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The Trinomial Transform Triangle

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Abstract

The trinomial transform of a sequence is a generalization of the well-known binomial transform, replacing binomial coefficients with trinomial coefficients. We examine Pascal-like triangles under the trinomial transform, focusing on ternary linear recurrent sequences. We determine the sums and alternating sums of the elements in columns, and we give some examples of the trinomial transform triangle.

1 Introduction

The binomial transform of a sequence $(a_i)_{i=0}^{\infty}$ is a sequence $(b_n)_{n=0}^{\infty}$ defined by $b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$. This transformation is invertible with formula $a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b_i$. Several studies [4, 6, 7, 10] examine the properties and the generalizations of the binomial transform. One generation is the *trinomial transform*. Let it be given by

$$b_n = \sum_{i=0}^{2n} \binom{n}{i}_2 a_i,$$

where for $0 \le i \le 2n$

$$\binom{n}{i}_2 = \sum_{j=0}^{i} \binom{n}{j} \binom{j}{i-j}$$

holds with the classical binomial coefficients. It is known that the trinomial triangle (Figure 2) determines the trinomial coefficients $\binom{n}{i}_2$ that arise in the expansion of $(1+x+x^2)^n$. (For example, Belbachir et al. [5] discussed some details of the trinomial coefficients, the trinomial tringle and their generalizations.)

To the best of our knowledge, this transformation has not been previously studied or even mentioned in journals, although there are eleven sequences in the OEIS [8], which are trinomial transforms of certain sequences, that do not appear in the literature We will refer to some of them in the last section.

In this article, we define an arithmetic structure similar to an infinite triangle, where the terms are arranged in rows and columns. Let the trinomial transform triangle $\mathcal{T}$ be defined the following way. Row 0 consists of the terms of a given sequence $(a_k)_{k=0}^{\infty}$, and any term in an other row is the sum of the three terms directly above it according to Figure 1. The exact definition is

$$\begin{aligned} a_0^k &= a_k, \\ a_n^k &= a_{n-1}^{k-1} + a_{n-1}^{k} + a_{n-1}^{k+1}, \quad (1 \le n \le k). \end{aligned} \tag{1}$$

Let the ℓ^{th} $(\ell \ge 0)$ diagonal sequence of $\mathcal{T}$ be the sequence $(a_n^{n+l})_{n=0}^{\infty}$. We shall give some properties of triangle $\mathcal{T}$ and show that the 0^{th} or main diagonal sequence of $\mathcal{T}$ is the trinomial transform sequence of (a_k). In particular, we show that the trinomial transform of a ternary linear recurrent sequence is also a ternary linear recurrent sequence. Moreover, we give some properties of $\mathcal{T}$, for example, sums and partial sums in the columns. The author [6] dealt with a generalized binomial transform triangle.

In the last section, we present some special examples for the trinomial transform triangles.

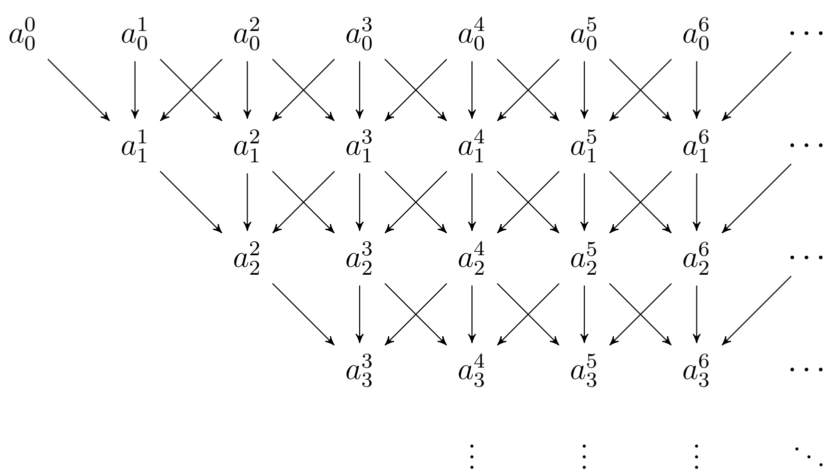

Figure 1: Construction of the trinomial transform triangle $\mathcal{T}$

2 Trinomial triangle and its partial sum triangle

Most of our proofs are based on the elements of the trinomial triangle (or Feinberg's triangle named by Anatriello and Vincenzi [2]) and its partial sum triangle, therefore first of all we define them and give some known basic properties [1, 5].

Let the trinomial coefficient $\binom{n}{k}_2$ be the k^{th} term of the n^{th} row in the trinomial triangle, where $0 \le n$ and $0 \le k \le 2n$ (see Figure 2 and A027907 in the OEIS [8]). The terms of the triangle satisfy the relations

$$\binom{n}{0}_2 = \binom{n}{2n}_2 = 1 \qquad (0 \le n),$$

$$\binom{n}{k}_2 = \binom{n-1}{k-2}_2 + \binom{n-1}{k-1}_2 + \binom{n-1}{k}_2 \qquad (1 \le n).$$

We use the convention $\binom{n}{k}_2 = 0$ for $k \notin \{0, 1, \ldots, 2n\}$. (For more details and some generalization of binomial and trinomial coefficients see [5].) Now, we take the partial sums of terms in columns of the trinomial triangle. Let $\left[\binom{n}{k}\right]_2$ denote the partial sum of values of $\binom{n}{k}_2$ and all the terms above it in its column, so that

$$\left[\binom{n}{k}\right]_2 = \sum_{j=0}^{n} \binom{n-j}{k-j}_2.$$

Without zero elements, we also have

$$\left[\binom{n}{k}\right]_2 = \sum_{i=|n-k|}^{n} \binom{i}{i-(n-k)}_2 = \sum_{i=|n-k|}^{n} \sum_{j=0}^{i-n+k} \binom{i}{j} \binom{j}{i-n+k-j}.$$

						1						
					1	1	1					
				1	2	3	2	1				
			1	3	6	7	6	3	1			
		1	4	10	16	19	16	10	4	1		
	1	5	15	30	45	51	45	30	15	5	1	
1	6	21	50	90	126	141	126	90	50	21	6	1

Figure 2: Trinomial triangle (A027907)

By arranging them into an infinite triangle, we gain the partial sum triangle of trinomial triangle (or partial sum trinomial triangle — see Figure 3). It is not in the OEIS yet, but the partial sum sequences of the central trinomial coefficients and of the two neighbouring column sequences do — see A097893, A097861, and A097894.

