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Explicit Formulas for the p-adic Valuations of Fibonomial Coefficients

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Abstract

We obtain explicit formulas for the p-adic valuations of Fibonomial coefficients which extend some results in the literature.

1 Introduction

The *Fibonacci sequence* $(F_n)_{n\geq 1}$ is given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial values $F_1 = F_2 = 1$. For each $m \geq 1$ and $1 \leq k \leq m$, the *Fibonomial coefficients* $\binom{m}{k}_F$ are defined by

$$\binom{m}{k}_F = \frac{F_1F_2F_3\cdots F_m}{(F_1F_2F_3\cdots F_k)(F_1F_2F_3\cdots F_{m-k})} = \frac{F_{m-k+1}F_{m-k+2}\cdots F_m}{F_1F_2F_3\cdots F_k},$$

where F_n is the nth Fibonacci number. If $k = 0$, we define $\binom{m}{k}_F = 1$ and if $k > m$, we define $\binom{m}{k}_F = 0$. It is well known that $\binom{m}{k}_F$ is an integer for all positive integers m and k. So it

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is natural to consider the divisibility properties and the p-adic valuation of $\binom{m}{k}_F$. As usual, p always denotes a prime and the *p-adic valuation (or p-adic order)* of a positive integer n, denoted by $\nu_p(n)$, is the exponent of p in the prime factorization of n. In addition, the *order (or the rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer k such that $n \mid F_k$. The Fibonacci sequence and the triangle of Fibonomial coefficients are, respectively, A000045 and A010048 in OEIS [25]. Also see A055870 and A003267 for signed Fibonomial triangle and central Fibonomial coefficients, respectively.

In 1989, Knuth and Wilf [8] gave a short description of the p-adic valuation of $\binom{m}{k}_C$ where C is a regularly divisible sequence. However, this does not give explicit formulas for $\binom{m}{k}_F$. Then recently, there has been some interest in explicitly evaluating the p-adic valuation of Fibonomial coefficients of the form $\binom{p^b}{p^a}_F$. For example, Marques and Trojovský [10, 11] and Marques, Sellers, and Trojovský [12] deal with the case $b = a + 1$, $a \geq 1$. Ballot [2, Theorem 4.2] extends the Kummer-like theorem of Knuth and Wilf [8, Theorem 2], which gives the p-adic valuation of Fibonomials, to all Lucasnomials, and, in particular, uses it to determine explicitly the p-adic valuation of Lucasnomials of the form $\binom{p^b}{p^a}_U$, for all nondegenerate fundamental Lucas sequences U and all integers $b > a \geq 0$, [2, Theorem 7.1].

Note that in the formula given by Marques and Trojovský [11, Theorem 1] for $U = F$ and $b = a + 1$, only the case of a even is actually explicitly computed. It appears, using the theorem of Ballot [1, Theorem 7.1], that their stated result for a odd is correct only for primes p for which p^2 does not divide $F_{z(p)}$, where $z(p)$ is the rank of appearance of p in the Fibonacci sequence. Also see Examples 16 and 18 in this article.

Our purpose is to extend Ballot's theorem, Theorem 7.1, in the case $U = F$ and $b \geq a > 0$ and obtain explicit formulas for $\binom{\ell_1 p^b}{\ell_2 p^a}_F$, where ℓ_1 and ℓ_2 are arbitrary positive integers such that $\ell_1 p^b > \ell_2 p^a$. This leads us to study the p-adic valuations of integers of the forms

$$\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \quad \text{or} \quad \left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor!, \tag{1}$$

where $p \equiv \pm 1 \pmod{m}$. For instance, we obtain in Example 17 the following result: for positive integers a, b, ℓ with $b \geq a$, and a prime p distinct from 2 and 5, if $p \equiv \pm 1 \pmod 5$, then

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \begin{cases} b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } z(p) \mid \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $p \equiv \pm 2 \pmod 5$, then

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \begin{cases} 0, & \text{if } \ell \equiv 1 - 2\varepsilon \pmod{z(p)}; \\ b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is odd}, \end{cases}$$

where $\varepsilon = 1$ if a and b have different parity and $\varepsilon = 0$ otherwise. We also obtain the corresponding results for $p \in \{2, 5\}$ in Examples 15 and 19. These extend all the main results in [10, 11, 12] and Ballot's theorem, Theorem 7.1, in the case $U = F$.

Recall that for each $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\}$ is the *fractional part* of x given by $\{x\} = x - \lfloor x \rfloor$, and $\lceil x \rceil$ is the smallest integer larger than or equal to x. In addition, we write $a \bmod m$ to denote the least nonnegative residue of a modulo m. We also use the *Iverson notation*: if P is a mathematical statement, then

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

For example, $[5 \equiv -1 \pmod 4] = 0$ and $[3 \equiv -1 \pmod 4] = 1$.

We organize this article as follows. In Section 2, we give some preliminaries and useful results which are needed in the proof of the main theorems. In Section 3, we give exact formulas for the p-adic valuations of integers (1). In Section 4, we apply the results obtained in Section 3 to Fibonomial coefficients. Our most general theorem is Theorem 13. Finally, in Section 5, we give the p-adic valuations of some specific sub-families of Fibonomial coefficients of type (1), since generally, the more specific the family, the shortest the formula becomes.

For more information related to Fibonacci numbers, we invite the readers to visit the second author's Researchgate account [23] which contains some freely downloadable versions of his publications [5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

2 Preliminaries and lemmas

Recall that for each odd prime p and $a \in \mathbb{Z}$, the *Legendre symbol* $(\frac{a}{p})$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a; \\ 1, & \text{if } a \text{ is a quadratic residue of } p; \\ -1, & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Then we have the following result.

Lemma 1. *Let $p \neq 5$ be a prime and let m and n be positive integers. Then the following statements hold.*

(i) *If $p > 2$, then $F_{p-(\frac{5}{p})} \equiv 0 \pmod p$.*

(ii) *$n \mid F_m$ if and only if $z(n) \mid m$.*

(iii) *$z(p) \mid p+1$ if and only if $p \equiv 2$ or $-2 \pmod 5$, and $z(p) \mid p-1$ otherwise.*

(iv) $\gcd(z(p), p) = 1$.

Proof. These are well known results. For example, (i) and (ii) can be found in [4, p. 410] and [26], respectively. Then (iii) follows from (i) and (ii). By (iii), $z(p) \mid p \pm 1$. Since $\gcd(p, p \pm 1) = 1$, we obtain $\gcd(z(p), p) = 1$. This proves (iv). $\square$

Lengyel's result and Legendre's formula given in the following lemmas are important tools in evaluating the p-adic valuation of Fibonomial coefficients. We also refer the reader to [10, 11, 12, 15] for other similar applications of Lengyel's result.

Lemma 2. (Lengyel [9]) *For $n \geq 1$, we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, *and if p is a prime distinct from 2 and 5, then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}: \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}, \end{cases}$$

Lemma 3. (Legendre's formula) *Let n be a positive integer and let p be a prime. Then*

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

In the proof of the main results, we will deal with a lot of calculation involving the floor function. So it is useful to recall the following results.

Lemma 4. *For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, the following holds*

(i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$,

(ii) $\{n + x\} = \{x\}$,

(iii) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$

(iv) $\{-x\} = \begin{cases} 1 - \{x\}, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$

(v) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \geq 1, \end{cases}$

(vi) $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$ *for $n \geq 1$.*

Proof. These are well-known results and can be proved easily. For more details, see in [1, Exercise 13, p. 72] or in [3, Chapter 3]. We also refer the reader to [14] for a nice application of (v). $\square$

The next lemma is used often in counting the number of positive integers $n \le x$ lying in a residue class a mod q, see for instance in [24, Proof of Lemma 2.6].

Lemma 5. *For $x \in [1, \infty)$, a, $q \in \mathbb{Z}$ and $q \ge 1$, we have*

$$\sum_{\substack{1 \le n \le x \\ n \equiv a \pmod{q}}} 1 = \left\lfloor \frac{x-a}{q} \right\rfloor - \left\lfloor -\frac{a}{q} \right\rfloor. \tag{2}$$

Proof. Replacing a by $a+q$ and applying Lemma 4, we see that the value on the right-hand side of (2) is not changed. Obviously, the left-hand side is also invariant when we replace a by $a+q$. So it is enough to consider only the case $1 \le a \le q$. Since $n \equiv a \pmod{q}$, we write $n = a + kq$ where $k \ge 0$ and $a + kq \le x$. So $k \le \frac{x-a}{q}$. Therefore

$$\sum_{\substack{1 \le n \le x \\ n \equiv a \pmod{q}}} 1 = \sum_{0 \le k \le \frac{x-a}{q}} 1 = \left\lfloor \frac{x-a}{q} \right\rfloor + 1 = \left\lfloor \frac{x-a}{q} \right\rfloor - \left\lfloor -\frac{a}{q} \right\rfloor.$$

$\square$

It is convenient to use the Iverson notation and to denote the least nonnegative residue of a modulo m by $a \bmod m$. Therefore we will do so from this point on.

Lemma 6. *Let n and k be integers, m a positive integer, $r = n \bmod m$, and $s = k \bmod m$. Then*

$$\left\lfloor \frac{n-k}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor - [r < s].$$

Proof. By Lemma 4(i) and the fact that $0 \le r < m$, we obtain

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{n-r}{m} + \frac{r}{m} \right\rfloor = \frac{n-r}{m} + \left\lfloor \frac{r}{m} \right\rfloor = \frac{n-r}{m}.$$

Similarly, $\left\lfloor \frac{k}{m} \right\rfloor = \frac{k-s}{m}$. Therefore $\left\lfloor \frac{n-k}{m} \right\rfloor$ is equal to

$$\left\lfloor \frac{n-r}{m} - \frac{k-s}{m} + \frac{r-s}{m} \right\rfloor = \frac{n-r}{m} - \frac{k-s}{m} + \left\lfloor \frac{r-s}{m} \right\rfloor = \begin{cases} \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, & \text{if } r \ge s; \\ \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor - 1, & \text{if } r < s. \end{cases}$$

$\square$

3 The p-adic valuation of integers in special forms

In this section, we calculate the p-adic valuation of $\left\lfloor \frac{\ell p^a}{m} \right\rfloor!$ and other integers in similar forms.

Theorem 7. *Let p be a prime and let $a \geq 0$, $\ell \geq 0$, and $m \geq 1$ be integers. Assume that $p \equiv \pm 1 \pmod{m}$ and let $\delta = [\ell \not\equiv 0 \pmod{m}]$. Then*

$$\nu_p\left(\left\lfloor \frac{\ell p^a}{m} \right\rfloor!\right) = \begin{cases} \frac{\ell(p^a-1)}{m(p-1)} - a\left\{\frac{\ell}{m}\right\} + \nu_p\left(\left\lfloor \frac{\ell}{m}\right\rfloor!\right), & \text{if } p \equiv 1 \pmod{m}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a}{2}\delta + \nu_p\left(\left\lfloor \frac{\ell}{m}\right\rfloor!\right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is even;} \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a-1}{2}\delta - \left\{\frac{\ell}{m}\right\} + \nu_p\left(\left\lfloor \frac{\ell}{m}\right\rfloor!\right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is odd.} \end{cases}$$

We remark that if $m = 1$ or 2, then the expressions in each case of this theorem are all equal.

