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Algebraic Generating Functions for Languages Avoiding Riordan Patterns

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Abstract

We study the languages $\mathfrak{L}^{[\mathfrak{p}]} \subset \{0,1\}^*$ of binary words w avoiding a given pattern $\mathfrak{p}$ such that $|w|_0 \leq |w|_1$ for every $w \in \mathfrak{L}^{[\mathfrak{p}]}$, where $|w|_0$ and $|w|_1$ correspond to the number of 0-bits and 1-bits in the word w, respectively. In particular, we concentrate on patterns $\mathfrak{p}$ related to the concept of Riordan arrays. These languages are not regular, but can be enumerated by algebraic generating functions corresponding to many integer sequences that were previously unlisted in the On-Line Encyclopedia of Integer Sequences. We give explicit formulas for these generating functions, expressed in terms of the autocorrelation polynomial of $\mathfrak{p}$, and also give explicit formulas for the coefficients of some particular patterns, algebraically and combinatorially.

1 Introduction

In this paper we study the languages $\mathfrak{L}^{[\mathfrak{p}]} \subset \{0,1\}^*$ of binary words avoiding a given binary pattern $\mathfrak{p}$, having the property that $|w|_0 \leq |w|_1$ for every word $w \in \mathfrak{L}^{[\mathfrak{p}]}$, where $|w|_0$ and $|w|_1$ correspond to the number of 0-bits and 1-bits in the word w, respectively. The notion of a pattern can be formalized in several ways. In this paper we consider *factor patterns*, that is, patterns whose letters must appear in exact order and contiguously in the sequence under observation. The set of binary words avoiding a pattern, without the restriction $|w|_0 \leq |w|_1$, is defined by a regular language, and can be enumerated in terms of the number of 1-bits and 0-bits by using classical results (see, e.g., Guibas and Odlyzko [4, 5] and Sedgewick and Flajolet [11]). However, when we consider the additional restriction that the words have no more 0-bits than 1-bits, the language is no longer regular and enumerating it is a harder problem.

In this paper we are interested in *Riordan patterns*, a concept defined by Sprugnoli and the first author [9] in terms of the *autocorrelation polynomial* $C^{[\mathfrak{p}]}(x,y)$ of pattern $\mathfrak{p} = p_0 \cdots p_{h-1}$. The coefficients of this polynomial are given by the *autocorrelation vector* associated to $\mathfrak{p}$, that is, the vector $c = (c_0, \ldots, c_{h-1})$ of bits defined in terms of Iverson's bracket notation (for a predicate P, the expression $[\![P]\!]$ has value 1 if P is true and 0 otherwise) as follows:

$$c_i = [\![p_0 p_1 \cdots p_{h-1-i} = p_i p_{i+1} \cdots p_{h-1}]\!];$$

or in words, the bit c_i is determined by shifting $\mathfrak{p}$ to the right by i positions, setting $c_i = 1$ if and only if the remaining letters match the original. For example, when $\mathfrak{p} = 10101$ the autocorrelation vector is $c = (1,0,1,0,1)$, as illustrated in Table 1, and $C^{[\mathfrak{p}]}(x,y) = 1 + xy + x^2y^2$, namely we add a term $x^j y^i$ for each tail of the pattern with j 1-bits and i 0-bits, where $c_{j+i} = 1$.

1	0	1	0	1	Tails				c_i
1	0	1	0	1					1
	1	0	1	0	1				0
		1	0	1	0	1			1
			1	0	1	0	1		0
				1	0	1	0	1	1

Table 1: The autocorrelation vector for $\mathfrak{p} = 10101$.

For each pattern $\mathfrak{p}$, we can compute the *complement* pattern $\bar{\mathfrak{p}}$ by changing every 1 to 0 and every 0 to 1; for example, if $\mathfrak{p} = 10101$ then $\bar{\mathfrak{p}} = 01010$, therefore $C^{[\mathfrak{p}]}(x,y) = C^{[\bar{\mathfrak{p}}]}(y,x)$.