						1						
					1	2	1					
				1	3	5	3	1				
			1	4	9	12	9	4	1			
		1	5	14	25	31	25	14	5	1		
	1	6	20	44	70	82	70	44	20	6	1	
1	7	27	70	134	196	223	196	134	70	27	7	1

Figure 3: Partial sum triangle of trinomial triangle

Our major aim in this paper is not to examine the partial sum trinomial triangle, but we give some of its properties that we use in the following sections.

From the vertical symmetry of the trinomial triangle, $\binom{n}{k}_2 = \binom{n}{2n-k}_2$, the partial sum trinomial triangle also has a symmetry axis and

$$\left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \left[\!\!\left[\binom{n}{2n-k}\right]\!\!\right]_2.$$

Moreover, if $n \geq k$, then

$$\left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \sum_{i=0}^{k} \binom{n-i}{k-i}_2 = \sum_{i=0}^{k}\sum_{j=0}^{n-i} \binom{n-i}{j}\binom{j}{k-i-j}.$$

Furthermore, this triangle evidently satisfies the recursive relations below.

$$\left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \begin{cases} 1, & \text{if } k=0 \text{ or } k=2n; \\ \left[\!\!\left[\binom{n-1}{k-1}\right]\!\!\right]_2 + \binom{n}{k}_2, & \text{otherwise.} \end{cases} \tag{2}$$

$$\left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \begin{cases} 1, & \text{if } k=0 \text{ or } k=2n; \\ n+1, & \text{if } k=1 \text{ or } k=2n-1; \\ \left[\!\!\left[\binom{n-1}{n-2}\right]\!\!\right]_2 + \left[\!\!\left[\binom{n-1}{n-1}\right]\!\!\right]_2 + \left[\!\!\left[\binom{n-1}{n}\right]\!\!\right]_2 + 1, & \text{if } 2 \leq n = k; \\ \left[\!\!\left[\binom{n-1}{k-2}\right]\!\!\right]_2 + \left[\!\!\left[\binom{n-1}{k-1}\right]\!\!\right]_2 + \left[\!\!\left[\binom{n-1}{k}\right]\!\!\right]_2, & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 1 provides summation identities for the partial sum trinomial triangle, as a corollary of the last section.

Theorem 1. *The partial sum trinomial triangle satisfies the following summation expressions*

$$\sum_{k=0}^{2n} \left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \frac{3^{n+1}-1}{2}, \qquad \sum_{k=0}^{2n} (-1)^k \left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \begin{cases} 0, & \textit{if } n \textit{ is odd}; \\ 1, & \textit{otherwise}, \end{cases}$$

$$\sum_{k=0}^{2n} k \left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = n\frac{3^{n+1}-1}{2}, \qquad \sum_{k=0}^{2n} (-1)^k k \left[\!\!\left[\binom{n}{k}\right]\!\!\right]_2 = \begin{cases} 0, & \textit{if } n \textit{ is odd or } n=0; \\ n, & \textit{otherwise}. \end{cases}$$

3 Trinomial transform triangle

Any term of $\mathcal{T}$ can be determined by the terms of sequence (a_k) with trinomial coefficients. Recall $a_0^k = a_k$.

Theorem 2. *For any $0 \le n \le k$, we have*

$$a_n^k = \sum_{i=k-n}^{k+n} \binom{n}{i-k+n}_2 a_0^i. \tag{4}$$

Proof. The proof is by induction on n. For $n = 0$ the statement is trivial. For clarity in case $n = 1$, we have

$$a_1^k = a_0^{k-1} + a_0^k + a_0^{k+1} = \binom{1}{0}_2 a_0^{k-1} + \binom{1}{1}_2 a_0^k + \binom{1}{2}_2 a_0^{k+1} = \sum_{i=k-1}^{k+1} \binom{1}{i-k+1}_2 a_0^i.$$

Let us suppose that the result is true for $n-1$, so, for example,

$$a_{n-1}^k = \sum_{i=k-n+1}^{k+n-1} \binom{n-1}{i-k+n-1}_2 a_0^i.$$

Using (3), we have

$$\begin{aligned}
a_n^k &= a_{n-1}^{k-1} + a_{n-1}^k + a_{n-1}^{k+1} \\
&= \sum_{i=k-n}^{k+n-2} \binom{n-1}{i-k+n}_2 a_0^i + \sum_{i=k-n+1}^{k+n-1} \binom{n-1}{i-k+n-1}_2 a_0^i + \sum_{i=k-n+2}^{k+n} \binom{n-1}{i-k+n-2}_2 a_0^i \\
&= \binom{n-1}{0}_2 a_0^{k-n} + \left(\binom{n-1}{1}_2 + \binom{n-1}{0}_2\right) a_0^{k-n+1} \\
&\quad + \left(\binom{n-1}{2}_2 + \binom{n-1}{1}_2 + \binom{n-1}{0}_2\right) a_0^{k-n+2} + \cdots \\
&\quad + \left(\binom{n-1}{i-k+n-2}_2 + \binom{n-1}{i-k+n-1}_2 + \binom{n-1}{i-k+n}_2\right) a_0^i + \cdots \\
&\quad + \left(\binom{n-1}{2n-3}_2 + \binom{n-1}{2n-2}_2\right) a_0^{k+n-1} + \binom{n-1}{2n-2}_2 a_0^{k+n} \\
&= \sum_{i=k-n}^{k+n} \binom{n}{i-k+n}_2 a_0^i.
\end{aligned}$$

$\square$

We let (b_n) denote the 0^{th} diagonal sequence (a_n^n) in $\mathcal{T}$. If $k = n$, then we obtain the next corollary from Theorem 2.

Corollary 3. *The diagonal sequence (b_n) of the trinomial transform triangle is the trinomial transform sequence of (a_k).*

Let $(s_n)_{n=0}^{\infty}$ be the sum sequence of the values of columns in $\mathcal{T}$, so that

$$s_n = \sum_{i=0}^{n} a_i^n. \tag{5}$$

Theorem 4.

$$s_n = \sum_{i=0}^{n} \sum_{j=n-i}^{n+i} \binom{i}{j-n+i}_2 a_0^j = \sum_{i=0}^{n} \sum_{k=0}^{2i} \binom{i}{k}_2 a_0^{n+k-i}.$$

Proof. Substitute (4) into (5) and put $k = i + j - n$. ◻

Let us express the term s_n by the help of the terms of the partial sum trinomial triangle and for it, first, we have to prove the theorem below.