Proof. The result is easily verified when $a = 0$ or $\ell = 0$. So we assume throughout that $a \geq 1$ and $\ell \geq 1$. We also use Lemmas 4(i), 4(vi), and 5 repeatedly without reference. By Legendre's formula, we obtain

$$\nu_p\left(\left\lfloor \frac{\ell p^a}{m} \right\rfloor!\right) = \sum_{j=1}^{\infty} \left\lfloor \frac{\ell p^a}{mp^j} \right\rfloor = \sum_{j=1}^{a} \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor + \sum_{j=a+1}^{\infty} \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor = \sum_{j=1}^{a} \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor + \nu_p\left(\left\lfloor \frac{\ell}{m} \right\rfloor!\right). \tag{3}$$

From (3), it is immediate that for $m = 1$, we obtain

$$\nu_p\left((\ell p^a)!\right) = \frac{\ell(p^a-1)}{p-1} + \nu_p(\ell!).$$

So we assume throughout that $m \geq 2$.

Case 1. $p \equiv 1 \pmod{m}$. Then, for every $k \geq 0$, $p^k \equiv 1 \pmod{m}$ and

$$\left\lfloor \frac{\ell p^k}{m} \right\rfloor = \left\lfloor \frac{\ell(p^k-1)}{m} + \frac{\ell}{m} \right\rfloor = \frac{\ell(p^k-1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor.$$

Therefore the sum $\sum_{j=1}^{a} \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor$ appearing in (3) is equal to

$$\sum_{j=1}^{a} \left(\frac{\ell(p^{a-j}-1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor\right) = \left(\frac{\ell}{m}\sum_{j=1}^{a} p^{a-j}\right) - a\left(\frac{\ell}{m} - \left\lfloor \frac{\ell}{m} \right\rfloor\right) = \frac{\ell(p^a-1)}{m(p-1)} - a\left\{\frac{\ell}{m}\right\}.$$

Case 2. $p \equiv -1 \pmod{m}$. Then for $k \geq 0$, we have $p^k \equiv 1 \pmod{m}$ if k is even, and $p^k \equiv -1 \pmod{m}$ if k is odd. Therefore

$$\left\lfloor \frac{\ell p^k}{m} \right\rfloor = \left\lfloor \frac{\ell(p^k-1)}{m} + \frac{\ell}{m} \right\rfloor = \frac{\ell(p^k-1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor \qquad \text{if } k \geq 0 \text{ and } k \text{ is even,}$$

$$\left\lfloor \frac{\ell p^k}{m} \right\rfloor = \left\lfloor \frac{\ell(p^k+1)}{m} - \frac{\ell}{m} \right\rfloor = \frac{\ell(p^k+1)}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor \qquad \text{if } k \geq 0 \text{ and } k \text{ is odd.}$$

Therefore the sum $\sum_{j=1}^{a}\left\lfloor\frac{\ell p^{a-j}}{m}\right\rfloor$ appearing in (3) is equal to

$$\begin{aligned}
&\sum_{\substack{1\le j\le a\\ a-j\equiv 0 \pmod 2}}\left(\frac{\ell(p^{a-j}-1)}{m}+\left\lfloor\frac{\ell}{m}\right\rfloor\right)+\sum_{\substack{1\le j\le a\\ a-j\equiv 1 \pmod 2}}\left(\frac{\ell(p^{a-j}+1)}{m}+\left\lfloor-\frac{\ell}{m}\right\rfloor\right)\\
&=\frac{\ell}{m}\sum_{1\le j\le a}p^{a-j}-\sum_{\substack{1\le j\le a\\ j\equiv a \pmod 2}}\left(\frac{\ell}{m}-\left\lfloor\frac{\ell}{m}\right\rfloor\right)+\sum_{\substack{1\le j\le a\\ j\equiv a-1 \pmod 2}}\left(\frac{\ell}{m}+\left\lfloor-\frac{\ell}{m}\right\rfloor\right)\\
&=\frac{\ell(p^a-1)}{m(p-1)}+\left\lfloor-\frac{a}{2}\right\rfloor\left(\frac{\ell}{m}-\left\lfloor\frac{\ell}{m}\right\rfloor\right)-\left\lfloor-\frac{a-1}{2}\right\rfloor\left(\frac{\ell}{m}+\left\lfloor-\frac{\ell}{m}\right\rfloor\right). \qquad (4)
\end{aligned}$$

By Lemma 4(iii), we see that

$$\left\lfloor\frac{\ell}{m}\right\rfloor+\left\lfloor-\frac{\ell}{m}\right\rfloor=-[\ell\not\equiv 0 \pmod m]=-\delta.$$

Therefore if a is even, then (4) is equal to

$$\begin{aligned}
&\frac{\ell(p^a-1)}{m(p-1)}-\frac{a}{2}\left(\frac{\ell}{m}-\left\lfloor\frac{\ell}{m}\right\rfloor\right)+\frac{a}{2}\left(\frac{\ell}{m}+\left\lfloor-\frac{\ell}{m}\right\rfloor\right)\\
&\quad=\frac{\ell(p^a-1)}{m(p-1)}+\frac{a}{2}\left(\left\lfloor\frac{\ell}{m}\right\rfloor+\left\lfloor-\frac{\ell}{m}\right\rfloor\right)=\frac{\ell(p^a-1)}{m(p-1)}-\frac{a}{2}\delta,
\end{aligned}$$

and if a is odd, then (4) is equal to

$$\begin{aligned}
&\frac{\ell(p^a-1)}{m(p-1)}-\frac{a+1}{2}\left(\frac{\ell}{m}-\left\lfloor\frac{\ell}{m}\right\rfloor\right)+\frac{a-1}{2}\left(\frac{\ell}{m}+\left\lfloor-\frac{\ell}{m}\right\rfloor\right)\\
&=\frac{\ell(p^a-1)}{m(p-1)}+\frac{a-1}{2}\left(\left\lfloor\frac{\ell}{m}\right\rfloor+\left\lfloor-\frac{\ell}{m}\right\rfloor\right)+\left\lfloor\frac{\ell}{m}\right\rfloor-\frac{\ell}{m}\\
&=\frac{\ell(p^a-1)}{m(p-1)}-\left(\frac{a-1}{2}\right)\delta-\left\{\frac{\ell}{m}\right\}.
\end{aligned}$$

This completes the proof. $\square$

We can combine every case in Theorem 7 into a single form as given in the next corollary.

Corollary 8. *Assume that p, a, ℓ, m, and δ satisfy the same assumptions as in Theorem 7. Then the p-adic valuation of $\left\lfloor\frac{\ell p^a}{m}\right\rfloor!$ is*

$$\begin{aligned}
&\frac{\ell(p^a-1)}{m(p-1)}-\left\lfloor\frac{a}{2}\right\rfloor\delta-\left\{\frac{\ell}{m}\right\}[a\equiv 1 \pmod 2]\\
&+\delta\left\lfloor\frac{a}{2}\right\rfloor\left(1-2\left\{\frac{\ell}{m}\right\}\right)[p\equiv 1 \pmod m]+\nu_p\left(\left\lfloor\frac{\ell}{m}\right\rfloor!\right). \qquad (5)
\end{aligned}$$

Proof. This is merely a combination of each case from Theorem 7. For example, when $p \equiv -1 \pmod{m}$, the right-hand side of (5) reduces to

$$\begin{aligned}
&\frac{\ell(p^a-1)}{m(p-1)} - \left\lfloor \frac{a}{2} \right\rfloor \delta - \left\{ \frac{\ell}{m} \right\} [a \equiv 1 \pmod{2}] + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor ! \right) \\
&= \begin{cases} \frac{\ell(p^a-1)}{m(p-1)} - \frac{a}{2}\delta + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor ! \right), & \text{if } a \text{ is even;} \\ \frac{\ell(p^a-1)}{m(p-1)} - \left(\frac{a-1}{2} \right) \delta - \left\{ \frac{\ell}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor ! \right), & \text{if } a \text{ is odd,} \end{cases}
\end{aligned}$$

which is the same as Theorem 7. The other cases are similar. We leave the details to the reader. ☐

Next we deal with the p-adic valuation of an integer of the form $\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor !$ where a, b, ℓ_1, ℓ_2, and m are positive integers. It is natural to assume $\ell_1 p^b - \ell_2 p^a > 0$. In addition, if $a = b$, then the above expression is reduced to $\left\lfloor \frac{(\ell_1 - \ell_2) p^b}{m} \right\rfloor !$, which can be evaluated by using Theorem 7. We consider the case $b \geq a$ in Theorem 9 and the other case in Theorem 10.

Theorem 9. *Let p be a prime, let a be a nonnegative integer, and let b, m, ℓ_1, ℓ_2 be positive integers satisfying $b \geq a$ and $\ell_1 p^b - \ell_2 p^a > 0$. Assume that $p \equiv \pm 1 \pmod{m}$. Then the following statements hold.*

(i) *If $p \equiv 1 \pmod{m}$, then*

$$\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor ! \right) = \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - a \left\{ \frac{\ell_1 - \ell_2}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor ! \right).$$

(ii) *If $p \equiv -1 \pmod{m}$ and $a \equiv b \pmod{2}$, then*

$$\begin{aligned}
\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor ! \right) &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ \frac{\ell_1 - \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] \\
&\quad - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor ! \right).
\end{aligned}$$

(iii) *If $p \equiv -1 \pmod{m}$ and $a \not\equiv b \pmod{2}$, then*

$$\begin{aligned}
\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor ! \right) &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ -\frac{\ell_1 + \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] \\
&\quad - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor ! \right).
\end{aligned}$$

We remark that if $m = 1$, the expressions in each case of this theorem are equal.

