the *q-binomial theorem* expanding the q-Pochhammer symbols, $(a;q)_n$, for each $n \geq 0$ [11, *cf.* §17.2].

Example 5 (Stirling Numbers and Euler Sums)**.** For all integers $p \geq 3$ and fixed $n \in \mathbb{Z}^+$, we have the following identity relating the successive differences of the p-order harmonic numbers and the Stirling numbers of the first kind:

$$\begin{aligned}
\frac{1}{n^p} &= \frac{1}{n^{p-1}} + \frac{(-1)^{p-1}}{n!} \left(\begin{bmatrix} n \\ p \end{bmatrix} + \begin{bmatrix} n \\ p-1 \end{bmatrix} \right) + \begin{bmatrix} n+1 \\ p \end{bmatrix} \frac{(-1)^p}{n \cdot n!} \\
&\qquad + \sum_{j=0}^{p-3} \begin{bmatrix} n+1 \\ j+2 \end{bmatrix} \frac{(-1)^{j+1}(n-1)}{n^{p-1-j} \cdot n!}.
\end{aligned} \tag{10}$$

The relation in (10) certainly implies new finite sum identities between the p-order harmonic numbers and the Stirling numbers of the first kind, though the generating functions and limiting cases of these sums provide more information on infinite sums considered in several of the references.

With this in mind, we define the *Nielsen generalized polylogarithm*, $S_{t,k}(z)$, by the infinite generating series over the t-power-scaled Stirling numbers as [1, *cf.* §5]

$$S_{t,k}(z) := \sum_{n\geq 1} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n^t \cdot n!}.$$

We see immediately that (10) provides strictly enumerative relations between the polylogarithm function generating functions, $\mathrm{Li}_p(z)/(1-z)$, for the p-order harmonic numbers and the Nielsen polylogarithms. Perhaps more interestingly, we also find new identities between the Riemann zeta functions, $\zeta(p)$ and $\zeta(p-1)$, and the special classes of *Euler sums* given by $S_{t,k}(1)$ for $t \in [2, p-1]$ and $k \in [2, p]$ defined as in the reference [1, §5].

3 Coefficient identities and generalized forms of the Stirling convolution polynomials

3.1 Generalized Coefficient Identities and Relations

There are several formulas involving small-indexed columns of the generalized triangle defined by (2) that follow easily from an inductive argument. In particular, the next identities given in (11) are given for general lower column index $k \geq 1$ by

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_{f(t)} &= [w^{k-1}] \left(\prod_{j=1}^{n-1} (w + f(j)\ t^{-j}) \right) [n \geq 1]_\delta + [n = k = 0]_\delta \\ &= \sum_{0 < i_1 < \cdots < i_{n-k} < n} f(i_1) \cdots f(i_{n-k}) \cdot t^{-(i_1 + \cdots + i_{n-k})}, \end{aligned} \tag{11}$$

which follows immediately by considering the first products of the form $\prod_i (z + x_i)$ in the context of elementary symmetric polynomials for these specific x_i.

Proposition 6 (Horizontal and Vertical Column Recurrences)**.** *The generalized Stirling numbers of the first kind over the first several special case columns for the shifted upper index of $n+1$ in the expansions of (2) are given by the next recurrence relations for all $n \geq 0$ and any $k \geq 2$.*

$$\begin{aligned} \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{f(t)} &= \frac{n!_f}{t^{n(n+1)/2}} \\ \begin{bmatrix} n+1 \\ k \end{bmatrix}_{f(t)} &= \frac{n!_f}{t^{n(n+1)/2}} \sum_{j=1}^{n} \begin{bmatrix} j \\ k-1 \end{bmatrix}_{f(t)} \frac{t^{j(j+1)/2}}{j!_f}, \quad \textit{if } k \geq 2 \end{aligned} \tag{12}$$

Proof. We begin by observing that by (2) when $k \equiv 1$, we have that

$$\begin{aligned} \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{f(t)} &= \frac{f(n)}{t^n} \begin{bmatrix} n \\ 1 \end{bmatrix}_{f(t)} + \begin{bmatrix} n \\ 0 \end{bmatrix}_{f(t)} \\ &= \frac{f(n)}{t^n} \begin{bmatrix} n \\ 1 \end{bmatrix}_{f(t)} + [n=0]_\delta \,, \end{aligned}$$

which implies the first claim by induction since $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]_{f(t)} = 1$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]_{f(t)} = 1$. To prove the column-wise recurrence relation given in (12), we notice again by induction that for any functions $g(n)$ and $b(n) \neq 0$, the sequence, $f_k(n)$, defined recursively by

$$f_k(n) = \begin{cases} b(n) \cdot f_k(n-1) + g(n-1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0; \end{cases}$$

has a closed-form solution given by

$$f_k(n) = \left(\prod_{j=1}^{n-1} b(j) \right) \times \sum_{0 \leq j < n} \frac{g(j)}{\prod_{i=1}^{j} b(j)}.$$