Theorem 5. *If $n \le k$, then*

$$\sum_{i=0}^{n} a_i^k = \sum_{\ell=k-n}^{k+n} \left[\binom{n}{\ell-k+n}\right]_2 a_0^{\ell}. \tag{6}$$

Proof. We prove again by induction. The case $n = 0$ is clear, and for clarity in case $n = 1$, we have $\sum_{i=0}^{1} a_i^k = a_0^k + a_1^k = a_0^{k-1} + 2a_0^k + a_0^{k+1}$. The hypothesis of induction for $n - 1$ is

$$\sum_{i=0}^{n-1} a_i^k = \sum_{\ell=k-n+1}^{k+n-1} \left[\binom{n-1}{\ell-k+n+1}\right]_2 a_0^{\ell}.$$

Now, using (2), we obtain

$$
\begin{aligned}
\sum_{i=0}^{n} a_i^k &= \sum_{i=0}^{n-1} a_i^k + a_n^k \\
&= \sum_{\ell=k-n+1}^{k+n-1} \left[\!\!\binom{n-1}{\ell-k+n+1}\!\!\right]_2 a_0^\ell + \sum_{i=k-n}^{k+n} \binom{n}{i-k+n}_2 a_0^i \\
&= \sum_{\ell=k-n+1}^{k+n-1} \left[\!\!\binom{n-1}{\ell-k+n+1}\!\!\right]_2 a_0^\ell + \binom{n}{0}_2 a_0^{k-n} + \binom{n}{2n}_2 a_0^{k+n} + \sum_{\ell=k-n+1}^{k+n-1} \binom{n}{\ell-k+n}_2 a_0^\ell \\
&= \binom{n}{0}_2 a_0^{k-n} + \sum_{\ell=k-n+1}^{k+n-1} \left(\left[\!\!\binom{n-1}{\ell-k+n+1}\!\!\right]_2 + \binom{n}{\ell-k+n}_2 \right) a_0^\ell + \binom{n}{2n}_2 a_0^{k+n} \\
&= \sum_{\ell=k-n}^{k+n} \left[\!\!\binom{n}{\ell-k+n}\!\!\right]_2 a_0^\ell.
\end{aligned}
$$

□

The same induction method yields the following theorem.

Theorem 6. *If $0 \leq j \leq n$, then we have*

$$\sum_{i=0}^{k} a_i^k = \sum_{\ell=k-n+j}^{k+n-j} \left[\!\!\binom{n-j}{\ell-k+n-j}\!\!\right]_2 a_j^\ell.$$

Applying Theorem 5 for the case $k = n$, we obtain the main theorem of this section.

Theorem 7. *The column sum sequence can be given with the expression*

$$s_n = \sum_{\ell=0}^{2n} \left[\!\!\binom{n}{\ell}\!\!\right]_2 a_0^\ell.$$

4 Trinomial transform triangles generated by ternary homogeneous linear recurrent sequences

From this point on we examine the case $a_0^k = a_k$, where $(a_k)_{k=0}^{\infty}$ is a ternary linear recursive sequence with initial values $a_0, a_1, a_2 \in \mathbb{Z}$ ($|a_0| + |a_1| + |a_2| \neq 0$). And it is defined for $3 \leq k$ by

$$a_k = \alpha a_{k-1} + \beta a_{k-2} + \gamma a_{k-3}, \tag{7}$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$, $\gamma \neq 0$. (In general, the results of this section hold for an integral domain.)

From now on, we will use three important variables

$$\begin{aligned}
\mathcal{A} &= \alpha^2 + \alpha + 2\beta + 3, \\
\mathcal{B} &= -2\alpha^2 + \alpha\beta + 2\alpha\gamma - \beta^2 - 2\alpha - 3\beta + 3\gamma - 3, \\
\mathcal{C} &= \alpha^2 - \alpha\beta - \alpha\gamma + \beta^2 - \beta\gamma + \gamma^2 + \alpha + \beta - 2\gamma + 1.
\end{aligned}$$

Theorem 8. *The terms in the rows satisfy the same ternary relation,*

$$a_n^k = \alpha a_n^{k-1} + \beta a_n^{k-2} + \gamma a_n^{k-3} \qquad (0 \leq n \leq k-3). \tag{8}$$

Proof. We prove by induction on n. If $n = 0$, then (8) is the definition of a_k. We suppose that it holds for up to row $n-1$. Then

$$\begin{aligned}
a_n^k &= a_{n-1}^{k-1} + a_{n-1}^k + a_{n-1}^{k+1} \\
&= \alpha a_{n-1}^{k-2} + \beta a_{n-1}^{k-3} + \gamma a_{n-1}^{k-4} + \alpha a_{n-1}^{k-1} + \beta a_{n-1}^{k-2} + \gamma a_{n-1}^{k-3} + \alpha a_{n-1}^{k} + \beta a_{n-1}^{k-1} + \gamma a_{n-1}^{k-2} \\
&= \alpha(a_{n-1}^{k} + a_{n-1}^{k-1} + a_{n-1}^{k-2}) + \beta(a_{n-1}^{k-1} + a_{n-1}^{k-2} + a_{n-1}^{k-3}) + \gamma(a_{n-1}^{k-2} + a_{n-1}^{k-3} + a_{n-1}^{k-4}) \\
&= \alpha a_n^{k-1} + \beta a_n^{k-2} + \gamma a_n^{k-3}.
\end{aligned}$$

□

Remark 9. Let us extend the sequence (a_k) for negative indices, so that let $a_k = 0$, if $k < 0$. From definition (1), the initial values of sequences of rows are $a_n^{-n} = a_0$, $a_n^{-n+1} = na_0 + a_1$, and $a_n^{-n+2} = \binom{n+1}{2} a_0 + na_1 + a_2$ with extension for negative indices. It can be easily proved by induction.

Theorem 10. *The terms in the diagonals can be described by the same ternary recurrence relation,*

$$a_n^{n+\ell} = \mathcal{A} a_{n-1}^{n+\ell-1} + \mathcal{B} a_{n-2}^{n+\ell-2} + \mathcal{C} a_{n-3}^{n+\ell-3} \qquad (3 \leq n,\ 0 \leq \ell). \tag{9}$$

Proof. We prove it by induction first on ℓ and second on n.