Addition of constraints to the nature of a pattern $\mathfrak{p}$ yields the following definition:

Definition 1 (Riordan pattern)**.** We say that $\mathfrak{p} = p_0 \cdots p_{h-1}$ is a Riordan pattern if and only if

$$C^{[\mathfrak{p}]}(x,y) = C^{[\bar{\mathfrak{p}}]}(y,x) = \sum_{i=0}^{\lfloor (h-1)/2 \rfloor} c_{2i} x^i y^i, \quad \text{with} \quad |n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]}| \in \{0,1\}$$

where $n_1^{[\mathfrak{p}]}$ and $n_0^{[\mathfrak{p}]}$ correspond to the number of 1-bits and 0-bits in the pattern, respectively.

For example, Table 1 corresponds to a Riordan pattern and $\mathfrak{p} = 1100110110011000$ is another Riordan pattern having $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]} = 8$ and $C^{[\mathfrak{p}]}(x,y) = 1$. Moreover, in Table 2 we give all the Riordan patterns of length 7 with first bit equal to 1 and their correlation polynomials, the corresponding complement patterns can be easily determined.

$\mathfrak{p}$	$C^{[\mathfrak{p}]}(x,y)$
$1010100, 1011000$ $1011100, 1100010$ $1100100, 1101000$ $1101010, 1101100$ $1110000, 1110010$ $1110100, 1111000$	1
$1001100, 1100110$	$1 + x^2y^2$
$1000111, 1001011$ $1001101, 1010011$ $1011001, 1100101$ $1101001, 1110001$	$1 + x^3y^3$
1010101	$1 + xy + x^2y^2 + x^3y^3$

Table 2: The Riordan patterns of length 7 with first bit equal to 1 and their correlation polynomials

The name *Riordan* in the above definition is due to the connection with the well-known concept of *Riordan arrays*. We briefly recall that a Riordan array is an infinite lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$, defined by a pair of formal power series $(d(t), h(t))$, such that $d(0) \neq 0, h(0) = 0, h'(0) \neq 0$ and the generic element $d_{n,k}$ is the coefficient of monomial t^n in the series expansion of $d(t)h(t)^k$. Formally,

$$d_{n,k} = [t^n]d(t)h(t)^k, \qquad n, k \geq 0$$

where $d_{n,k} = 0$ for $k > n$. These arrays were introduced in 1991 by Shapiro, Getu, Woan and Woodson [12], with the aim of defining a class of infinite lower triangular arrays with properties analogous to those of the Pascal triangle. Since then they have attracted, and continue to attract, much attention in the literature. Some of their structural properties were studied by Rogers, Sprugnoli, Verri and the first author [8], and additional properties were recently analyzed by Luzón, Morón, Sprugnoli and the first author [7]. In particular, we recall that the bivariate generating function enumerating the sequence $(d_{n,k})_{n,k\in\mathbb{N}}$ is

$$R(t, w) = \sum_{n,k\in\mathbb{N}} d_{n,k}t^n w^k = \frac{d(t)}{1 - wh(t)} \tag{1}$$

An important property of Riordan array concerns the computation of combinatorial sums. A first paper in this direction is due to Sprugnoli [13], while the case dealing with implicit Riordan arrays is treated by Sprugnoli, Verri and the first author [10]. In particular we have the following result:

$$\sum_{k=0}^{n} d_{n,k}f_k = [t^n]d(t)f(h(t)) \quad \text{or} \quad (d(t), h(t)) * f(t) = d(t)f(h(t)), \tag{2}$$

that is, every combinatorial sum involving a Riordan array can be computed by extracting the coefficient of t^n from the series expansion of $d(t)f(h(t))$, where $f(t) = \mathcal{G}(f_k) = \sum_{k\geq 0} f_k t^k$ is the generating function of the sequence $(f_k)_{k\in\mathbb{N}}$ and the symbol $\mathcal{G}$ denotes the generating function operator. Due to its importance, relation (2) is often called the *fundamental rule* of Riordan arrays. In this paper, the notation $(f_k)_k$ will be used as an abbreviation of $(f_k)_{k\in\mathbb{N}}$.