Proof. The result is easily checked when $a = 0$, and as discussed above, if $b = a$, then the result can be verified using Theorem 7. So we assume throughout that $a \geq 1$ and $b > a$. Similar to the proof of Theorem 7, we use Lemmas 4(i), 4(vi), and 5 repeatedly without reference. Then, as for (3), we obtain

$$\begin{aligned}
\nu_p\left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor !\right) &= \sum_{j=1}^{a} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor + \sum_{j=a+1}^{\infty} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor \\
&= \sum_{j=1}^{a} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor + \nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor !\right). \qquad (6)
\end{aligned}$$

We see that when $m = 1$, (6) becomes

$$\nu_p\left((\ell_1 p^b - \ell_2 p^a)!\right) = \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{p-1} + \nu_p\left((\ell_1 p^{b-a} - \ell_2)!\right).$$

So assume throughout that $m \geq 2$. We begin with the proof of (i). Suppose that $p \equiv 1 \pmod{m}$. For each $1 \leq j \leq a$, we have

$$\begin{aligned}
\left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor &= \left\lfloor \frac{\ell_1 p^{b-j} - \ell_1}{m} - \frac{\ell_2 p^{a-j} - \ell_2}{m} + \frac{\ell_1 - \ell_2}{m} \right\rfloor \\
&= \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{\ell_1 - \ell_2}{m} + \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor.
\end{aligned}$$

Then the sum $\sum_{j=1}^{a} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor$ appearing in (6) is equal to

$$\begin{aligned}
&\frac{\ell_1}{m} \sum_{1 \leq j \leq a} p^{b-j} - \frac{\ell_2}{m} \sum_{1 \leq j \leq a} p^{a-j} - a\left(\frac{\ell_1 - \ell_2}{m} - \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor\right) \\
&= \frac{\ell_1}{m}\left(\frac{p^{b-a}(p^a - 1)}{p-1}\right) - \frac{\ell_2}{m}\left(\frac{p^a - 1}{p-1}\right) - a\left\{\frac{\ell_1 - \ell_2}{m}\right\} \\
&= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - a\left\{\frac{\ell_1 - \ell_2}{m}\right\}.
\end{aligned}$$

This proves (i). So from this point on, we assume that $p \equiv -1 \pmod{m}$. For each $1 \leq j \leq a$, we have

$$\begin{aligned}
\left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor &= \left\lfloor \frac{\ell_1 p^{b-j} - (-1)^{b-j}\ell_1}{m} - \frac{\ell_2 p^{a-j} - (-1)^{a-j}\ell_2}{m} + \frac{(-1)^{b-j}\ell_1 - (-1)^{a-j}\ell_2}{m} \right\rfloor \\
&= \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{(-1)^{b-j}\ell_1 - (-1)^{a-j}\ell_2}{m} + \left\lfloor \frac{(-1)^{b-j}\ell_1 - (-1)^{a-j}\ell_2}{m} \right\rfloor \\
&= \begin{cases} \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{(-1)^{b-j}(\ell_1 - \ell_2)}{m} + \left\lfloor \frac{(-1)^{b-j}(\ell_1 - \ell_2)}{m} \right\rfloor, & \text{if } a \equiv b \pmod{2}; \\ \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} + \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor, & \text{if } a \not\equiv b \pmod{2}. \end{cases}
\end{aligned}$$

Case 1. $a \equiv b \pmod 2$. Then the sum $\sum_{j=1}^{a} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor$ appearing in (6) is equal to

$$\begin{aligned}
&\frac{\ell_1}{m} \sum_{1\le j\le a} p^{b-j} - \frac{\ell_2}{m} \sum_{1\le j\le a} p^{a-j} - \left(\frac{\ell_1-\ell_2}{m}\right) \sum_{1\le j\le a} (-1)^{b-j} + \sum_{1\le j\le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1-\ell_2)}{m} \right\rfloor \\
&= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left(\frac{\ell_1-\ell_2}{m}\right) \sum_{1\le j\le a} (-1)^{b-j} + \sum_{1\le j\le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1-\ell_2)}{m} \right\rfloor. \qquad (7)
\end{aligned}$$

Observe that

$$\sum_{1\le j\le a} (-1)^{b-j} = \begin{cases} 0, & \text{if } a \text{ is even;} \\ 1, & \text{if } a \text{ is odd.} \end{cases}$$

So we have

$$\left(\frac{\ell_1-\ell_2}{m}\right) \sum_{1\le j\le a} (-1)^{b-j} = \left(\frac{\ell_1-\ell_2}{m}\right) [a \equiv 1 \pmod 2].$$

It remains to calculate the last term in (7). If $\ell_1 \equiv \ell_2 \pmod m$, then we obtain by Lemma 4(iii) that

$$\begin{aligned}
\sum_{1\le j\le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1-\ell_2)}{m} \right\rfloor &= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \left\lfloor \frac{\ell_1-\ell_2}{m} \right\rfloor, & \text{if } a \text{ is odd;} \end{cases} \\
&= \left\lfloor \frac{\ell_1-\ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2].
\end{aligned}$$

Similarly, if $\ell_1 \not\equiv \ell_2 \pmod m$, then we obtain by Lemma 4(iii) that

$$\begin{aligned}
\sum_{1\le j\le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1-\ell_2)}{m} \right\rfloor &= \begin{cases} -\frac{a}{2}, & \text{if } a \text{ is even;} \\ \left\lfloor \frac{\ell_1-\ell_2}{m} \right\rfloor - \frac{a-1}{2}, & \text{if } a \text{ is odd;} \end{cases} \\
&= \left\lfloor \frac{\ell_1-\ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor.
\end{aligned}$$

In any case,

$$\sum_{1\le j\le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1-\ell_2)}{m} \right\rfloor = \left\lfloor \frac{\ell_1-\ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod m].$$

Therefore (7) is equal to

$$\begin{aligned}
&\frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left(\frac{\ell_1-\ell_2}{m}\right) [a \equiv 1 \pmod 2] + \left\lfloor \frac{\ell_1-\ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2] \\
&- \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod m] \\
&= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{\frac{\ell_1-\ell_2}{m}\right\} [a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod m].
\end{aligned}$$

This proves (ii). Next we prove (iii).

Case 2. $a \not\equiv b \pmod 2$. Similar to Case 1, the sum $\sum_{j=1}^{a} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor$ appearing in (6) is equal to

$$\begin{aligned}
&\frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left(\frac{\ell_1 + \ell_2}{m}\right) \sum_{1 \le j \le a} (-1)^{b-j} + \sum_{1 \le j \le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor \\
&= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} + \left(\frac{\ell_1 + \ell_2}{m}\right) [a \equiv 1 \pmod 2] + \sum_{1 \le j \le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor. \qquad (8)
\end{aligned}$$

If $\ell_1 \equiv -\ell_2 \pmod m$, then we obtain by Lemma 4(iii) that

$$\begin{aligned}
\sum_{1 \le j \le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor &= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor, & \text{if } a \text{ is odd;} \end{cases} \\
&= \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2].
\end{aligned}$$

Similarly, if $\ell_1 \not\equiv -\ell_2 \pmod m$, then we obtain by Lemma 4(iii) that

$$\begin{aligned}
\sum_{1 \le j \le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor &= \begin{cases} -\frac{a}{2}, & \text{if } a \text{ is even;} \\ \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor - \frac{a-1}{2}, & \text{if } a \text{ is odd;} \end{cases} \\
&= \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor.
\end{aligned}$$

In any case, $\sum_{1 \le j \le a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor = \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod m]$. Therefore (8) is equal to

$$\begin{aligned}
&\frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} + \left(\frac{\ell_1 + \ell_2}{m}\right) [a \equiv 1 \pmod 2] + \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod 2] \\
&- \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod m] \\
&= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ -\frac{\ell_1 + \ell_2}{m} \right\} [a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod m].
\end{aligned}$$

This completes the proof. □

Next we replace the assumption $b \ge a$ in Theorem 9 by $b < a$. The calculation follows from the same idea so we skip the details of the proof. Although we do not use it in this article, it may be useful for future reference. So we record it in the next theorem.

Theorem 10. *Let p be a prime, let b be a nonnegative integer, and let a, m, ℓ_1, ℓ_2 be positive integers satisfying $b < a$ and $\ell_1 p^b - \ell_2 p^a > 0$. Assume that $p \equiv \pm 1 \pmod m$. Then the following statements hold.*

(i) *If* $p \equiv 1 \pmod{m}$*, then*

$$\nu_p\left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor !\right) = \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - b\left\{\frac{\ell_1 - \ell_2}{m}\right\} + \nu_p\left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor !\right).$$

(ii) *If* $p \equiv -1 \pmod{m}$ *and* $a \equiv b \pmod{2}$*, then*

$$\begin{aligned}\nu_p\left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor !\right) &= \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - \left\{\frac{\ell_1 - \ell_2}{m}\right\}[b \equiv 1 \pmod{2}] \\ &\quad - \left\lfloor \frac{b}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p\left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor !\right).\end{aligned}$$

(iii) *If* $p \equiv -1 \pmod{m}$ *and* $a \not\equiv b \pmod{2}$*, then*

$$\begin{aligned}\nu_p\left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor !\right) &= \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - \left\{\frac{\ell_1 + \ell_2}{m}\right\}[b \equiv 1 \pmod{2}] \\ &\quad - \left\lfloor \frac{b}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p\left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor !\right).\end{aligned}$$

Proof. We begin by writing $\nu_p\left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor !\right)$ as

$$\sum_{j=1}^{b} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor + \sum_{j=b+1}^{\infty} \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor.$$

The second sum above is $\nu_p\left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor !\right)$. The first sum can be evaluated in the same way as in Theorem 9. We leave the details to the reader. $\square$

When we put more restrictions on the range of ℓ_1 and ℓ_2, the expression $\nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor !\right)$ appearing in Theorems 9 and 10 can be evaluated further. Nevertheless, since we do not need it in our application, we do not give them here. In the future, we plan to put it in the second author's Researchgate account. So the interested reader can find it there.

4 The p-adic valuations of Fibonomial coefficients

Recall that the *binomial coefficients* $\binom{m}{k}$ is defined by

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!}, & \text{if } 0 \le k \le m; \\ 0, & \text{if } k < 0 \text{ or } k > m. \end{cases}$$

A classical result of Kummer states that for $0 \le k \le m$, $\nu_p\left(\binom{m}{k}\right)$ is equal to the number of carries when we add k and $m-k$ in base p. From this, it is not difficult to show that for all primes p and positive integers k, b, a with $b \ge a$, we have

$$\nu_p\left(\binom{p^b}{p^a}\right) = b - a, \quad \text{or more generally,} \quad \nu_p\left(\binom{p^a}{k}\right) = a - \nu_p(k).$$

Knuth and Wilf [8] also obtain the result analogous to that of Kummer for a C-nomial coefficient. However, our purpose is to obtain $\nu_p\left(\binom{m}{k}_F\right)$ is an explicit form. So we first express $\nu_p\left(\binom{m}{k}_F\right)$ in terms of the p-adic valuation of some binomial coefficients in Theorem 11. Then we write it in a form which is easy to use in Corollary 12. Then we apply it to obtain the p-adic valuation of Fibonomial coefficients of the form $\binom{\ell_1 p^b}{\ell_2 p^a}_F$.

Theorem 11. *Let $0 \le k \le m$ be integers. Then the following statements hold.*

(i) *Let $m' = \left\lfloor \frac{m}{6} \right\rfloor$, $k' = \left\lfloor \frac{k}{6} \right\rfloor$, and let $r = m \bmod 6$ and $s = k \bmod 6$ be the least nonnegative residues of m and k modulo 6, respectively. Then*

$$\begin{aligned}\nu_2\left(\binom{m}{k}_F\right) = \nu_2\left(\binom{m'}{k'}\right) + \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor - 3\left\lfloor \frac{r-s}{6} \right\rfloor \\ + [r < s]\nu_2\left(\left\lfloor \frac{m-k+6}{6} \right\rfloor\right).\end{aligned}$$

(ii) $\nu_5\left(\binom{m}{k}_F\right) = \nu_5\left(\binom{m}{k}\right)$.