First, let $n = 3$. In case $\ell = 0, 1, 3$, we can check the recurrence (9) by computer. (For example the expression of a_3^3 contains more than hundred characters. Figure 4 shows a small part of the triangle based on initial elements x, y and z.) Now, we suppose that (9) holds for up to $\ell - 1$. We obtain

$$\begin{aligned}
a_3^{6+\ell} &= \alpha a_3^{5+\ell} + \beta a_3^{4+\ell} + \gamma a_3^{3+\ell} \\
&= \alpha\left(\mathcal{A} a_2^{4+\ell} + \mathcal{B} a_1^{3+\ell} + \mathcal{C} a_0^{2+\ell}\right) + \beta\left(\mathcal{A} a_2^{3+\ell} + \mathcal{B} a_1^{2+\ell} + \mathcal{C} a_0^{1+\ell}\right) \\
&\qquad + \gamma\left(\mathcal{A} a_2^{2+\ell} + \mathcal{B} a_1^{1+\ell} + \mathcal{C} a_0^{\ell}\right) \\
&= \mathcal{A}\left(\alpha a_2^{4+\ell} + \beta a_2^{3+\ell} + \gamma a_2^{2+\ell}\right) + \mathcal{B}\left(\alpha a_1^{3+\ell} + \beta a_1^{2+\ell} + \gamma a_1^{1+\ell}\right) \\
&\qquad + \mathcal{C}\left(\alpha a_0^{2+\ell} + \beta a_0^{1+\ell} + \gamma a_0^{\ell}\right) \\
&= \mathcal{A} a_2^{5+\ell} + \mathcal{B} a_1^{4+\ell} + \mathcal{C} a_0^{3+\ell}.
\end{aligned}$$

Second, we suppose in case any ℓ that (9) holds for up to $4 \le n-1$, and thus we have

$$\begin{aligned}
a_n^{n+\ell} &= a_{n-1}^{n+\ell-1} + a_{n-1}^{n+\ell} + a_{n-1}^{n+\ell+1} \\
&= \mathcal{A}a_{n-2}^{n+\ell-2} + \mathcal{B}a_{n-3}^{n+\ell-3} + \mathcal{C}a_{n-4}^{n+\ell-4} + \mathcal{A}a_{n-2}^{n+\ell-1} + \mathcal{B}a_{n-3}^{n+\ell-2} + \mathcal{C}a_{n-4}^{n+\ell-3} \\
&\qquad + \mathcal{A}a_{n-2}^{n+\ell} + \mathcal{B}a_{n-3}^{n+\ell-1} + \mathcal{C}a_{n-4}^{n+\ell-2} \\
&= \mathcal{A}\left(a_{n-2}^{n+\ell-2} + a_{n-2}^{n+\ell-1} + a_{n-2}^{n+\ell}\right) + \mathcal{B}\left(a_{n-3}^{n+\ell-3} + a_{n-3}^{n+\ell-2} + a_{n-3}^{n+\ell-1}\right) \\
&\qquad + \mathcal{C}\left(a_{n-4}^{n+\ell-4} + a_{n-4}^{n+\ell-3} + a_{n-4}^{n+\ell-2}\right) \\
&= \mathcal{A}a_{n-1}^{n+\ell-1} + \mathcal{B}a_{n-2}^{n+\ell-2} + \mathcal{C}a_{n-3}^{n+\ell-3}.
\end{aligned}$$

□

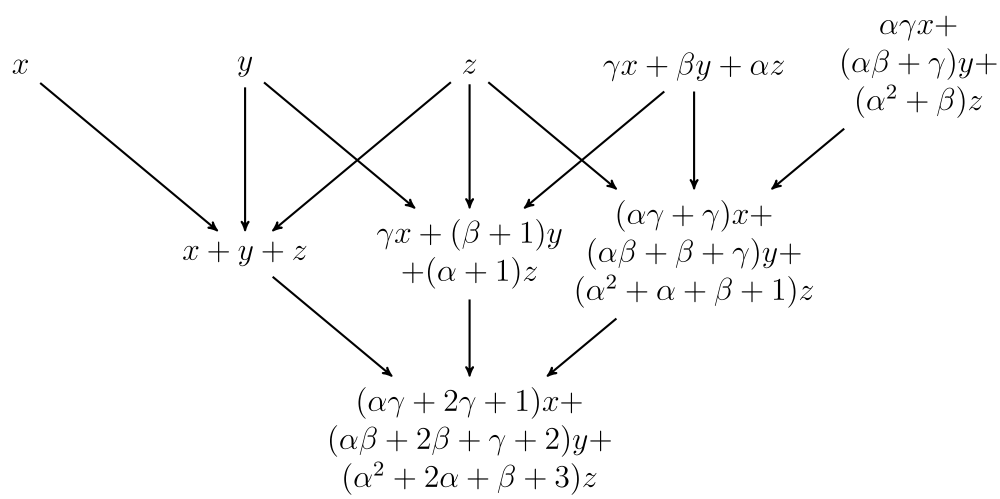

Figure 4: A part of the growing of $\mathcal{T}$

Theorem 10 for case $\ell = 0$ and Figure 4 provide the main statement of this section.

Corollary 11 (Main corollary)**.** *The trinomial transform sequence of a given ternary linear recurrent sequence (a_k) defined by (7) is the ternary linear recurrent sequence (b_k) defined by*

$$b_n = \mathcal{A}b_{n-1} + \mathcal{B}b_{n-2} + \mathcal{C}b_{n-3} \qquad (3 \le n,\ 0 \le \ell),$$

with initial values $b_0 = a_0$, $b_1 = a_0 + a_1 + a_2$ and $b_2 = (\alpha\gamma + 2\gamma + 1)a_0 + (\alpha\beta + 2\beta + \gamma + 2)a_1 + (\alpha^2 + 2\alpha + \beta + 3)a_2$.

Moreover, we can give a general statement for the linear homogenous recurrence sequences according to Barbero et al. [3].