Coming back to the languages $\mathfrak{L}^{[\mathfrak{p}]} \subset \{0,1\}^*$ of binary words avoiding a pattern $\mathfrak{p}$, let $R^{[\mathfrak{p}]}_{n,k}$ be the number of words avoiding $\mathfrak{p}$ and having n 1-bits and $n-k$ 0-bits; additionally, let $\mathcal{R}^{[\mathfrak{p}]} = \left(R^{[\mathfrak{p}]}_{n,k}\right)_{n,k\in\mathbb{N}}$ the enclosing matrix. The following theorem, which was proved by Sprugnoli and the first author [9], shows the importance of Riordan patterns:

Theorem 2. *Matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ are Riordan arrays if and only if $\mathfrak{p}$ is a Riordan pattern.*

By the previous theorem, matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ can be defined as

$$\mathcal{R}^{[\mathfrak{p}]} = (d^{[\mathfrak{p}]}(t), h^{[\mathfrak{p}]}(t)) \text{ and } \mathcal{R}^{[\bar{\mathfrak{p}}]} = (d^{[\bar{\mathfrak{p}}]}(t), h^{[\bar{\mathfrak{p}}]}(t))$$

for the appropriate $d^{[\mathfrak{p}]}$, $h^{[\mathfrak{p}]}$, $d^{[\bar{\mathfrak{p}}]}$, $h^{[\bar{\mathfrak{p}}]}$, given a Riordan pattern $\mathfrak{p}$; moreover, they represent the lower and upper part of the array $\mathcal{F}^{[\mathfrak{p}]} = (F^{[\mathfrak{p}]}_{n,k})_{n,k\in\mathbb{N}}$, where $F^{[\mathfrak{p}]}_{n,k}$ denotes the number of words avoiding pattern $\mathfrak{p}$ and having n 1-bits and k 0-bits . For the sake of clarity, Tables 3, 4 and 5 illustrate some rows for the matrices $\mathcal{F}^{[\mathfrak{p}]}$, $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$, where $\mathfrak{p} = 10101$.

Remark 3. Riordan patterns are not the only patterns related to Riordan arrays; for example, given the pattern $\mathfrak{p} = 0100100$ corresponding to $C^{[\mathfrak{p}]}(x,y) = 1 + xy^2 + x^2y^4$, matrix $\mathcal{R}^{[\mathfrak{p}]}$ is still a Riordan array but $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ is not, as illustrated by Baccherini, Sprugnoli and the first author [2, Example 5.4]. However, in these situations it is not easy to find functions $d^{[\mathfrak{p}]}(t)$ and $h^{[\mathfrak{p}]}(t)$, while for Riordan patterns it is always possible, as shown in Theorems 2 and 4.

n/k	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28	36
3	1	4	9	18	32	52	79	114
4	1	5	13	30	60	109	184	293
5	1	6	18	46	102	204	377	654
6	1	7	24	67	163	354	708	1324
7	1	8	31	94	248	580	1245	2490

Table 3: The matrix $\mathcal{F}^{[\mathfrak{p}]}$ for $\mathfrak{p} = 10101$

n/k	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	6	3	1					
3	18	9	4	1				
4	60	30	13	5	1			
5	204	102	46	18	6	1		
6	708	354	163	67	24	7	1	
7	2490	1245	580	248	94	31	8	1

Table 4: $\mathcal{R}^{[\mathfrak{p}]}$ for $\mathfrak{p} = 10101$

n/k	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	6	3	1					
3	18	10	4	1				
4	60	32	15	5	1			
5	204	109	52	21	6	1		
6	708	377	184	79	28	7	1	
7	2490	1324	654	293	114	36	8	1

Table 5: $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ for $\mathfrak{p} = 10101$