(iii) *Suppose that p is a prime, $p \ne 2$, and $p \ne 5$. Let $m' = \left\lfloor \frac{m}{z(p)} \right\rfloor$, $k' = \left\lfloor \frac{k}{z(p)} \right\rfloor$, and let $r = m \bmod z(p)$, and $s = k \bmod z(p)$ be the least nonnegative residues of m and k modulo $z(p)$, respectively. Then*

$$\nu_p\left(\binom{m}{k}_F\right) = \nu_p\left(\binom{m'}{k'}\right) + [r < s]\left(\nu_p\left(\left\lfloor \frac{m-k+z(p)}{z(p)} \right\rfloor\right) + \nu_p(F_{z(p)})\right).$$

Proof. We will use Lemmas 4(i) and 5 repeatedly without reference. In addition, it is useful to recall that for every $a, b \in \mathbb{N}$, $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and if $b \mid a$, then $\nu_p(\frac{a}{b}) = \nu_p(a) - \nu_p(b)$. Since the formulas to prove clearly hold when $k = 0$ or m, we assume $m \ge 2$ and $1 \le k < m$.

By Lemma 2, we obtain, for every $\ell \geq 1$,

$$\begin{aligned}
\nu_2(F_1F_2F_3\cdots F_\ell) &= \sum_{\substack{1\le n\le \ell \\ n\equiv 3 \pmod 6}} \nu_2(F_n) + \sum_{\substack{1\le n\le \ell \\ n\equiv 0 \pmod 6}} \nu_2(F_n) \\
&= \sum_{\substack{1\le n\le \ell \\ n\equiv 3 \pmod 6}} 1 + \sum_{\substack{1\le n\le \ell \\ n\equiv 0 \pmod 6}} (\nu_2(n)+2) \\
&= \left\lfloor \frac{\ell+3}{6} \right\rfloor + 2\left\lfloor \frac{\ell}{6} \right\rfloor + \sum_{1\le j\le \frac{\ell}{6}} \nu_2(6j) \\
&= \left\lfloor \frac{\ell+3}{6} \right\rfloor + 3\left\lfloor \frac{\ell}{6} \right\rfloor + \sum_{1\le j\le \frac{\ell}{6}} \nu_2(j) \\
&= \left\lfloor \frac{\ell+3}{6} \right\rfloor + 3\left\lfloor \frac{\ell}{6} \right\rfloor + \nu_2\left(\left\lfloor \frac{\ell}{6} \right\rfloor ! \right). \qquad (9)
\end{aligned}$$

Then we obtain from the definition of $\binom{m}{k}_F$ and from (9) that

$$\begin{aligned}
\nu_2\left(\binom{m}{k}_F\right) &= \nu_2(F_1F_2\cdots F_m) - \nu_2(F_1F_2\cdots F_{m-k}) - \nu_2(F_1F_2\cdots F_k) \\
&= \left(\left\lfloor \frac{m+3}{6} \right\rfloor - \left\lfloor \frac{m-k+3}{6} \right\rfloor - \left\lfloor \frac{k+3}{6} \right\rfloor \right) + 3\left(\left\lfloor \frac{m}{6} \right\rfloor - \left\lfloor \frac{m-k}{6} \right\rfloor - \left\lfloor \frac{k}{6} \right\rfloor \right) \\
&\quad + \nu_2\left(\left\lfloor \frac{m}{6} \right\rfloor ! \right) - \nu_2\left(\left\lfloor \frac{m-k}{6} \right\rfloor ! \right) - \nu_2\left(\left\lfloor \frac{k}{6} \right\rfloor ! \right). \qquad (10)
\end{aligned}$$

The expression in the first parenthesis in (10) is equal to

$$\begin{aligned}
&\left\lfloor \frac{m-r}{6} + \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{(m-r)-(k-s)}{6} + \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{k-s}{6} + \frac{s+3}{6} \right\rfloor \\
&= \frac{m-r}{6} + \left\lfloor \frac{r+3}{6} \right\rfloor - \frac{(m-r)-(k-s)}{6} - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \frac{k-s}{6} - \left\lfloor \frac{s+3}{6} \right\rfloor \\
&= \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor.
\end{aligned}$$

Similarly, the expression in the second parenthesis is

$$3\left(\left\lfloor \frac{r}{6} \right\rfloor - \left\lfloor \frac{r-s}{6} \right\rfloor - \left\lfloor \frac{s}{6} \right\rfloor \right) = -3\left\lfloor \frac{r-s}{6} \right\rfloor.$$

Therefore (10) becomes

$$\nu_2\left(\binom{m}{k}_F\right) = \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor - 3\left\lfloor \frac{r-s}{6} \right\rfloor + \nu_2\left(\frac{\lfloor x+y \rfloor !}{\lfloor x \rfloor ! \lfloor y \rfloor !} \right) \qquad (11)$$

where $x = \frac{m-k}{6}$ and $y = \frac{k}{6}$. By Lemma 4(v), we see that

$$\begin{aligned}
\frac{\lfloor x+y\rfloor!}{\lfloor x\rfloor!\lfloor y\rfloor!} &= \begin{cases} \binom{\lfloor x+y\rfloor}{\lfloor y\rfloor}, & \text{if } \{x\}+\{y\}<1;\\ \binom{\lfloor x+y\rfloor}{\lfloor y\rfloor}(\lfloor x\rfloor+1), & \text{if } \{x\}+\{y\}\geq 1;\end{cases}\\
&= \begin{cases} \binom{m'}{k'}, & \text{if } \{x\}+\{y\}<1;\\ \binom{m'}{k'}\left(\left\lfloor\frac{m-k+6}{6}\right\rfloor\right), & \text{if } \{x\}+\{y\}\geq 1.\end{cases}
\end{aligned}$$

By Lemma 4(ii), we obtain

$$\{x\} = \left\{\frac{(m-r)-(k-s)}{6}+\frac{r-s}{6}\right\} = \left\{\frac{r-s}{6}\right\} \text{ and } \{y\} = \left\{\frac{k-s}{6}+\frac{s}{6}\right\} = \frac{s}{6}.$$

If $r \geq s$, then $\{x\}+\{y\} = \left\{\frac{r-s}{6}\right\}+\frac{s}{6} = \frac{r-s}{6}+\frac{s}{6} = \frac{r}{6} < 1$. If $r < s$, then we obtain by Lemma 4(iv) that $\{x\}+\{y\} = \left\{-\frac{s-r}{6}\right\}+\frac{s}{6} = 1-\frac{s-r}{6}+\frac{s}{6} = 1+\frac{r}{6} \geq 1$. Therefore

$$\frac{\lfloor x+y\rfloor!}{\lfloor x\rfloor!\lfloor y\rfloor!} = \begin{cases} \binom{m'}{k'}, & \text{if } r\geq s;\\ \binom{m'}{k'}\left(\left\lfloor\frac{m-k+6}{6}\right\rfloor\right), & \text{if } r<s.\end{cases} \tag{12}$$

Substituting (12) in (11), we obtain part (i) of this theorem. The calculation in parts (ii) and (iii) are similar, so we give fewer details than given in part (i). By Lemma 2, for every $\ell \geq 1$, we have

$$\nu_5(F_1F_2\cdots F_\ell) = \sum_{1\leq n\leq \ell}\nu_5(F_n) = \sum_{1\leq n\leq\ell}\nu_5(n) = \nu_5(\ell!),$$

which implies

$$\nu_5\left(\binom{m}{k}_F\right) = \nu_5(m!)-\nu_5(k!)-\nu_5((m-k)!) = \nu_5\left(\binom{m}{k}\right).$$

For (iii), we apply Lemmas 2 and 1(iv) to obtain

$$\begin{aligned}
\nu_p(F_1F_2\cdots F_\ell) &= \sum_{\substack{1\leq n\leq\ell\\ n\equiv 0 \pmod{z(p)}}}\nu_p(F_n) = \sum_{\substack{1\leq n\leq\ell\\ n\equiv 0 \pmod{z(p)}}}(\nu_p(n)+\nu_p(F_{z(p)}))\\
&= \sum_{1\leq k\leq\frac{\ell}{z(p)}}\nu_p(kz(p)) + \left\lfloor\frac{\ell}{z(p)}\right\rfloor\nu_p(F_{z(p)})\\
&= \nu_p\left(\left\lfloor\frac{\ell}{z(p)}\right\rfloor!\right)+\left\lfloor\frac{\ell}{z(p)}\right\rfloor\nu_p(F_{z(p)}).
\end{aligned}$$

As in part (i), the above implies that

$$\nu_p\left(\binom{m}{k}_F\right) = \nu_p\left(\frac{\lfloor x+y\rfloor!}{\lfloor x\rfloor!\lfloor y\rfloor!}\right)+(\lfloor x+y\rfloor-\lfloor x\rfloor-\lfloor y\rfloor)\nu_p(F_{z(p)}), \tag{13}$$

where $x = \frac{m-k}{z(p)}$ and $y = \frac{k}{z(p)}$. In addition, if $r \geq s$, then $\{x\} + \{y\} < 1$ and if $r < s$, then $\{x\} + \{y\} \geq 1$. Therefore (13) can be simplified to the desired result. This completes the proof. $\square$

By Theorem 11(ii), we see that the 5-adic valuations of Fibonomial and binomial coefficients are the same. So we focus our investigation only on the p-adic valuations of Fibonomial coefficients when $p \neq 5$. Calculating r and s in Theorem 11(i) in every case and writing Theorem 11(iii) in another form, we obtain the following corollary.

Corollary 12. *Let m, k, r, and s be as in Theorem 11. Let*

$$A_2 = \nu_2\left(\left\lfloor \frac{m}{6} \right\rfloor !\right) - \nu_2\left(\left\lfloor \frac{k}{6} \right\rfloor !\right) - \nu_2\left(\left\lfloor \frac{m-k}{6} \right\rfloor !\right),$$

and for each prime $p \neq 2, 5$, let $A_p = \nu_p\left(\left\lfloor \frac{m}{z(p)} \right\rfloor !\right) - \nu_p\left(\left\lfloor \frac{k}{z(p)} \right\rfloor !\right) - \nu_p\left(\left\lfloor \frac{m-k}{z(p)} \right\rfloor !\right)$. Then the following statements hold.

(i) $$\nu_2\left(\binom{m}{k}_F\right) = \begin{cases} A_2, & \text{if } r \geq s \text{ and } (r,s) \neq (3,1), (3,2), (4,2); \\ A_2 + 1, & \text{if } (r,s) = (3,1), (3,2), (4,2); \\ A_2 + 3, & \text{if } r < s \text{ and } (r,s) \neq (0,3), (1,3), (2,3), \\ & (1,4), (2,4), (2,5); \\ A_2 + 2, & \text{if } (r,s) = (0,3), (1,3), (2,3), (1,4), (2,4), \\ & (2,5). \end{cases}$$

(ii) *For $p \neq 2, 5$, we have*

$$\nu_p\left(\binom{m}{k}_F\right) = \begin{cases} A_p, & \text{if } r \geq s; \\ A_p + \nu_p(F_{z(p)}), & \text{if } r < s. \end{cases}$$

Proof. For (i), we have $0 \leq r \leq 5$ and $0 \leq s \leq 5$, so we can directly consider every case and reduce Theorem 11(i) to the result in this corollary. In addition, (ii) follows directly from (13). $\square$

In a series of papers (see [11] and references therein), Marques and Trojovský obtain a formula for $\nu_p\left(\binom{p^b}{p^a}_F\right)$ only when $b = a + 1$. Then Ballot [2] extends it to any case $b > a$. Corollary 12 enables us to compute $\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right)$. We illustrate this in the next theorem.