Theorem 12. *Let (a_k) be a linear recurrent sequence of degree r with characteristic polynomial given by $p(t)$, and zeros $\omega_1, \omega_2, \ldots, \omega_r$ are all different. Then its trinomial transform sequence (b_k) is a linear recurrent sequence of degree r, and zeros of the characteristic polynomial of (b_k) are $\omega_1^2+\omega_1+1$, $\omega_2^2+\omega_2+1$, $\ldots$, $\omega_r^2+\omega_r+1$.*

Proof. Our proof is similar to the proof of Barbero et al. [3, Thm. 10]. If $p(t)$ has distinct complex zeros $\omega_1, \ldots, \omega_r$, then by Binet formula

$$a_n = \sum_{j=1}^{r} A_j \omega_j^n,$$

for some complex A_i derived from initial conditions [9, Thm. C.1. p. 33]. For all terms of (b_k), we have

$$b_n = \sum_{i=0}^{2n} \binom{n}{i}_2 a_i = \sum_{i=0}^{2n} \binom{n}{i}_2 \sum_{j=1}^{r} A_j \omega_j^i = \sum_{j=1}^{r} A_j (\omega_j^2+\omega_j+1)^n .$$

□

It is surprising that a ternary recurrence relation with rational coefficients holds for the finite column sequences $(a_i^k)_{i=0}^k$. The following theorem formulates it precisely.

Theorem 13. *The terms in the column k can be described by the ternary recurrence relation*

$$a_n^k = \frac{\mathcal{P}}{\gamma} a_{n-1}^k + \frac{\mathcal{Q}}{\gamma} a_{n-2}^k + \frac{\mathcal{C}}{\gamma} a_{n-3}^k \qquad (3 \le n \le k),$$

where $\mathcal{P} = \alpha\gamma - \beta + 3\gamma$ and $\mathcal{Q} = \alpha\beta - 2\alpha\gamma + \beta\gamma - \alpha + 2\beta$.

Proof. The proof is very similar to that of Theorem 10. We use the induction method.

First, let $n = 3$. For cases $k = 3, 4, 5$ we can check the equation

$$\gamma a_3^k = \mathcal{P} a_2^k + \mathcal{Q} a_1^k + \mathcal{C} a_0^k. \tag{10}$$

Now, we suppose that the equation (10) holds up to $k-1$. So the condition of induction yields the equations

$$\begin{aligned}
\alpha(\gamma a_3^{k-1}) &= \alpha(\mathcal{P} a_2^{k-1} + \mathcal{Q} a_1^{k-1} + \mathcal{C} a_0^{k-1}),\\
\beta(\gamma a_3^{k-2}) &= \beta(\mathcal{P} a_2^{k-2} + \mathcal{Q} a_1^{k-2} + \mathcal{C} a_0^{k-2}),\\
\gamma(\gamma a_3^{k-3}) &= \gamma(\mathcal{P} a_2^{k-3} + \mathcal{Q} a_1^{k-3} + \mathcal{C} a_0^{k-3}).
\end{aligned}$$

Summing the equations and applying Theorem 8 we obtain that (10) holds for any k.

Second, if we suppose that

$$\gamma a_{n-1}^k = \mathcal{P} a_{n-2}^k + \mathcal{Q} a_{n-3}^k + \mathcal{C} a_{n-4}^k \tag{11}$$

holds for any $4 \le n \le k$, then we have

$$\begin{aligned}
\gamma a_{n-1}^{k-1} &= \mathcal{P} a_{n-2}^{k-1} + \mathcal{Q} a_{n-3}^{k-1} + \mathcal{C} a_{n-4}^{k-1}, \\
\gamma a_{n-1}^{k} &= \mathcal{P} a_{n-2}^{k} + \mathcal{Q} a_{n-3}^{k} + \mathcal{C} a_{n-4}^{k}, \\
\gamma a_{n-1}^{k+1} &= \mathcal{P} a_{n+1}^{k-3} + \mathcal{Q} a_{n+1}^{k-3} + \mathcal{C} a_{n-4}^{k+1}.
\end{aligned}$$

Using (1) for the sum of the equations we obtain

$$\gamma a_n^k = \mathcal{P} a_{n-1}^k + \mathcal{Q} a_{n-2}^k + \mathcal{C} a_{n-3}^k.$$

□

4.1 Sums and alternating sums of columns

We give (s_n) as a linear recurrent sequence with coefficients given by the coefficients of sequences (a_n) and (b_n). Recall (s_n) was defined in (5).

Theorem 14. *If $n \ge 6$, then the sequence (s_n) can be described by the sixth order homogeneous linear recurrence relation*

$$\begin{aligned}
s_n = (\alpha + \mathcal{A}) s_{n-1} + (\mathcal{B} - \alpha\mathcal{A} + \beta) s_{n-2} - (\alpha\mathcal{B} + \beta\mathcal{A} - \mathcal{C} - \gamma) s_{n-3} \\
- (\alpha\mathcal{C} + \beta\mathcal{B} + \gamma\mathcal{A}) s_{n-4} - (\beta\mathcal{C} + \gamma\mathcal{B}) s_{n-5} - \gamma\mathcal{C} s_{n-6}. \quad (12)
\end{aligned}$$

Proof. First, we prove that the recurrence relations of sequences (a_n^k), $(a_n^{n+\ell})$ $(\ell \ge 0)$ are the same as relation (12). Extend relation (8) to sixth order homogeneous linear recurrence relation by sum of a_n^k, $\mathcal{A} a_n^{k-1}$, $\mathcal{B} a_n^{k-2}$ and $\mathcal{C} a_n^{k-3}$, moreover extend (9) also by sum of $a_n^{n+\ell}$, $\alpha a_{n-1}^{n+\ell-1}$, $\beta a_{n-2}^{n+\ell-2}$ and $\gamma a_{n-3}^{n+\ell-3}$. Then we obtain the same coefficients as in (12).

Second, we give the connection between s_n and s_{n-2}. It is from

$$\begin{aligned}
s_{n-1} - a_0^{n-1} &= \sum_{i=1}^{n-1} a_i^{n-1} = \sum_{i=1}^{n-1} \left(a_{i-1}^{n-2} + a_{i-1}^{n-1} + a_{i-1}^{n} \right) = \sum_{i=0}^{n-2} a_i^{n-2} + \sum_{i=0}^{n-2} a_i^{n-1} + \sum_{i=0}^{n-2} a_i^{n} \\
&= s_{n-2} + s_{n-1} - a_{n-1}^{n-1} + s_n - a_{n-1}^{n} - a_n^n.
\end{aligned}$$