Thus by (2) the second claim is true. □

3.2 Generalized forms of the Stirling convolution polynomials

Definition 7 (Stirling Polynomial Analogs). For $x, n, x-n \geq 1$, we suggest the next two variants of the generalized *Stirling convolution polynomials*, denoted by $\sigma_{f(t),n}(x)$ and $\widetilde{\sigma}_{f(t),n}(x)$, respectively, as the right-hand-side coefficient definitions in the following equations:

$$\begin{aligned} \sigma_{f(t),n}(x) &:= \begin{bmatrix} x \\ x-n \end{bmatrix}_{f(t)} \frac{(x-n-1)!}{x!_f} \quad \iff \quad \begin{bmatrix} n+1 \\ k \end{bmatrix}_{f(t)} = \frac{(n+1)!_f}{(k-1)!} \sigma_{f(t),n+1-k}(n+1) \\ \widetilde{\sigma}_{f(t),n}(x) &:= \begin{bmatrix} x \\ x-n \end{bmatrix}_{f(t)} \frac{(x-n-1)!}{x!} \quad \iff \quad \begin{bmatrix} n+1 \\ k \end{bmatrix}_{f(t)} = \frac{(n+1)!}{(k-1)!} \widetilde{\sigma}_{f(t),n+1-k}(n+1). \end{aligned} \tag{13}$$

Proposition 8 (Recurrence Relations). *For integers $x, n, x-n \geq 1$, the analogs of the Stirling convolution polynomial sequences defined by (13) each satisfy a respective recurrence relation stated in the next equations.*

$$\begin{aligned} f(x+1)\sigma_{f(t),n}(x+1) &= (x-n)\sigma_{f(t),n}(x) + f(x)\ t^{-x} \cdot \sigma_{f(t),n-1}(x) + [n=0]_\delta \\ (x+1)\widetilde{\sigma}_{f(t),n}(x+1) &= (x-n)\widetilde{\sigma}_{f(t),n}(x) + f(x)\ t^{-x} \cdot \widetilde{\sigma}_{f(t),n-1}(x) + [n=0]_\delta \end{aligned} \tag{14}$$

Proof. We give a proof of the second identity since the first recurrence follows almost immediately from this result. Let $x, n, x-n \geq 1$ and consider the expansion of the left-hand-side of (14) according to Definition 7 as follows:

$$\begin{aligned}
(x+1)\widetilde{\sigma}_{f(t),n}(x+1) &= \begin{bmatrix} x+1 \\ x+1-n \end{bmatrix}_{f(t)} \frac{(x-n)!}{x!} \\
&= \left(f(x)t^{-x}\begin{bmatrix} x \\ x+1-n \end{bmatrix}_{f(t)} + \begin{bmatrix} x \\ x-n \end{bmatrix}_{f(t)}\right)(x-n)\cdot\frac{(x-n-1)!}{x!} \\
&= (x-n)\widetilde{\sigma}_{f(t),n}(x) + f(x)t^{-x}\cdot\widetilde{\sigma}_{f(t),n-1}(x).
\end{aligned}$$

For any non-negative integer x, when $n=0$, we see that $\left[\begin{smallmatrix} x+1 \\ x+1 \end{smallmatrix}\right]_{f(t)} \equiv 1$, which implies the result. □