As already observed, the enumeration of the set of binary words avoiding a pattern, without the restriction about the number of 1-bits and 0-bits can be done by using classical results and gives the following rational bivariate generating function for the sequence $(F^{[\mathfrak{p}]}_{n,k})_{n,k\in\mathbb{N}}$:

$$F^{[\mathfrak{p}]}(x,y) = \frac{C^{[\mathfrak{p}]}(x,y)}{(1-x-y)C^{[\mathfrak{p}]}(x,y) + x^{n_1^{[\mathfrak{p}]}} y^{n_0^{[\mathfrak{p}]}}},$$

where $n_1^{[\mathfrak{p}]}$ and $n_0^{[\mathfrak{p}]}$ correspond to the number of 1-bits and 0-bits, respectively, and $C^{[\mathfrak{p}]}(x,y)$ is the autocorrelation polynomial, all relative to pattern $\mathfrak{p}$. Consequently, $F^{[\mathfrak{p}]}(t,1)$ and $F^{[\mathfrak{p}]}(t,t)$ count the words avoiding $\mathfrak{p}$ according to the number of 1-bits and to length of each word, respectively.

Using the theory of Riordan arrays and the results by Sprugnoli and the first author [9], we give explicit algebraic generating functions enumerating the set of binary words avoiding a Riordan pattern with the restriction $|w|_0 \le |w|_1$ according to various parameters, in particular to the number of 1-bits and to the words length. Most of the corresponding sequences are new to the On-Line Encyclopedia of Integer Sequences (OEIS)[1] [6]; moreover, we also give explicit formulas for the coefficients of some particular patterns by providing algebraic and combinatorial proofs.

Finally, our results can be interpreted in the theory of paths and codes in light of the bijection among binary words and paths, which maps a 0-bit to a south-east step $\diagdown$ and a 1-bit to a north-east step $\diagup$. From this point of view, a coefficient $R^{[\mathfrak{p}]}_{n,k} \in \mathcal{R}^{[\mathfrak{p}]}$ counts the number of paths containing n steps of $\diagup$ and $n-k$ steps of $\diagdown$, avoiding the subpath corresponding to pattern $\mathfrak{p}$, allowed to cross the x-axis but required to end at coordinate $(2n-k,k)$ such that $0 \le k \le n$. In particular, $d^{[\mathfrak{p}]}(t)$ is the generating function of paths that avoid $\mathfrak{p}$ and end on the x-axis.

[1] We attach a label $Axxxxxx$ to a sequence if it appears in the OEIS with that identifier.

2 Riordan arrays for Riordan patterns

We start with a theorem that is a direct consequence of the results by Sprugnoli and the first author [9, Theorems 2.3 and 3.3].

Theorem 4. *Let $R_{n,k}^{[\mathfrak{p}]}$ be the number of binary words with n 1-bits and $n-k$ 0-bits, avoiding a Riordan pattern $\mathfrak{p}$. Then the triangle $\mathcal{R}^{[\mathfrak{p}]} = (R_{n,k}^{[\mathfrak{p}]})$ is a Riordan array $\mathcal{R}^{[\mathfrak{p}]} = (d^{[\mathfrak{p}]}(t), h^{[\mathfrak{p}]}(t))$. In particular, if $n_1^{[\mathfrak{p}]}$ and $n_0^{[\mathfrak{p}]}$ correspond to the number of 1-bits and 0-bits in the pattern, $C^{[\mathfrak{p}]}(x,y)$ is the autocorrelation polynomial of $\mathfrak{p}$ and $C^{[\mathfrak{p}]}(t) = C^{[\mathfrak{p}]}(\sqrt{t}, \sqrt{t})$, then*

- *if $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 1$ we have*

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_0^{[\mathfrak{p}]}})}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) - \sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_0^{[\mathfrak{p}]}})}}{2C^{[\mathfrak{p}]}(t)};$$