Reordering the equation, we have

$$s_n = -s_{n-2} + a_{n-1}^{n-1} + a_{n-1}^{n} + a_n^n - a_0^{n-1}.$$

Third, we have finished the preparations to prove the theorem by induction on n. Because of the complexity in case $n = 6$, we can check the recurrence (12) by computer. Then we suppose that (12) holds for cases up to $n-1$. We obtain

$$\begin{aligned}
s_n &= -s_{n-2} + a_{n-1}^{n-1} + a_{n-1}^{n} + a_n^n - a_0^{n-1} \\
&= -((\alpha+\mathcal{A})s_{n-3} - (\mathcal{B}-\alpha\mathcal{A}+\beta)s_{n-4} - (\alpha\mathcal{B}+\beta\mathcal{A}-\mathcal{C}-\gamma)s_{n-5} \\
&\qquad -(\alpha\mathcal{C}+\beta\mathcal{B}+\gamma\mathcal{A})s_{n-6} - (\beta\mathcal{C}+\gamma\mathcal{B})s_{n-7} - \gamma\mathcal{C}s_{n-8}) \\
&\quad +(\alpha+\mathcal{A})a_{n-2}^{n-2} + (\mathcal{B}-\alpha\mathcal{A}+\beta)a_{n-3}^{n-3} + (\alpha\mathcal{B}+\beta\mathcal{A}-\mathcal{C}-\gamma)a_{n-4}^{n-4} \\
&\qquad +(\alpha\mathcal{C}+\beta\mathcal{B}+\gamma\mathcal{A})a_{n-5}^{n-5} + (\beta\mathcal{C}+\gamma\mathcal{B})a_{n-6}^{n-6} + \gamma\mathcal{C}a_{n-7}^{n-7} \\
&\quad + \cdots \\
&\quad -(\alpha+\mathcal{A})a_0^{n-2} - (\mathcal{B}-\alpha\mathcal{A}+\beta)a_0^{n-3} - (\alpha\mathcal{B}+\beta\mathcal{A}-\mathcal{C}-\gamma)a_0^{n-4} \\
&\qquad -(\alpha\mathcal{C}+\beta\mathcal{B}+\gamma\mathcal{A})a_0^{n-5} - (\beta\mathcal{C}+\gamma\mathcal{B})a_0^{n-6} - \gamma\mathcal{C}a_0^{n-7} \\
&= (\alpha+\mathcal{A})(-s_{n-3} + a_{n-2}^{n-2} + a_{n-2}^{n-1} + a_{n-1}^{n-1} - a_0^{n-2}) \\
&\quad + (\mathcal{B}-\alpha\mathcal{A}+\beta)(-s_{n-4} + a_{n-3}^{n-3} + a_{n-3}^{n-2} + a_{n-2}^{n-2} - a_0^{n-3}) \\
&\quad - (\alpha\mathcal{B}+\beta\mathcal{A}-\mathcal{C}-\gamma)(-s_{n-5} + a_{n-4}^{n-4} + a_{n-4}^{n-3} + a_{n-3}^{n-3} - a_0^{n-4}) \\
&\quad - (\alpha\mathcal{C}+\beta\mathcal{B}+\gamma\mathcal{A})(-s_{n-6} + a_{n-5}^{n-5} + a_{n-5}^{n-4} + a_{n-4}^{n-4} - a_0^{n-5}) \\
&\quad - (\beta\mathcal{C}+\gamma\mathcal{B})(-s_{n-7} + a_{n-6}^{n-6} + a_{n-6}^{n-5} + a_{n-5}^{n-5} - a_0^{n-6}) \\
&\quad - \gamma\mathcal{C}(-s_{n-8} + a_{n-7}^{n-7} + a_{n-7}^{n-6} + a_{n-6}^{n-6} - a_0^{n-7}) \\
&= (\alpha+\mathcal{A})s_{n-1} + (\mathcal{B}-\alpha\mathcal{A}+\beta)s_{n-2} - (\alpha\mathcal{B}+\beta\mathcal{A}-\mathcal{C}-\gamma)s_{n-3} \\
&\quad -(\alpha\mathcal{C}+\beta\mathcal{B}+\gamma\mathcal{A})s_{n-4} - (\beta\mathcal{C}+\gamma\mathcal{B})s_{n-5} - \gamma\mathcal{C}s_{n-6}.
\end{aligned}$$

□

Let $(\bar{s}_n)$ be the alternating sum of the values of columns n, so that

$$\bar{s}_n = \sum_{i=0}^{n} (-1)^i a_i^n = \sum_{i=0}^{n} \sum_{k=0}^{2i} (-1)^i \binom{i}{k}_2 a_0^{n+k-i}.$$

Theorem 15. *If $n \geq 6$, then the alternating sum sequences $(\bar{s}_n)$ can be described by the sixth order homogeneous linear recurrence relation*

$$\bar{s}_n = (\alpha-\mathcal{A})\bar{s}_{n-1} + (\mathcal{B}+\alpha\mathcal{A}+\beta)\bar{s}_{n-2} - (\alpha\mathcal{B}-\beta\mathcal{A}+\mathcal{C}-\gamma)\bar{s}_{n-3} + (\alpha\mathcal{C}-\beta\mathcal{B}+\gamma\mathcal{A})\bar{s}_{n-4} + (\beta\mathcal{C}-\gamma\mathcal{B})\bar{s}_{n-5} + \gamma\mathcal{C}\bar{s}_{n-6}.$$

Proof. Row by row the signs of the terms in the alternating sums change in directions parallel to the diagonal, hence we have to change the sign of $\mathcal{A}$ and $\mathcal{C}$ in the summation relation (12). They have influence on the signs of every next and third terms, respectively. In a row the signs of the terms do not change. □

5 Examples

In this section, we give some examples for the trinomial transform triangles generated by ternary linear recurrent sequences.

5.1 Fibonacci sequence

Let the base sequence be the Fibonacci sequence $(F_k)_{k=0}^{\infty}$ defined by $F_0 = 0$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$, $k \geq 2$ (A000045). If we extend it to a ternary recurrence (with sum of F_k and F_{k-1}), then we obtain that $F_k = 2F_{k-1} - F_{k-3}$, $k \geq 3$. So let $a_0^k = F_k$ for any k, moreover $\alpha = 2$, $\beta = 0$ and $\gamma = -1$.

When we substitute the initial values into $\mathcal{T}$, then we have the Fibonacci trinomial transform triangle, see Table 1.