Theorem 13. *Let a, b, ℓ_1, and ℓ_2 be positive integers and $b \geq a$. Let $p \neq 5$ be a prime. Assume that $\ell_1 p^b > \ell_2 p^a$ and let $m_p = \left\lfloor \frac{\ell_1 p^{b-a}}{z(p)} \right\rfloor$ and $k_p = \left\lfloor \frac{\ell_2}{z(p)} \right\rfloor$. Then the following statements hold.*

(i) *If* $a \equiv b \pmod 2$, *then* $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ *is equal to*

$$\begin{cases} \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv \ell_2 \pmod 3 \text{ or } \ell_2 \equiv 0 \pmod 3; \\ a + 2 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod 3 \text{ and } \ell_2 \not\equiv 0 \pmod 3; \\ \left\lceil \frac{a}{2} \right\rceil + 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod 3 \text{ and } \ell_2 \equiv 2 \pmod 3; \\ \left\lceil \frac{a+1}{2} \right\rceil + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod 3 \text{ and } \ell_2 \equiv 1 \pmod 3, \end{cases}$$

and if $a \not\equiv b \pmod 2$, *then* $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ *is equal to*

$$\begin{cases} \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv -\ell_2 \pmod 3 \text{ or } \ell_2 \equiv 0 \pmod 3; \\ a + 2 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod 3 \text{ and } \ell_2 \not\equiv 0 \pmod 3; \\ \left\lceil \frac{a+1}{2} \right\rceil + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod 3 \text{ and } \ell_2 \equiv 1 \pmod 3; \\ \left\lceil \frac{a}{2} \right\rceil + 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod 3 \text{ and } \ell_2 \equiv 2 \pmod 3. \end{cases}$$

(ii) *Let* $p \neq 5$ *be an odd prime and let* $r = \ell_1 p^b \bmod z(p)$ *and* $s = \ell_2 p^a \bmod z(p)$. *If* $p \equiv \pm 1 \pmod 5$, *then*

$$\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right) = [r < s]\left(a + \nu_p(m_p - k_p) + \nu_p(F_{z(p)})\right) + \nu_p\left(\binom{m_p}{k_p}\right),$$

and if $p \equiv \pm 2 \pmod 5$, *then* $\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right)$ *is equal to*

$$\begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r = s \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and} \\ & \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r > s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even;} \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r < s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even;} \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r > s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd;} \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r < s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd.} \end{cases}$$

Remark 14. In the proof of this theorem, we also show that the condition $r = s$ in Theorem 13(ii) is equivalent to $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod 2] \pmod{z(p)}$. It seems more natural to write $r = s$ in the statement of the theorem, but it is more convenient in the proof to use the condition $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod 2] \pmod{z(p)}$.

Proof of Theorem 13. We apply Corollary 12 to calculate $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ with $m = \ell_1 2^b$, $k = \ell_2 2^a$, $r = \ell_1 2^b \mod 6$, and $s = \ell_2 2^a \mod 6$. For convenience, we also let $r' = \ell_1 \mod 3$, and $s' = \ell_2 \mod 3$. Therefore A_2 given in Corollary 12 is

$$A_2 = \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-1}}{3} \right\rfloor !\right) - \nu_2\left(\left\lfloor \frac{\ell_2 2^{a-1}}{3} \right\rfloor !\right) - \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-1} - \ell_2 2^{a-1}}{3} \right\rfloor !\right). \tag{14}$$

By Corollary 8, the first term on the right-hand side of (14) is equal to

$$\begin{aligned}
&\frac{\ell_1(2^{b-1}-1)}{3} - \left\lfloor \frac{b-1}{2} \right\rfloor [\ell_1 \not\equiv 0 \pmod 3] - \left\{ \frac{\ell_1}{3} \right\} [b \equiv 0 \pmod 2] + \nu_2\left(\left\lfloor \frac{\ell_1}{3} \right\rfloor !\right) \\
&= \frac{\ell_1(2^{b-1}-1)}{3} - \left\lfloor \frac{b-1}{2} \right\rfloor [r' \neq 0] - \frac{r'}{3}[b \equiv 0 \pmod 2] + \nu_2\left(\left\lfloor \frac{\ell_1}{3} \right\rfloor !\right).
\end{aligned} \tag{15}$$

Similarly, the second term is

$$\frac{\ell_2(2^{a-1}-1)}{3} - \left\lfloor \frac{a-1}{2} \right\rfloor [s' \neq 0] - \frac{s'}{3}[a \equiv 0 \pmod 2] + \nu_2\left(\left\lfloor \frac{\ell_2}{3} \right\rfloor !\right). \tag{16}$$

To evaluate the third term on the right-hand side of (14), we divide the proof into two cases according to the parity of a and b.

Case 1. $a \equiv b \pmod 2$. Observe that $\ell_1 \equiv \ell_2 \pmod 3$ if and only if $r' = s'$. In addition, $\left\{\frac{\ell_1 - \ell_2}{3}\right\} = \left\{\frac{r'-s'}{3}\right\}$ and $\left\lfloor \frac{r'-s'}{3} \right\rfloor = -[r' < s']$. Then by Theorem 9, the third term on the right-hand side of (14) is equal to

$$\begin{aligned}
&\frac{(\ell_1 2^{b-a} - \ell_2)(2^{a-1}-1)}{3} - \left\{ \frac{r'-s'}{3} \right\} [a \equiv 0 \pmod 2] - \left\lfloor \frac{a-1}{2} \right\rfloor [r' \neq s'] + \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor !\right) \\
&= \frac{(\ell_1 2^{b-a} - \ell_2)(2^{a-1}-1)}{3} - \left(\frac{r'-s'}{3} + [r' < s'] \right) [a \equiv 0 \pmod 2] - \left\lfloor \frac{a-1}{2} \right\rfloor [r' \neq s'] \\
&\quad + \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor !\right).
\end{aligned} \tag{17}$$

Recall that $m_2 = \left\lfloor \frac{\ell_1 2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{\ell_2}{3} \right\rfloor$. Since $b - a$ is even, $2^{b-a} \equiv 1 \pmod 3$ and we obtain by Lemma 6 that

$$\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor = m_2 - k_2 - [r' < s'].$$

Therefore $\nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a}-\ell_2}{3}\right\rfloor !\right)$ is equal to

$$\nu_2(m_2!) - [r' < s']\nu_2(m_2-k_2) - \nu_2\left(\frac{m_2!}{(m_2-k_2)!}\right).$$

By Corollary 8, $\nu_2(m_2!)$ is equal to

$$\frac{\ell_1(2^{b-a}-1)}{3} - \frac{b-a}{2}[r' \neq 0] + \nu_2\left(\left\lfloor \frac{\ell_1}{3}\right\rfloor !\right).$$

We substitute the value of $\nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a}-\ell_2}{3}\right\rfloor !\right)$ in (17) and then substitute (15), (16), and (17) in (14) to obtain A_2. We see that there are some cancellations. For instance,

$$\frac{r'}{3}\Big([a \equiv 0 \pmod 2] - [b \equiv 0 \pmod 2]\Big) = 0.$$

and

$$\nu_2\left(\frac{m_2!}{(m_2-k_2)!}\right) - \nu_2\left(\left\lfloor \frac{\ell_2}{3}\right\rfloor !\right) = \nu_2\left(\binom{m_2}{k_2}\right)$$

Then we obtain

$$\begin{aligned} A_2 = &-\left\lfloor \frac{b-1}{2}\right\rfloor [r' \neq 0] + \left\lfloor \frac{a-1}{2}\right\rfloor [s' \neq 0] + [r' < s'][a \equiv 0 \pmod 2] \\ &+ \left\lfloor \frac{a-1}{2}\right\rfloor [r' \neq s'] + \frac{b-a}{2}[r' \neq 0] + [r' < s']\nu_2(m_2-k_2) + \nu_2\left(\binom{m_2}{k_2}\right). \end{aligned} \tag{18}$$

Next we divide the calculation of A_2 into 4 cases:

- Case 1.1. $\ell_1 \equiv \ell_2 \pmod 3$ or $\ell_2 \equiv 0 \pmod 3$,
- Case 1.2. $\ell_1 \equiv 0 \pmod 3$ and $\ell_2 \not\equiv 0 \pmod 3$,
- Case 1.3. $\ell_1 \equiv 1 \pmod 3$ and $\ell_2 \equiv 2 \pmod 3$,
- Case 1.4. $\ell_1 \equiv 2 \pmod 3$ and $\ell_2 \equiv 1 \pmod 3$.

Since the calculation in each case is similar, we only show the details in Case 1.1 and Case 1.2. So assume that $\ell_1 \equiv \ell_2 \pmod 3$. Then $r' = s'$ and (18) becomes

$$A_2 = -\left\lfloor \frac{b-1}{2}\right\rfloor [r' \neq 0] + \left\lfloor \frac{a-1}{2}\right\rfloor [r' \neq 0] + \frac{b-a}{2}[r' \neq 0] + \nu_2\left(\binom{m_2}{k_2}\right)$$

Since $-\left\lfloor \frac{b-1}{2}\right\rfloor + \left\lfloor \frac{a-1}{2}\right\rfloor + \frac{b-a}{2} = 0$, we see that $A_2 = \nu_2\left(\binom{m_2}{k_2}\right)$. Next if $\ell_2 \equiv 0 \pmod 3$, then $s' = 0$ and the same calculation leads to $A_2 = \nu_2\left(\binom{m_2}{k_2}\right)$. Next assume that $\ell_1 \equiv 0 \pmod 3$ and $\ell_2 \not\equiv 0 \pmod 3$. Then $r' = 0$, $s' \neq 0$, and (18) becomes

$$A_2 = \left\lfloor \frac{a-1}{2}\right\rfloor + [a \equiv 0 \pmod 2] + \left\lfloor \frac{a-1}{2}\right\rfloor + \nu_2(m_2-k_2) + \nu_2\left(\binom{m_2}{k_2}\right).$$

Observing that the sum of the first three terms above is equal to $a-1$, we obtain $A_2 = a - 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right)$. The other cases are similar. Therefore A_2 is

$$\begin{cases} \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv \ell_2 \pmod 3 \text{ or } \ell_2 \equiv 0 \pmod 3; \\ a - 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod 3 \text{ and } \ell_2 \not\equiv 0 \pmod 3; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod 3 \text{ and } \ell_2 \equiv 2 \pmod 3; \\ \left\lfloor \frac{a-1}{2} \right\rfloor + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod 3 \text{ and } \ell_2 \equiv 1 \pmod 3. \end{cases}$$