- *if $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 0$ we have*

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{(C^{[\mathfrak{p}]}(t) + t^{n_0^{[\mathfrak{p}]}})^2 - 4tC^{[\mathfrak{p}]}(t)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) + t^{n_0^{[\mathfrak{p}]}} - \sqrt{(C^{[\mathfrak{p}]}(t) + t^{n_0^{[\mathfrak{p}]}})^2 - 4tC^{[\mathfrak{p}]}(t)^2}}{2C^{[\mathfrak{p}]}(t)};$$

- *if $n_0^{[\mathfrak{p}]} - n_1^{[\mathfrak{p}]} = 1$ we have*

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{[\mathfrak{p}]}})}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) - \sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{[\mathfrak{p}]}})}}{2(C^{[\mathfrak{p}]}(t) - t^{n_1^{[\mathfrak{p}]}})}.$$

If $R^{[\mathfrak{p}]}(t,w)$ denotes the bivariate generating function of the Riordan array $\mathcal{R}^{[\mathfrak{p}]}$, as already mentioned in the Introduction, we have

$$R^{[\mathfrak{p}]}(t,w) = \sum_{n,k\in\mathbb{N}} R_{n,k}^{[\mathfrak{p}]} t^n w^k = \frac{d^{[\mathfrak{p}]}(t)}{1 - wh^{[\mathfrak{p}]}(t)},$$

and Theorem 4 allow us to state the following results.

Theorem 5. *Let $\mathfrak{p}$ be a Riordan pattern and $S^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} S_n^{[\mathfrak{p}]} t^n$ the generating function enumerating the set of binary words $\{w \in \{0,1\}^* : |w|_0 \leq |w|_1\}$ avoiding a Riordan pattern $\mathfrak{p}$ according to the number of 1-bits. Then we have*

- *if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]} + 1$*
$$S^{[\mathfrak{p}]}(t) = \frac{2C^{[\mathfrak{p}]}(t)}{\sqrt{Q(t)}\left(\sqrt{C^{[\mathfrak{p}]}(t)} + \sqrt{Q(t)}\right)},$$
where $Q(t) = (1-4t)C^{[\mathfrak{p}]}(t)^2 + 4t^{n_1^{[\mathfrak{p}]}}$;

- *if $n_0^{[\mathfrak{p}]} = n_1^{[\mathfrak{p}]} + 1$*
$$S^{[\mathfrak{p}]}(t) = \frac{2C^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{[\mathfrak{p}]}})}{\sqrt{Q(t)}\left(C^{[\mathfrak{p}]}(t) - 2t^{n_1^{[\mathfrak{p}]}} + \sqrt{Q(t)}\right)},$$
where $Q(t) = (1-4t)C^{[\mathfrak{p}]}(t)^2 + 4t^{n_0^{[\mathfrak{p}]}} C^{[\mathfrak{p}]}(t)$;

- *if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]}$*
$$S^{[\mathfrak{p}]}(t) = \frac{2C^{[\mathfrak{p}]}(t)^2}{\sqrt{Q(t)}\left(C^{[\mathfrak{p}]}(t) - t^{n_0^{[\mathfrak{p}]}} + \sqrt{Q(t)}\right)},$$
where $Q(t) = (1-4t)C^{[\mathfrak{p}]}(t)^2 + 2t^{n_0^{[\mathfrak{p}]}} C^{[\mathfrak{p}]}(t) + t^{2n_0^{[\mathfrak{p}]}}$.