	0	1	2	3	4	5	6	7	8	9
0	0	1	1	2	3	5	8	13	21	34
1		2	4	6	10	16	26	42	68	110
2			12	20	32	52	84	136	220	356
3				64	104	168	272	440	712	1152
4					336	544	880	1424	2304	3728
5						1760	2848	4608	7456	12064
6							9216	14912	24128	39040
7								48256	78080	126336
8									252672	408832
9										1323008
s_n	**0**	**3**	**17**	**92**	**485**	**2545**	**13334**	**69831**	**365661**	**1914660**
$\bar{s}_n$	0	-1	9	-48	257	-1343	7042	-36861	193029	-1010680

Table 1: Fibonacci trinomial transform triangle

Having observed the rows of Table 1, we have found that they are the well-known i-Fibonacci sequences, defined by $F_0^{[i]} = 0$, $F_1^{[i]} = i$ and $F_j^{[i]} = F_{j-1}^{[i]} + F_{j-2}^{[i]}$, $j \geq 2$. This result is expressed in the following theorem.

Theorem 16. *The terms of the rows in Table 1 are the terms of the 2^n-Fibonacci sequences, so that*

$$a_n^k = F_{k+n}^{[2^n]}.$$

Proof. Obviously, for case $n = 1$ the theorem is true. Let us suppose that the result is also true for $n-1$. As the relation between the Fibonacci and i-Fibonacci sequences is $i \cdot F_k = F_k^{[i]}$, we gain

$$\begin{aligned} a_n^k &= a_{n-1}^{k-1} + a_{n-1}^k + a_{n-1}^{k+1} = F_{k-1+n-1}^{[2^{n-1}]} + F_{k+n-1}^{[2^{n-1}]} + F_{k+1+n-1)}^{[2^{n-1}]} \\ &= 2F_{k-n}^{[2^{n-1}]} = F_{k-n}^{[2^n]}. \end{aligned}$$

□

For all the directions parallel to the diagonal of Table 1 and for the trinomial transform sequence, we obtain from (9) and Theorem 11 the following corollaries.

Corollary 17. *If $2 \le n \le k$, then*

$$a_n^k = 6a_{n-1}^{k-1} - 4a_{n-2}^{k-2}.$$

Corollary 18. *The trinomial transform sequence (b_n) of the Fibonacci sequence is the binary sequence $b_n = 6b_{n-1} - 4b_{n-2}$ with initial elements $b_0 = 0$, $b_1 = 2$. (The main diagonal of Table 1.)*

In OEIS the trinomial transform of the Fibonacci numbers is the sequence A082761 $(1, 4, 20, 104, \ldots)$, which can be seen in Table 1 as the second diagonal.

We obtain the following corollaries from the theorems of the previous section.

Corollary 19. *If $2 \le n \le k$, then*

$$\begin{aligned} F_{k+n}^{[2^n]} &= 2F_{k+n-1}^{[2^{n-1}]} + 4F_{k+n-2}^{[2^{n-2}]}, \\ F_{k+n}^{[2^n]} &= 6F_{k+n-2}^{[2^{n-1}]} - 4F_{k+n-4}^{[2^{n-2}]}. \end{aligned}$$

Corollary 20. *For the sum and alternating sum of columns, we have*

$$\begin{aligned} s_n &= \sum_{i=0}^{n} F_{i+n}^{[2^n]} = \sum_{i=0}^{2n} \left[\binom{n}{i}\right]_2 F_i = \sum_{i=0}^{n}\sum_{k=0}^{2i} \binom{i}{k}_2 F_{n+k-i}, \\ \bar{s}_n &= \sum_{i=0}^{n} (-1)^i F_{i+n}^{[2^n]} = \sum_{i=0}^{n} (-1)^i \sum_{k=0}^{2i} \binom{i}{k}_2 F_{n+k-i}. \end{aligned}$$

Moreover, if $n \ge 4$, then

$$\begin{aligned} s_k &= 7s_{k-1} - 9s_{k-2} - 2s_{k-3} + 4s_{k-4}, \\ \bar{s}_k &= -5\bar{s}_{k-1} + 3\bar{s}_{k-2} + 10\bar{s}_{k-3} + 4\bar{s}_{k-4}. \end{aligned}$$

with initial values $s_0 = 0$, $s_1 = 3$, $s_2 = 17$, $s_3 = 92$, and $\bar{s}_0 = 0$, $\bar{s}_1 = -1$, $\bar{s}_2 = 9$, $\bar{s}_3 = -48$.

5.2 Tribonacci sequence

Let the base sequence a_0^k be the Tribonacci sequence (A000073), defined by $t_k = t_{k-1} + t_{k-2} + t_{k-3}$, $k \ge 3$ with initial values $t_0 = 0$, $t_1 = 0$, $t_2 = 1$. So let $a_0^k = t_k$ and $\alpha = \beta = \gamma = 1$. The Tribonacci trinomial transform triangle is depicted in Table 2.

Easy to see that $a_n^k = a_{n-1}^{k+2} = \cdots = a_0^{k+2n} = t_{k+2n}$, and from it with the statements in the previous section, we have the following corollaries.

Corollary 21. *In the case $3 \le n \le k$, we obtain*

$$\begin{aligned} a_n^k &= 7a_{n-1}^{k-1} - 5a_{n-2}^{k-2} + a_{n-3}^{k-3}, \\ a_n^k &= 3a_{n-1}^k + a_{n-2}^k + a_{n-3}^k \end{aligned}$$

Moreover, the terms of sequence $(a_n^k)_{n=0}^k$ are every second terms of the Tribonacci numbers, so that $a_n^n = t_{3n}$.

	0	1	2	3	4	5	6	7	8	9
0	0	0	1	1	2	4	7	13	24	44
1		1	2	4	7	13	24	44	81	149
2			7	13	24	44	81	149	274	504
3				44	81	149	274	504	927	1705
4					274	504	927	1705	3136	5768
5						1705	3136	5768	10609	19513
6							10609	19513	35890	66012
7								66012	121415	223317
8									410744	755476
9										2555757
s_k	**0**	**1**	**10**	**62**	**388**	**2419**	**15058**	**93708**	**583100**	**3628245**
$\bar{s}_k$	0	-1	6	-34	212	-1315	8190	-50948	317036	-1972637

Table 2: Tribonacci trinomial transform triangle

Corollary 22. *The trinomial transform sequence (b_n) of the Tribonacci sequence is the ternary sequence $b_n = 6b_{n-1} - 4b_{n-2} + b_{n-3}$ with initial terms $b_0 = 0$, $b_1 = 1$, $b_2 = 7$. (The main diagonal of Table 2.)*

In OEIS the trinomial transform of the Tribonacci numbers is the sequence A192806 $(1, 1, 4, 24, 149, 927, \ldots)$, which is the third diagonal in Table 1 with an additional first term.