Recall that $r = \ell_1 2^b \mod 6$ and $s = \ell_2 2^a \mod 6$. Therefore

$$r = \begin{cases} 0, & \text{if } \ell_1 \equiv 0 \pmod 3; \\ 2, & \text{if } b \text{ is even and } \ell_1 \equiv 2 \pmod 3 \text{ or if } b \text{ is odd and } \ell_1 \equiv 1 \pmod 3; \\ 4 & \text{if } b \text{ is even and } \ell_1 \equiv 1 \pmod 3 \text{ or if } b \text{ is odd and } \ell_1 \equiv 2 \pmod 3, \end{cases}$$

and

$$s = \begin{cases} 0, & \text{if } \ell_2 \equiv 0 \pmod 3; \\ 2, & \text{if } a \text{ is even and } \ell_2 \equiv 2 \pmod 3 \text{ or if } a \text{ is odd and } \ell_2 \equiv 1 \pmod 3; \\ 4 & \text{if } a \text{ is even and } \ell_2 \equiv 1 \pmod 3 \text{ or if } a \text{ is odd and } \ell_2 \equiv 2 \pmod 3. \end{cases}$$

To obtain the formula for $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$, we divide the calculation into 4 cases: Case 1.1 to Case 1.4 as before. Then we consider the values of r and s in each case, and substitute A_2 in Corollary 12. This leads to the desired result. Since the calculation in each case is similar, we only give the details in Case 1.3. In this case, $A_2 = \left\lfloor \frac{a}{2} \right\rfloor + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right)$, $(r, s) = (2, 4)$ if a and b are odd, and $(r, s) = (4, 2)$ if a and b are even. By Corollary 12, we obtain

$$\begin{aligned} \nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right) &= \begin{cases} A_2 + 2, & \text{if } a \text{ and } b \text{ are odd;} \\ A_2 + 1, & \text{if } a \text{ and } b \text{ are even,} \end{cases} \\ &= \left\lceil \frac{a}{2} \right\rceil + 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), \end{aligned}$$

as required. The other cases are similar.

Case 2. $a \not\equiv b \pmod 2$. The calculation in this case is similar to Case 1, so we omit some details. By Theorem 9, the third term on the right-hand side of (14) is equal to

$$\begin{aligned} &\frac{(\ell_1 2^{b-a} - \ell_2)(2^{a-1} - 1)}{3} - \left\{ -\frac{r' + s'}{3} \right\} [a \equiv 0 \pmod 2] - \left\lfloor \frac{a-1}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod 3] \\ &+ \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor ! \right). \end{aligned} \tag{19}$$

Since $b-a$ is odd, $\ell_1 2^{b-a} \equiv -r' \pmod 3$ and we obtain by Lemma 6 that

$$\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor = m_2 - k_2 - B$$

where $B = [(r', s') \in \{(0,1), (0,2), (2,2)\}]$. Similar to Case 1, $\nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a}-\ell_2}{3} \right\rfloor !\right)$ is

$$\nu_2(m_2!) - B\nu_2(m_2-k_2) - \nu_2\left(\frac{m_2!}{(m_2-k_2)!}\right).$$

Then we evaluate $\nu_2(m_2!)$ by Corollary 8, and substitute all of these in (14) to obtain that A_2 is equal to

$$\begin{aligned}
&\left(\frac{b-a-1}{2} - \left\lfloor \frac{b-1}{2} \right\rfloor\right)[r' \neq 0] - \frac{r'}{3}[b \equiv 0 \pmod 2] + \left\lfloor \frac{a-1}{2} \right\rfloor [s' \neq 0] \\
&+ \left(\frac{s'}{3} + \left\{-\frac{r'+s'}{3}\right\}\right)[a \equiv 0 \pmod 2] + \left\lfloor \frac{a-1}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod 3] + \frac{r'}{3} \\
&+ B\nu_2(m_2-k_2) + \nu_2\left(\binom{m_2}{k_2}\right).
\end{aligned}$$

Then we divide the calculation into 4 cases and obtain that A_2 is

$$\begin{cases}
\nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv -\ell_2 \pmod 3 \text{ or } \ell_2 \equiv 0 \pmod 3; \\
a-1+\nu_2(m_2-k_2)+\nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod 3 \text{ and } \ell_2 \not\equiv 0 \pmod 3; \\
\left\lfloor \frac{a-1}{2} \right\rfloor + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod 3 \text{ and } \ell_2 \equiv 1 \pmod 3; \\
\left\lfloor \frac{a}{2} \right\rfloor + \nu_2(m_2-k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod 3 \text{ and } \ell_2 \equiv 2 \pmod 3.
\end{cases}$$

We illustrate the calculation of A_2 above only for the case $\ell_2 \equiv 0 \pmod 3$ since the other cases are similar. So suppose $\ell_2 \equiv 0 \pmod 3$. So $s' = 0$. If $r' = 0$, then it is easy to see that A_2 is equal to $\nu_2\left(\binom{m_2}{k_2}\right)$. So assume that $r' \neq 0$. Then A_2 is equal to $x + y + \nu_2\left(\binom{m_2}{k_2}\right)$, where

$$x = \frac{b-a-1}{2} - \left\lfloor \frac{b-1}{2} \right\rfloor + \left\lfloor \frac{a-1}{2} \right\rfloor = \begin{cases} 0, & \text{if } a \text{ is odd;} \\ -1, & \text{if } a \text{ is even,} \end{cases}$$

$$y = \frac{-r'}{3}[b \equiv 0 \pmod 2] + \left\{-\frac{r'}{3}\right\}[a \equiv 0 \pmod 2] + \frac{r'}{3} = \begin{cases} 0, & \text{if } a \text{ is odd;} \\ 1, & \text{if } a \text{ is even.} \end{cases}$$

Therefore $A_2 = \nu_2\left(\binom{m_2}{k_2}\right)$, as required. As in Case 1, we divide the calculation of $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ into 4 cases according to the value of A_2, which leads to the desired result. This proves (i).

For (ii), we apply Corollary 12 with $m = \ell_1 p^b$ and $k = \ell_2 p^a$. For convenience, we let $r' = \ell_1 \mod z(p)$ and $s' = \ell_2 \mod z(p)$. The calculation of this part is similar to that of part (i), so we omit some details. We have

$$A_p = \nu_p\left(\left\lfloor \frac{\ell_1 p^b}{z(p)} \right\rfloor!\right) - \nu_p\left(\left\lfloor \frac{\ell_2 p^a}{z(p)} \right\rfloor!\right) - \nu_p\left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{z(p)} \right\rfloor!\right). \tag{20}$$

Case 1. $p \equiv \pm 1 \pmod 5$. Then by Lemma 1(iii), $p \equiv 1 \pmod{z(p)}$. By Corollary 8, the first term on the right-hand side of (20) is equal to

$$\frac{\ell_1(p^b-1)}{z(p)(p-1)} + \nu_p\left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor!\right) - \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] - \frac{r'}{z(p)}[b \equiv 1 \pmod 2] + \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0]\left(1 - \frac{2r'}{z(p)}\right)$$
$$= \frac{\ell_1(p^b-1)}{z(p)(p-1)} - \frac{br'}{z(p)} + \nu_p\left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor!\right),$$

and similarly, the second term is

$$\frac{\ell_2(p^a-1)}{z(p)(p-1)} - \frac{as'}{z(p)} + \nu_p\left(\left\lfloor \frac{\ell_2}{z(p)} \right\rfloor!\right).$$

By Theorem 9, the third term is

$$\frac{(\ell_1 p^{b-a} - \ell_2)(p^a-1)}{z(p)(p-1)} - a\left(\frac{r'-s'}{z(p)} + [r' < s']\right) + \nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor!\right).$$

Since $p \equiv 1 \pmod{z(p)}$, we obtain by Lemma 6 that

$$\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor = m_p - k_p - [r' < s'].$$

Therefore $\nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor!\right)$ is equal to

$$\nu_p(m_p!) - [r' < s']\nu_p(m_p - k_p) - \nu_p\left(\frac{m_p!}{(m_p-k_p)!}\right).$$

As usual, the first term above can be evaluated by Corollary 8 and is equal to

$$\frac{\ell_1(p^{b-a}-1)}{z(p)(p-1)} - (b-a)\frac{r'}{z(p)} + \nu_p\left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor!\right).$$

We substitute all of these in (20) to obtain

$$A_p = [r' < s']\left(a + \nu_p(m_p - k_p)\right) + \nu_p\left(\binom{m_p}{k_p}\right).$$

Since $p \equiv 1 \pmod{z(p)}$, $r = r'$ and $s = s'$. Substituting A_p and applying Corollary 12, we obtain the desired result.

Case 2. $p \equiv \pm 2 \pmod 5$. Then by Lemma 1(iii), $p \equiv -1 \pmod{z(p)}$. By Corollary 8, the first term on the right-hand side of (20) is equal to

$$\frac{\ell_1(p^b-1)}{z(p)(p-1)} - \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] - \frac{r'}{z(p)}[b \equiv 1 \pmod 2] + \nu_p\left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor !\right).$$

Similarly, the second term is

$$\frac{\ell_2(p^a-1)}{z(p)(p-1)} - \left\lfloor \frac{a}{2} \right\rfloor [s' \neq 0] - \frac{s'}{z(p)}[a \equiv 1 \pmod 2] + \nu_p\left(\left\lfloor \frac{\ell_2}{z(p)} \right\rfloor !\right).$$

For the third term, we divide the proof into two cases according to the parity of a and b.

Case 2.1. $a \equiv b \pmod 2$. Then by Theorem 9, the third term on the right-hand side of (20) is equal to

$$\begin{aligned}&\frac{(\ell_1 p^{b-a}-\ell_2)(p^a-1)}{z(p)(p-1)} - \left(\frac{r'-s'}{z(p)} + [r' < s']\right)[a \equiv 1 \pmod 2] - \left\lfloor \frac{a}{2} \right\rfloor [r' \neq s'] \\ &+ \nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a}-\ell_2}{z(p)} \right\rfloor !\right).\end{aligned}$$

As in Case 1, we apply Lemma 6 to write

$$\left\lfloor \frac{\ell_1 p^{b-a}-\ell_2}{z(p)} \right\rfloor = m_p - k_p - [r' < s'],$$

and then use Corollary 8 to show that $\nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a}-\ell_2}{z(p)} \right\rfloor !\right)$ is equal to

$$\frac{\ell_1(p^{b-a}-1)}{z(p)(p-1)} - \frac{b-a}{2}[r' \neq 0] + \nu_p\left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor !\right) - [r' < s']\nu_p(m_p - k_p) - \nu_p\left(\frac{m_p!}{(m_p-k_p)!}\right).$$

Substituting all of these in (20), we see that A_p is equal to

$$\begin{aligned}&-\left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] + \left\lfloor \frac{a}{2} \right\rfloor [s' \neq 0] + [r' < s'][b \equiv 1 \pmod 2] + \left\lfloor \frac{a}{2} \right\rfloor [r' \neq s'] + \frac{b-a}{2}[r' \neq 0] \\ &+ [r' < s']\nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right) \\ &= \begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv \ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' > s'; \\ \left\lceil \frac{a}{2} \right\rceil + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' < s'. \end{cases}\end{aligned}$$

Recall that $r = \ell_1 p^b \mod z(p)$ and $s = \ell_2 p^a \mod z(p)$. If a and b are even, then $p^b \equiv p^a \equiv 1 \pmod{z(p)}$, $r = r'$, and $s = s'$, and we can obtain $\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right)$ by substituting A_p in Corollary 12. Suppose a and b are odd. Then $r \equiv -r' \pmod{z(p)}$ and $s \equiv -s' \pmod{z(p)}$ and thus when r and s are both nonzero or are both zero, we have

$$r \geq s \quad \text{if and only if} \quad r' \leq s'.$$

Similar to the above, we can obtain $\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right)$ by the substitution of A_p in Corollary 12. We see that $\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}\right)$ is equal to

$$\begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv \ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r > s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r < s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}; \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r > s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r < s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}. \end{cases}$$

Since $a \equiv b \pmod 2$, we see that $p^a \equiv p^b \pmod{z(p)}$ and therefore

$$\ell_1 \equiv \ell_2 \pmod{z(p)} \Leftrightarrow r = s \tag{21}$$

So the condition $\ell_1 \equiv \ell_2 \pmod{z(p)}$ can be replaced by $r = s$.