Proof. For the proof we can observe that $S^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} S_n^{[\mathfrak{p}]} t^n = R^{[\mathfrak{p}]}(t,1)$, or, equivalently, that $S_n^{[\mathfrak{p}]} = \sum_{k=0}^{n} R_{n,k}^{[\mathfrak{p}]}$ and apply the fundamental rule (2) with $f_k = 1$. The statement of the theorem can be found after some algebraic simplification. □

Theorem 6. *Let $\mathfrak{p}$ be a Riordan pattern and $L^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} L_n^{[\mathfrak{p}]} t^n$ the generating function enumerating the set of binary words $\{w \in \{0,1\}^* : |w|_0 \leq |w|_1\}$ avoiding a Riordan pattern $\mathfrak{p}$ according to the length. Then we have*

- *if $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]} + 1$*
$$L^{[\mathfrak{p}]}(t) = \frac{2tC^{[\mathfrak{p}]}(t^2)^2}{\sqrt{Q(t)}\left((2t-1)C(t^2) + \sqrt{Q(t)}\right)},$$
where $Q(t) = C^{[\mathfrak{p}]}(t^2)\left((1-4t^2)C^{[\mathfrak{p}]}(t^2) + 4t^{2n_1^{[\mathfrak{p}]}}\right)$;

- *if $n_0^{[\mathfrak{p}]} = n_1^{[\mathfrak{p}]} + 1$*
$$L^{[\mathfrak{p}]}(t) = \frac{2t\sqrt{C^{[\mathfrak{p}]}(t^2)}(t^{2n_1^{[\mathfrak{p}]}} - C^{[\mathfrak{p}]}(t^2))}{\sqrt{Q(t)}\left((1-2t)C^{[\mathfrak{p}]}(t^2) + 2t^{n_0^{[\mathfrak{p}]}+n_1^{[\mathfrak{p}]}} - \sqrt{C^{[\mathfrak{p}]}(t^2)Q(t)}\right)},$$
where $Q(t) = (1-4t^2)C^{[\mathfrak{p}]}(t^2) + 4t^{2n_0^{[\mathfrak{p}]}}$;

- *if* $n_1^{[\mathfrak{p}]} = n_0^{[\mathfrak{p}]}$

$$L^{[\mathfrak{p}]}(t) = \frac{2tC^{[\mathfrak{p}]}(t^2)^2}{\sqrt{Q(t)}\left((2t-1)C(t^2) - t^{2n_0^{[\mathfrak{p}]}} + \sqrt{Q(t)}\right)},$$

where $Q(t) = (1-4t^2)C^{[\mathfrak{p}]}(t^2)^2 + 2t^{2n_0^{[\mathfrak{p}]}}C^{[\mathfrak{p}]}(t^2) + t^{4n_0^{[\mathfrak{p}]}}$.

Proof. For the proof we can observe that the application of generating function $R^{[\mathfrak{p}]}(t, w)$ as

$$R^{[\mathfrak{p}]}\left(tw, \frac{1}{w}\right) = \sum_{n,k\in\mathbb{N}} R_{n,k}^{[\mathfrak{p}]} t^n w^{n-k}$$

entails that $[t^r w^s]R^{[\mathfrak{p}]}\left(tw, \frac{1}{w}\right) = R_{r,r-s}^{[\mathfrak{p}]}$, which is the number of binary words with r 1-bits and s 0-bits. To enumerate according to the length let $t = w$, therefore $L^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} L_n^{[\mathfrak{p}]} t^n = R^{[\mathfrak{p}]}(t^2, 1/t)$. The statement of the theorem can be found after some algebraic simplification. □

Theorems 5 and 6 allow us to find the generating functions $S^{[\mathfrak{p}]}(t)$ and $L^{[\mathfrak{p}]}(t)$ in terms of the autocorrelation polynomial for every Riordan pattern $\mathfrak{p}$. In what follows, we study some special classes of patterns characterized by an autocorrelation polynomial that can be easily computed, as in the case $C^{[\mathfrak{p}]}(x, y) = 1$. For such particular patterns, Theorem 4 can be simplified as follows:

Corollary 7. *Let* $\mathcal{R}^{[\mathfrak{p}]} = (R_{n,k}^{[\mathfrak{p}]})_{n,k\in\mathbb{N}} = (d^{[\mathfrak{p}]}(t), h^{[\mathfrak{p}]}(t))$ *be the Riordan array corresponding to the number of binary words with* n *1-bits and* $n-k$ *0-bits that avoid the Riordan pattern* $\mathfrak{p}$*. Then we have the following particular cases:*