Corollary 23. *For the sum and alternating sum of columns, we have*

$$\begin{aligned} s_n &= \sum_{i=0}^{n} t_{n+2i} = \sum_{i=0}^{2n} \left[\binom{n}{i}\right]_2 t_i = \sum_{i=0}^{n}\sum_{k=0}^{2i} \binom{i}{k}_2 t_{n+k-i}, \\ \bar{s}_n &= \sum_{i=0}^{n} (-1)^i t_{n+2i} = \sum_{i=0}^{n} (-1)^i \sum_{k=0}^{2i} \binom{i}{k}_2 t_{n+k-i}. \end{aligned}$$

In addition,

$$\begin{aligned} s_n &= 8s_{n-1} - 11s_{n-2} - 3s_{n-4} + 4s_{n-5} - s_{n-6} \qquad (n \geq 6), \\ \bar{s}_n &= -6\bar{s}_{n-1} + 3\bar{s}_{n-2} + 12\bar{s}_{n-2} + 13\bar{s}_{n-4} + 6\bar{s}_{n-5} + \bar{s}_{n-6} \qquad (n \geq 6), \end{aligned}$$

with initial values $s_0 = 0$, $s_1 = 1$, $s_2 = 10$, $s_3 = 62$, $s_4 = 388$, $s_5 = 2419$, and $\bar{s}_0 = 0$, $\bar{s}_1 = -1$, $\bar{s}_2 = 6$, $\bar{s}_3 = -34$, $\bar{s}_4 = 212$, $\bar{s}_5 = -1315$.

5.3 The constant sequence 1

In this subsection, we give Table 3 generated by the constant sequence 1 (A000012) and the expressions that prove Theorem 1.

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		3	3	3	3	3	3	3	3	3
2			9	9	9	9	9	9	9	9
3				27	27	27	27	27	27	27
4					81	81	81	81	81	81
5						243	243	243	243	243
6							729	729	729	729
7								2187	2187	2187
8									6561	6561
9										19683
s_k	**1**	**4**	**13**	**40**	**121**	**364**	**1093**	**3280**	**9841**	**29524**
$\bar{s}_k$	1	-2	7	-20	61	-182	547	-1640	4921	-14762

Table 3: Trinomial transform triangle generated by the constant sequence 1

One can easily see that $a_n^k = 3^n$ (A000244). It implies that the trinomial transform sequence of the constant sequence 1 is the sequence 3^n. Moreover, the columns also form geometric sequences with common ratio 3. Thus

$$\begin{aligned} s_n &= \sum_{i=0}^{2n} \left[\binom{n}{i}\right]_2 = \sum_{i=0}^{n}\sum_{k=0}^{2i}\binom{i}{k}_2 = \sum_{i=0}^{n} 3^i = \frac{3^{n+1}-1}{2} \qquad (\text{A003462}), \\ \bar{s}_n &= \sum_{i=0}^{n}\sum_{k=0}^{2i}\binom{i}{k}_2 (-1)^i = \sum_{i=0}^{n}(-3)^i = \frac{3(-3)^n+1}{4} \qquad (\text{A014983}), \end{aligned}$$

5.4 Natural numbers

Let $a_0^k = k$, the non-negative integers (A001477). We obtain that $a_n^k = k3^n$, and the trinomial transform of the natural sequence is the sequence $n3^n$ (A036290).

In this case, the columns also form geometric sequences with common ratio 3 (see Table 4). Thus

$$\begin{aligned} s_n &= \sum_{i=0}^{2n} i\left[\binom{n}{i}\right]_2 = \sum_{i=0}^{n}\sum_{k=0}^{2i} n\binom{i}{k}_2 = \sum_{i=0}^{n} n3^i = n\frac{3^{n+1}-1}{2}, \\ \bar{s}_n &= \sum_{i=0}^{n}\sum_{k=0}^{2i}\binom{i}{k}_2 (-1)^i n = \sum_{i=0}^{n} n(-3)^i = n\frac{3(-3)^n+1}{4}. \end{aligned}$$

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1		3	6	9	12	15	18	21	24	27
2			18	27	36	45	54	63	72	81
3				81	108	135	162	189	216	243
4					324	405	486	567	648	729
5						1215	1458	1701	1944	2187
6							4374	5103	5832	6561
7								15309	17496	19683
8									52488	59049
9										177147
s_k	**0**	**4**	**26**	**120**	**484**	**1820**	**6558**	**22960**	**78728**	**265716**
$\bar{s}_k$	0	-2	14	-60	244	-910	3282	-11480	39368	-132858

Table 4: Trinomial transform triangle generated by natural numbers

5.5 Two other sequences

In this subsection, we give two cases without tables, whose trinomial transform sequences are the all 1's sequence and the natural numbers mentioned in the previous subsections. We gain the results with very easy calculations.

First, let $a_0^k = (-1)^k$ (A033999). Then $a_n^k = (-1)^{n+k}$ and $b_n = 1$.

$$s_n = \sum_{i=0}^{2n} (-1)^i \left[\binom{n}{i}\right]_2 = \sum_{i=0}^{n}\sum_{k=0}^{2i} (-1)^{n+k-i} \binom{i}{k}_2 = \sum_{i=0}^{n} (-1)^{n+i} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

$$\bar{s}_n = \sum_{i=0}^{n}\sum_{k=0}^{2i} \binom{i}{k}_2 (-1)^i = \sum_{i=0}^{n} 1 = n+1.$$

Second, let $a_0^k = (-1)^k k$ (A038608). Then $a_n^k = (-1)^{n+k} k$ and $b_n = n$.

$$s_n = \sum_{i=0}^{2n} (-1)^i i \left[\binom{n}{i}\right]_2 = \sum_{i=0}^{n}\sum_{k=0}^{2i} (-1)^{n+k-i} n \binom{i}{k}_2 = \sum_{i=0}^{n} (-1)^{n+i} n$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd or } n = 0; \\ n, & \text{otherwise.} \end{cases}$$

$$\bar{s}_n = n \sum_{i=0}^{n}\sum_{k=0}^{2i} \binom{i}{k}_2 (-1)^i = \sum_{i=0}^{n} n = n^2 + n \qquad \text{(A002378).}$$

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