Case 2.2. $a \not\equiv b \pmod 2$. The calculation in this case is similar to that given before. So we skip some details. By Theorem 9, the third term on the right-hand side of (20) is equal to

$$\frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{z(p)(p-1)} - [a \equiv 1 \pmod 2] B_1 - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{z(p)}] + \nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor !\right),$$

where $B_1 = \left\{-\frac{r'+s'}{z(p)}\right\} = -\frac{r'+s'}{z(p)} + [r' + s' > 0] + [r' + s' > z(p)]$. Since $p \equiv -1 \pmod{z(p)}$, we obtain by Lemma 6 and a straightforward verification that

$$\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor = m_p - k_p - \varepsilon,$$

where $\varepsilon = [-r' \mod z(p) < s'] = [r' = 0 \text{ and } s' \neq 0] + [r' + s' > z(p)]$. Then by Corollary 8, $\nu_p\left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor !\right)$ is equal to

$$\frac{\ell_1(p^{b-a} - 1)}{z(p)(p-1)} - \left\lfloor \frac{b-a}{2} \right\rfloor [r' \neq 0] - \frac{r'}{z(p)} + \nu_p\left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor !\right) - B_2 - \nu_p\left(\frac{m_p!}{(m_p - k_p)!}\right),$$

where $B_2 = \varepsilon\nu_p\left(m_p - k_p\right)$. Since $a \not\equiv b \pmod 2$, $[b \equiv 1 \pmod 2] = 1 - [a \equiv 1 \pmod 2]$ and $\left\lfloor \frac{b-a}{2} \right\rfloor + \left\lfloor \frac{a}{2} \right\rfloor - \left\lfloor \frac{b}{2} \right\rfloor + [a \equiv 1 \pmod 2]$. We substitute all of these in (20) to obtain that A_p is equal to

$$\begin{aligned}
&\left(\left\lfloor \frac{b-a}{2} \right\rfloor - \left\lfloor \frac{b}{2} \right\rfloor\right)[r' \neq 0] + \left\lfloor \frac{a}{2} \right\rfloor \left([s' \neq 0] + [\ell_1 \not\equiv -\ell_2 \pmod{z(p)}]\right) \\
&+ \left([r' + s' > 0] + [r' + s' > z(p)]\right)[a \equiv 1 \pmod 2] + B_2 + \nu_p\left(\binom{m_p}{k_p}\right) \\
&= \begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p\left(m_p - k_p\right) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' + s' < z(p); \\ \left\lceil \frac{a}{2} \right\rceil + \nu_p\left(m_p - k_p\right) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' + s' > z(p). \end{cases}
\end{aligned}$$

Recall that $r = \ell_1 p^b \mod z(p)$ and $s = \ell_2 p^a \mod z(p)$. Suppose that a is odd and b is even. Then $r = r'$ and $s \equiv -s' \pmod{z(p)}$. Moreover, if $s' \neq 0$, then $s = z(p) - s'$ and thus

$$r < s \Leftrightarrow r' + s' < z(p) \quad \text{and} \quad r > s \Leftrightarrow r' + s' > z(p).$$

Similarly, if a is even and b is odd, then $r \equiv -r' \pmod{z(p)}$ and $s = s'$, and for $r' \neq 0$, we have

$$r < s \Leftrightarrow r' + s' > z(p) \quad \text{and} \quad r > s \Leftrightarrow r' + s' < z(p).$$

From the above observation and the substitution of A_p in Corollary 12, we see that $\nu_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right)$ is equal to

$$\begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p\left(m_p - k_p\right) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a+1}{2} + \nu_p\left(m_p - k_p\right) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r > s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r < s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}; \\ \frac{a}{2} + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r > s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p\left(m_p - k_p\right) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } r < s,\ \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}. \end{cases}$$

Since $a \not\equiv b \pmod 2$, we see that $p^a \equiv -p^b \pmod{z(p)}$ and therefore

$$\ell_1 \equiv -\ell_2 \pmod{z(p)} \Leftrightarrow r = s.$$

Combining this with (21), we conclude that

$$\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod 2] \pmod{z(p)} \Leftrightarrow r = s.$$

This completes the proof. $\square$

5 Examples

In this last section, we give several examples to show applications of our main results. We also recall from Remark 14 that the condition $r = s$ in Theorem 13(ii) can be replaced by $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod 2] \pmod{z(p)}$. In the calculation given in this section, we will use this observation without further reference.

Example 15. Let a, b, and ℓ be positive integers and $b \geq a$. We assert that for $\ell \not\equiv 0 \pmod 3$, we have

$$\nu_2\left(\binom{\ell \cdot 2^b}{2^a}_F\right) = \left\lceil \frac{a+1}{2} \right\rceil (\varepsilon_1\varepsilon_2 + \varepsilon_1'\varepsilon_2'), \tag{22}$$

where $\varepsilon_1 = [\ell \equiv 2 \pmod 3]$, $\varepsilon_2 = [a \equiv b \pmod 2]$, $\varepsilon_1' = [\ell \equiv 1 \pmod 3]$, and $\varepsilon_2' = [a \not\equiv b \pmod 2]$. In addition, if $\ell \equiv 0 \pmod 3$, then

$$\nu_2\left(\binom{\ell \cdot 2^b}{2^a}_F\right) = b + 2 + \nu_2(\ell). \tag{23}$$

Proof. We apply Theorem 13 to verify our assertion. Here $m_2 = \left\lfloor \frac{\ell \cdot 2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{1}{3} \right\rfloor = 0$. So we immediately obtain the following: if $a \equiv b \pmod 2$, then

$$\nu_2\left(\binom{\ell \cdot 2^b}{2^a}_F\right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod 3; \\ a + 2 + \nu_2(m_2), & \text{if } \ell \equiv 0 \pmod 3; \\ \left\lceil \frac{a+1}{2} \right\rceil, & \text{if } \ell \equiv 2 \pmod 3, \end{cases}$$

and if $a \not\equiv b \pmod 2$, then

$$\nu_2\left(\binom{\ell \cdot 2^b}{2^a}_F\right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod 3; \\ a + 2 + \nu_2(m_2), & \text{if } \ell \equiv 0 \pmod 3; \\ \left\lceil \frac{a+1}{2} \right\rceil, & \text{if } \ell \equiv 1 \pmod 3. \end{cases}$$

This proves (22). If $\ell \equiv 0 \pmod 3$, then $m_2 = \frac{\ell}{3} \cdot 2^{b-a}$ and $\nu_2(m_2)$ is equal to

$$\nu_2(m_2) = \nu_2(\ell) + \nu_2(2^{b-a}) - \nu_2(3) = b - a + \nu_2(\ell),$$

which implies (23). □

Example 16. Substituting $\ell = 1$ in Example 15, we see that

$$\begin{aligned} \nu_2\left(\binom{2^b}{2^a}_F\right) &= \left\lceil \frac{a+1}{2} \right\rceil [a \not\equiv b \pmod 2] \\ &= \begin{cases} 0, & \text{if } a \equiv b \pmod 2; \\ \left\lceil \frac{a+1}{2} \right\rceil, & \text{if } a \not\equiv b \pmod 2. \end{cases} \end{aligned} \tag{24}$$

Our example also implies that (24) still holds for the 2-adic valuations of $\binom{2^{b+2c}}{2^a}_F$, $\binom{7 \cdot 2^b}{2^a}_F$, $\binom{5 \cdot 2^{b+1}}{2^a}_F$, $\binom{13 \cdot 2^b}{2^a}_F$, etc.

Example 17. Let a, b, and ℓ be positive integers, $b \geq a$, and p a prime distinct from 2 and 5. If $p \equiv \pm 1 \pmod 5$, then

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \big(b + \nu_p(F_{z(p)}) + \nu_p(\ell)\big)\,[\ell \equiv 0 \pmod{z(p)}],$$

and if $p \equiv \pm 2 \pmod 5$, then

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \begin{cases} 0, & \text{if } \ell \equiv 1 - 2\varepsilon \pmod{z(p)};\\ b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } \ell \equiv 0 \pmod{z(p)};\\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, 1-2\varepsilon \pmod{z(p)} \text{ and } a \text{ is even};\\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1-2\varepsilon \pmod{z(p)} \text{ and } a \text{ is odd},\end{cases}$$

where $\varepsilon = [a \not\equiv b \pmod 2]$.