Remark 9 (A comparison of polynomial generating functions). The generating functions for the Stirling convolution polynomials, $\sigma_n(x)$, and the α-factorial polynomials, $\sigma_n^{(\alpha)}(x)$, from [14] each have the comparatively simple special case closed-form generating functions given by

$$\begin{aligned}
x\sigma_n(x) &= \begin{bmatrix} x \\ x-n \end{bmatrix}\frac{(x-n-1)!}{(x-1)!} = [z^n]\left(\frac{ze^z}{e^z-1}\right)^x && \text{for } (f(n),t)\equiv(n,1) \qquad (15)\\
x\sigma_n^{(\alpha)}(x) &= \begin{bmatrix} x \\ x-n \end{bmatrix}_{\alpha}\frac{(x-n-1)!}{(x-1)!} = [z^n]e^{(1-\alpha)z}\left(\frac{\alpha ze^{\alpha z}}{e^{\alpha z}-1}\right)^x && \text{for } (f(n),t)\equiv(\alpha n+1-\alpha,1)\\
x\sigma_n^{(\alpha;\beta)}(x) &= \begin{bmatrix} x \\ x-n \end{bmatrix}_{(\alpha;\beta)}\frac{(x-n-1)!}{(x-1)!} = [z^n]e^{\beta z}\left(\frac{\alpha ze^{\alpha z}}{e^{\alpha z}-1}\right)^x && \text{for } (f(n),t)\equiv(\alpha n+\beta,1).
\end{aligned}$$

The Stirling polynomial sequence in (15) is a special case of a more general class of *convolution polynomial* sequences defined by Knuth in his article [7].

These polynomial sequences are defined by a general sequence of coefficients, s_n^* with $s_0^*=1$, such that the corresponding polynomials, $s_n(x)$, are enumerated by the power series over the original sequence as

$$\sum_{n=0}^{\infty} s_n(x)z^n := S(z)^x \equiv \left(1+\sum_{n=1}^{\infty} s_n^* z^n\right)^x.$$

Polynomial sequences of this form satisfy a number of interesting properties, and in particular, the next identity provides a generating function for a variant of the original convolution polynomial sequence over n when $t \in \mathbb{C}$ is fixed.

$$\mathcal{S}_t(z) := S\left(z\mathcal{S}_t(z)^t\right) \quad\Longrightarrow\quad \frac{xs_n(x+tn)}{(x+tn)} = [z^n]\mathcal{S}_t(z)^x \qquad (16)$$

This result is also useful in expanding many identities for the $t:=1$ case as given for the Stirling polynomial case in [5, §6.2] [7]. A related generalized class of polynomial sequences is considered in Roman's book defining the form of *Sheffer polynomial* sequences. The polynomial sequences of this particular type, say with sequence terms given by $s_n(x)$, satisfy the form in the following generating function identity where $A(z)$ and $B(z)$ are prescribed power series satisfying the initial conditions from the reference [13, *cf.* §2.3]:

$$\sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!} := A(z) e^{xB(z)}.$$

For example, the form of the generalized, or higher-order Bernoulli polynomials (numbers) is a parameterized sequence whose generating function yields the form of many other special case sequences, including the Stirling polynomial case defined in equation (15) [13, *cf.* §4.2.2] [14, *cf.* §5].

3.2.1 An experimental procedure towards evaluating the generalized polynomials

We expect that the generalized convolution polynomial analogs defined in (13) above form a sequence of finite-degree polynomials in x, for example, as in the Stirling polynomial case when we have that

$$\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k \geq 0} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$$

where $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ denotes the special triangle of *second-order Eulerian numbers* for $n, k \geq 0$ and where the binomial coefficient terms in the previous equations each have a finite-degree polynomial expansion in x [5, §6.2]. The previous identity also allows us to extend the Stirling numbers of the first kind to *arbitrary* real, or complex-valued inputs.

Given the relatively simple and elegant forms of the generating functions that enumerate the polynomial sequences of the special case forms in (15), it seems natural to attempt to extend these relations to the generalized polynomial sequence forms defined by (13). However, in this more general context we appear to have a stronger dependence of the form and ordinary generating functions of these polynomial sequences on the underlying function f. Specifically, for the form of the first sequence in (13), we suppose that the function $f(n)$ is arbitrary.

Based on the first several cases of these polynomials, it appears that the generating function for the sequence can be expanded as

$$f_n(x) := [z^n] F(z)^x \quad \text{where} \quad F(z) := \sum_{n=0}^{\infty} g_n(x) z^n \tag{17}$$
$$\implies g_n(x) = \frac{\sum_{j=0}^{n-1} f(x)^n \operatorname{num}_n(j;\ x) x^{n-1-j} (1+x)^j f(x+1)^j}{n!\ t^{nx}\ \sum_{j=0}^{2n-1} \operatorname{denom}_n(j;\ x) x^{2n-1-j} (1+x)^j f(x+1)^j}\ [n \geq 1]_\delta + [n = 0]_\delta$$

where the forms $\mathrm{num}_n(j;\ x)$ and $\mathrm{denom}_n(j;\ x)$ denote polynomial sequences of finite non-negative integral degree indexed over the natural numbers $n, j \geq 0$. Similarly it has been verified for the first 16 of each n and k that the following equation holds where the terms $g_n(x)$ involved in the series for $F(z)$ are defined through the form of the last equation.