- *for* $\mathfrak{p} = 1^{j+1}0^j$ *we have the Riordan array*

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+4t^{j+1}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1-\sqrt{1-4t+4t^{j+1}}}{2};$$

- $\mathfrak{p} = 0^{j+1}1^j$ *we have the Riordan array*

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+4t^{j+1}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1-\sqrt{1-4t+4t^{j+1}}}{2(1-t^j)};$$

- $\mathfrak{p} = 1^j0^j$ *and* $\mathfrak{p} = 0^j1^j$ *we have the Riordan array*

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+2t^j+t^{2j}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1+t^j-\sqrt{1-4t+2t^j+t^{2j}}}{2};$$

- $\mathfrak{p} = (10)^j 1$ *we have the Riordan array*

$$d^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i}{\sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i - \sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}}{2\sum_{i=0}^{j} t^i};$$

- $\mathfrak{p} = (01)^j 0$ *we have the Riordan array*

$$d^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i}{\sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i - \sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}}{2\sum_{i=0}^{j-1} t^i}.$$

As a peculiar instance, observe that when we instantiate a pattern from family $\mathfrak{p} = 1^j 0^j$ with $j = 1$ we get a Riordan array $\mathcal{R}^{[10]} = \left(d^{[10]}(t), h^{[10]}(t)\right)$ such that

$$d^{[10]}(t) = \frac{1}{1-t} \quad \text{and} \quad h^{[10]}(t) = t,$$

so the number $R_{n,0}^{[10]}$ of words containing n 1-bits and n 0-bits, avoiding pattern $\mathfrak{p} = 10$, is $[t^n]d^{[10]}(t) = 1$ for $n \in \mathbb{N}$. If we consider the combinatorial interpretation of $R_{n,0}^{[\mathfrak{p}]}$ as lattice paths as illustrated in the last paragraph of the Introduction, this corresponds to the fact that there is exactly one *valley*-shaped path having n steps of both kinds $\diagup$ and $\diagdown$, avoiding $\mathfrak{p} = 10$ and terminating at coordinate $(2n, 0)$ for each $n \in \mathbb{N}$, formally the path $0^n 1^n$.

By applying Theorem 5 to the same patterns as before, we get the following corollary.

Corollary 8. *Let $S^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} S_n^{[\mathfrak{p}]} t^n$ the generating function enumerating the set of binary words $\{w \in \{0,1\}^* : |w|_0 \leq |w|_1\}$ avoiding a Riordan pattern $\mathfrak{p}$ according to the number of 1-bits. We have the following particular cases:*

- *for $\mathfrak{p} = 1^{j+1}0^j$ we have*

$$S^{[\mathfrak{p}]}(t) = \frac{2}{\sqrt{1 - 4t + 4t^{j+1}}\left(1 + \sqrt{1 - 4t + 4t^{j+1}}\right)};$$

- *for $\mathfrak{p} = 0^{j+1}1^j$ we have*

$$S^{[\mathfrak{p}]}(t) = \frac{2(1 - t^j)}{\sqrt{1 - 4t + 4t^{j+1}}\left(1 - 2t^j + \sqrt{1 - 4t + 4t^{j+1}}\right)};$$

- *for* $\mathfrak{p} = 1^j0^j$ *and* $\mathfrak{p} = 0^j1^j$ *we have*

$$S^{[\mathfrak{p}]}(t) = \frac{2}{\sqrt{1-4t+2t^j+t^{2j}}\left(1-t^j+\sqrt{1-4t+2t^j+t^{2j}}\right)};$$

- *for* $\mathfrak{p} = (10)^j1$ *we have*

$$S^{[\mathfrak{p}]}(t) = \frac{2(1-t^{j+1})}{1-4t+3t^{j+1}+\sqrt{1-4t+2t^{j+1}+4t^{j+2}-3t^{2j+2}}};$$

- *for* $\mathfrak{p} = (01)^j0$ *we have*

$$S^{[\mathfrak{p}]}(t) = \frac{2(1-t^j-t^{j+1}+t^{2j+1})}{\sqrt{Q(t)}\left(1-2t^j+t^{j+1}+\sqrt{Q(t)}\right)},$$

  where $Q(t) = 1-4t+2t^{j+1}+4t^{j+2}-3t^{2j+2}$.