Proof. Similar to Example 15, we verify this by applying Theorem 13. Here $m_p = \left\lfloor \frac{\ell p^{b-a}}{z(p)} \right\rfloor$, $k_p = \left\lfloor \frac{1}{z(p)} \right\rfloor = 0$, $r = \ell p^b \bmod z(p)$, and $s = p^a \bmod z(p)$. We first assume that $p \equiv \pm 1 \pmod 5$. Then by Lemma 1, we have $p \equiv 1 \pmod{z(p)}$. Therefore $s = 1$, $r \equiv \ell \pmod{z(p)}$, and

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \big(a + \nu_p(m_p) + \nu_p(F_{z(p)})\big)\,[\ell \equiv 0 \pmod{z(p)}].$$

Similarly, if $p \equiv \pm 2 \pmod 5$ and $a \equiv b \pmod 2$, then we obtain by Lemma 1 and Theorem 13 that

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{z(p)};\\ a + \nu_p(m_p) + \nu_p(F_{z(p)}), & \text{if } \ell \equiv 0 \pmod{z(p)};\\ \frac{a}{2}, & \text{if } \ell \not\equiv \quad \bmod 0, 1z(p) \text{ and } a \text{ is even};\\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is odd}.\end{cases}$$

In addition, if $p \equiv \pm 2 \pmod 5$ and $a \not\equiv b \pmod 2$, then

$$\nu_p\left(\binom{\ell p^b}{p^a}_F\right) = \begin{cases} 0, & \text{if } \ell \equiv -1 \pmod{z(p)};\\ a + \nu_p(m_p) + \nu_p(F_{z(p)}), & \text{if } \ell \equiv 0 \pmod{z(p)};\\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is even};\\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is odd}.\end{cases}$$

It remains to calculate $\nu_p(m_p)$ when $\ell \equiv 0 \pmod{z(p)}$. In this case, we have

$$\nu_p(m_p) = \nu_p\left(\frac{\ell p^{b-a}}{z(p)}\right) = \nu_p(\ell) + \nu_p(p^{b-a}) - \nu_p(z(p)) = b - a + \nu_p(\ell).$$

This implies the desired result. $\square$

Example 18. Substituting $\ell = 1$ in Example 17, we see that for $p \neq 2, 5$, we have

$$\nu_p\left(\binom{p^b}{p^a}_F\right) = \begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod 5 \text{ or } a \equiv b \pmod 2; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod 5,\ a \not\equiv b \pmod 2, \text{ and } a \text{ is even;} \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod 5,\ a \not\equiv b \pmod 2, \text{ and } a \text{ is odd.} \end{cases} \tag{25}$$

Our example also implies that (25) still holds for the p-adic valuations of $\binom{p^{b+2c}}{p^a}_F$ and $\binom{(z(p)+1)\cdot p^b}{p^a}_F$. Similarly, for $p \neq 2, 5$, we have

$$\nu_p\left(\binom{2p^b}{p^a}_F\right) = \begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod 5; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod 5 \text{ and } a \text{ is even;} \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod 5 \text{ and } a \text{ is odd.} \end{cases} \tag{26}$$

In addition, (26) also holds when $\binom{2p^b}{p^a}_F$ is replaced by $\binom{\ell p^b}{p^a}_F$ for $\ell \not\equiv 0, \pm 1 \pmod{z(p)}$ and $p \neq 2, 5$. Furthermore, replacing $\binom{2p^b}{p^a}_F$ by $\binom{(z(p)-1)p^b}{p^a}_F$, the formula becomes

$$\begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod 5 \text{ or } a \not\equiv b \pmod 2; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod 5,\ a \equiv b \pmod 2, \text{ and } a \text{ is even;} \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod 5,\ a \equiv b \pmod 2, \text{ and } a \text{ is odd.} \end{cases}$$

Example 19. We know that the 5-adic valuations of Fibonomial coefficients are the same as those of binomial coefficients. For example, by Theorem 11(ii) and Kummer's theorem, we obtain

$$\nu_5\left(\binom{\ell \cdot 5^b}{5^a}_F\right) = \nu_5\left(\binom{\ell \cdot 5^b}{5^a}\right) = b - a + \nu_5(\ell),$$

for every $a, b, \ell \in \mathbb{N}$ with $b \geq a$. Similarly, $\nu_5\left(\binom{5^b}{\ell \cdot 5^a}_F\right) = b - a - \nu_5(\ell)$ for every $a, b, \ell \in \mathbb{N}$ such that $5^b > \ell \cdot 5^a$.

Example 20. Let a, b, and ℓ be positive integers and $2^b > \ell \cdot 2^a$. Let $m_2 = \left\lfloor \frac{2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{\ell}{3} \right\rfloor$. Then

$$\nu_2\left(\binom{2^b}{\ell \cdot 2^a}_F\right) = \nu_2\left(\binom{m_2}{k_2}\right) + \left(\left\lceil \frac{a+2}{2} \right\rceil + \nu_2(m_2 - k_2)\right)\varepsilon_1\varepsilon_2 + \left\lceil \frac{a+1}{2} \right\rceil \varepsilon_3\varepsilon_4, \tag{27}$$

where $\varepsilon_1 = [a \equiv b \pmod 2]$, $\varepsilon_2 = [\ell \equiv 2 \pmod 3]$, $\varepsilon_3 = [a \not\equiv b \pmod 2]$, and $\varepsilon_4 = [\ell \equiv 1 \pmod 3]$.

Proof. Similar to Example 15, this follows from the application of Theorem 13. So we leave the details to the reader. □

Example 21. Let $k \geq 2$. We observe that

$$\left\lfloor \frac{2^k}{3} \right\rfloor = \begin{cases} \frac{2^k-1}{3}, & \text{if } k \text{ is even;} \\ \frac{2(2^{k-1}-1)}{3}, & \text{if } k \text{ is odd,} \end{cases}$$

which implies,

$$\nu_2\left(\left\lfloor \frac{2^k}{3} \right\rfloor\right) = [k \equiv 1 \pmod 2]. \tag{28}$$

By a similar reason, we also see that for $k \geq 3$,

$$\nu_2\left(\left\lfloor \frac{2^k}{3} \right\rfloor - 1\right) = 2[k \equiv 0 \pmod 2]. \tag{29}$$

From (27), (28), and (29), we obtain the following results:

(i) if $b - a \geq 2$, then $\nu_2\left(\binom{2^b}{3\cdot 2^a}_F\right) = [a \not\equiv b \pmod 2]$,

(ii) if $b - a \geq 3$, then $\nu_2\left(\binom{2^b}{5\cdot 2^a}_F\right)$ is equal to

$$\begin{aligned} &[a \not\equiv b \pmod 2] + \left(\left\lceil \frac{a+2}{2} \right\rceil + 2[a \equiv b \pmod 2]\right)[a \equiv b \pmod 2] \\ &= 1 + \left\lceil \frac{a+4}{2} \right\rceil [a \equiv b \pmod 2], \end{aligned}$$

(iii) if $b - a \geq 3$, then $\nu_2\left(\binom{2^b}{6\cdot 2^a}_F\right) = [a \equiv b \pmod 2]$,

(iv) if $b - a \geq 4$, then $\nu_2\left(\binom{2^b}{7\cdot 2^a}_F\right) = [a \equiv b \pmod 2] + \left\lceil \frac{a+1}{2} \right\rceil [a \not\equiv b \pmod 2]$.

Example 22. Let $p \neq 5$ be an odd prime and let a, b, and ℓ be positive integers, $p^b > \ell p^a$, $m_p = \left\lfloor \frac{p^{b-a}}{z(p)} \right\rfloor$, and $k_p = \left\lfloor \frac{\ell}{z(p)} \right\rfloor$. Then the following statements hold.

(i) If $p \equiv \pm 1 \pmod 5$, then

$$\nu_p\left(\binom{p^b}{\ell p^a}_F\right) = \left(a + \nu_p\left(m_p - k_p\right) + \nu_p(F_{z(p)})\right)[\ell \not\equiv 0, 1 \pmod{z(p)}] + \nu_p\left(\binom{m_p}{k_p}\right),$$

(ii) If $p \equiv \pm 2 \pmod 5$, then $\nu_p\left(\binom{p^b}{\ell p^a}_F\right)$ is equal to

$$\nu_p\left(\binom{m_p}{k_p}\right) + \varepsilon_1\varepsilon_2\varepsilon_5\left(\left\lceil \frac{a}{2} \right\rceil + \nu_p(m_p - k_p) + \varepsilon_3\nu_p(F_z(p))\right) + \varepsilon_1\varepsilon_4(1-\varepsilon_5)\left(\left\lfloor \frac{a}{2} \right\rfloor + \varepsilon_3\nu_p(F_z(p))\right) \tag{30}$$

where $\varepsilon_1 = [\ell \not\equiv 0 \pmod{z(p)}]$, $\varepsilon_2 = [\ell \not\equiv 1 \pmod{z(p)}]$, $\varepsilon_3 = [b \equiv 0 \pmod 2]$, $\varepsilon_4 = [\ell \not\equiv -1 \pmod{z(p)}]$, and $\varepsilon_5 = [a \equiv b \pmod 2]$.

Proof. Similar to Example 17, this follows from the application of Lemma 1 and Theorem 13. Since (i) is easily verified, we only give the proof of (ii). The calculation is done in two cases. If $p \equiv \pm 2 \pmod 5$ and $a \equiv b \pmod 2$, then $\nu_p\left(\binom{p^b}{\ell p^a}_F\right)$ is equal to

$$\begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell \equiv 0, 1 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is odd}, \end{cases}$$

$$= \nu_p\left(\binom{m_p}{k_p}\right) + \varepsilon_1\varepsilon_2\left(\left\lceil \frac{a}{2} \right\rceil + \nu_p(m_p - k_p) + \varepsilon_3\nu_p(F_{z(p)})\right),$$

where $\varepsilon_1 = [\ell \not\equiv 0 \pmod{z(p)}]$, $\varepsilon_2 = [\ell \not\equiv 1 \pmod{z(p)}]$, and $\varepsilon_3 = [b \equiv 0 \pmod 2]$. If $p \equiv \pm 2 \pmod 5$ and $a \not\equiv b \pmod 2$, then

$$\nu_p\left(\binom{p^b}{\ell p^a}_F\right) = \begin{cases} \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell \equiv 0, -1 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p\left(\binom{m_p}{k_p}\right), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is odd}, \end{cases}$$

$$= \nu_p\left(\binom{m_p}{k_p}\right) + \varepsilon_1\varepsilon_4\left(\left\lfloor \frac{a}{2} \right\rfloor + \varepsilon_3\nu_p(F_{z(p)})\right),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are as above and $\varepsilon_4 = [\ell \not\equiv -1 \pmod{z(p)}]$. Let $\varepsilon_5 = [a \equiv b \pmod 2]$. Then both cases can be combined to obtain (ii). □

Example 23. Let $k \geq 2$. We observe that $z(7) = 8$ and

$$\left\lfloor \frac{7^k}{8} \right\rfloor = \begin{cases} \frac{7^k - 1}{8}, & \text{if } k \text{ is even}; \\ \frac{7(7^{k-1} - 1)}{8}, & \text{if } k \text{ is odd}. \end{cases}$$

Therefore

$$\nu_7\left(\left\lfloor \frac{7^k}{8} \right\rfloor\right) = [k \equiv 1 \pmod 2] \quad \text{and} \quad \nu_7\left(\left\lfloor \frac{7^k}{8} \right\rfloor - 1\right) = 0. \tag{31}$$

From (30) and (31), we obtain the following results:

(i) if $b - a \geq 2$, then $\nu_7\left(\binom{7^b}{8 \cdot 7^a}_F\right) = [a \not\equiv b \pmod 2]$,

(ii) if $b - a \geq 2$, then $\nu_7\left(\binom{7^b}{9 \cdot 7^a}_F\right) = \left(\left\lfloor \frac{a+2}{2} \right\rfloor + [b \equiv 0 \pmod 2]\right)[a \not\equiv b \pmod 2]$,

(iii) if $b - a \geq 2$, then $\nu_7\left(\binom{7^b}{15 \cdot 7^a}_F\right)$ is equal to

$$[a \not\equiv b \pmod 2] + \left(\left\lceil \frac{a}{2} \right\rceil + [b \equiv 0 \pmod 2]\right)[a \equiv b \pmod 2].$$

To keep this article not too lengthy, we plan to give more applications of our main results in the next article.

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