$$s_n(k) := f_{n-k}(n) \implies s_n(k) = [z^n]z^k F(z)^n = \sum_{j=1}^{n-k} \binom{n}{j} [z^{n-k}](F(z)-1)^j + [n=k]_\delta$$

Note that the coefficients defined through these implicit power series forms must also satisfy an implicit relation to the particular values of the polynomial parameter x as formed through the last equations, which is much different in construction than in the cases of the special polynomial sequence generating functions remarked on above. Other different expansions may result for special cases of the function $f(n)$ and explicit values of the parameter t.

4 Conclusions and future research

4.1 Summary

We have defined a generalized class of factorial product functions, $(x)_{f(t),n}$, that generalizes the forms of many special and symbolic factorial functions considered in the references. The coefficient-wise symbolic polynomial expansions of these f-factorial function variants define generalized triangles of Stirling numbers of the first kind which share many analogs of the combinatorial properties satisfied by the ordinary combinatorial triangle cases. Surprisingly, many inversion relations and other finite sum properties relating the ordinary Stirling number triangles are not apparent by inspection of these corresponding sums in the most general cases. A study of ordinary Stirling-number-like sums, inversion relations, and generating function transformations is not contained in this article. We pose formulating these analogs in the most general coefficient cases as a topic for future combinatorial work with the generalized Stirling number triangles defined in Section 1.2.

4.2 Topics suggested for future research

Another new avenue to explore with these sums and the generalized f-zeta series transformations motivated in [18, 17] is to consider finding new identities and expressions for the Euler-like sums suggested by the generalized identity in Proposition 4 and by the special case expansions for the Stirling numbers of the first kind given in Example 5. In particular, if we define a class of so-termed "*f-zeta*" functions, $\zeta_f(s) := \sum_{n\geq 1} f(n)^{-s}$, we seek analogs of these infinite Euler sum variants expanded through $\zeta_f(s)$ just as the Euler sums are expressed through sums and products of the *Riemann zeta function*, $\zeta(s)$, in the ordinary cases from [1].

For example, it is well known that for real-valued $r > 1$

$$\sum_{n\geq 1} \frac{H_n^{(r)}}{n^r} = \frac{1}{2}\left(\zeta(r)^2 + \zeta(2r)\right),$$

and moreover, summation by parts shows us that for any real $r > 1$ and any $t \in \mathbb{C}^*$ such that we have a convergent limiting zeta function series we have that

$$\begin{aligned}\sum_{n\geq 1} \frac{F_n^{(r)}(t^r)t^{rn}}{f(n)^r} &= \lim_{n\longrightarrow\infty} \left\{\left(F_n^{(r)}(t^r)\right)^2 - \sum_{0\leq j<n} \frac{F_j^{(r)}(t^r)t^{r(j+1)}}{f(j+1)^r}\right\} \\ &= \lim_{n\longrightarrow\infty} \left\{\left(F_n^{(r)}(t^r)\right)^2 - \sum_{0\leq j<n} \frac{F_{j+1}^{(r)}(t^r)t^{r(j+1)}}{f(j+1)^r} + \sum_{0\leq j<n} \frac{t^{2r(j+1)}}{f(j+1)^{2r}}\right\},\end{aligned}$$

which similarly implies that

$$\sum_{n\geq 1} \frac{F_n^{(r)}(1)}{f(n)^r} \quad\longmapsto\quad \frac{1}{2}\left(\zeta_f(r)^2 + \zeta_f(2r)\right).$$

Additionally, we seek other analogs of known identities for the infinite Euler-like-sum variants over the weighted f-harmonic number sums of the form

$$H_f\left(\varpi_1, \ldots, \varpi_k; s, t, z\right) := \sum_{n\geq 1} \frac{F_n^{(\varpi_1)}\left(t^{\varpi_1}\right)\cdots F_n^{(\varpi_k)}\left(t^{\varpi_k}\right)z^{sn}}{f(n)^s},$$

when $t = \pm 1$, or more generally for any fixed $t \in \mathbb{C}^*$, and where the right-hand-side series in the previous equation converges, say for $|z| \leq 1$.

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