We observe that the case $\mathfrak{p} = (10)^j1$ in Corollary 8 corresponds to the sequence studied by Bilotta, Grazzini and Pergola [3]; moreover, in Table 6, Table 7, Table 8, Table 9 and Table 10 we report some expansions and some set of words related to the $S^{[\mathfrak{p}]}(t)$ functions just defined, respectively.

j/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	3	7	15	31	63	127	255	511	1023	2047	4095
2	1	3	10	32	106	357	1222	4230	14770	51918	183472	651191
3	1	3	10	35	123	442	1611	5931	22010	82187	308427	1162218
4	1	3	10	35	126	459	1696	6330	23806	90068	342430	1307138
5	1	3	10	35	126	462	1713	6415	24205	91874	350406	1341782
6	1	3	10	35	126	462	1716	6432	24290	92273	352212	1349768
7	1	3	10	35	126	462	1716	6435	24307	92358	352611	1351574
8	1	3	10	35	126	462	1716	6435	24310	92375	352696	1351973

$$\begin{aligned}[t^3]S^{[110]}(t) = \big|\{&111, 0111, 1011, 00111, 01011, 10011, 10101, 000111,\\ &001011, 010011, 010101, 100011, 100101, 101001, 101010\}\big| = 15\end{aligned}$$

Table 6: Some expansions for $S^{[1^{j+1}0^j]}(t)$ and the set of words with $n = 3$ 1-bits, avoiding pattern $\mathfrak{p} = 110$, so $j = 1$ in the family; moreover, for $j = 1$ the sequence corresponds to A000225, for $j = 2$ the sequence corresponds to A261058.

j/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	3	8	20	48	112	256	576	1280	2816	6144	13312
2	1	3	10	33	111	378	1302	4525	15841	55783	197389	701286
3	1	3	10	35	124	447	1632	6015	22336	83439	313216	1180511
4	1	3	10	35	126	460	1701	6351	23890	90398	343713	1312108
5	1	3	10	35	126	462	1714	6420	24226	91958	350736	1343069
6	1	3	10	35	126	462	1716	6433	24295	92294	352296	1350098
7	1	3	10	35	126	462	1716	6435	24308	92363	352632	1351658
8	1	3	10	35	126	462	1716	6435	24310	92376	352701	1351994

$$[t^3]S^{[001]}(t) = \big|\{111, 0111, 1011, 1101, 1110, 01011, 01101, 01110, 10101, 10110, 11010, 11100, 010101, 010110, 011010, 011100, 101010, 101100, 110100, 111000\}\big| = 20$$

Table 7: Some expansions for $S^{[0^{j+1}1^j]}(t)$ and the set of words with $n = 3$ 1-bits, avoiding pattern $\mathfrak{p} = 001$, so $j = 1$ in the family; moreover, for $j = 1$ the sequence corresponds to A001792.

j/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	3	10	35	126	462	1716	6435	24310	92378	352716	1352078
1	1	2	3	4	5	6	7	8	9	10	11	12
2	1	3	9	27	82	253	791	2499	7960	25520	82248	266221
3	1	3	10	34	118	417	1493	5400	19684	72196	266122	985003
4	1	3	10	35	125	454	1671	6211	23261	87641	331821	1261398
5	1	3	10	35	126	461	1708	6390	24086	91328	347965	1331072
6	1	3	10	35	126	462	1715	6427	24265	92154	351666	1347326
7	1	3	10	35	126	462	1716	6434	24302	92333	352492	1351028
8	1	3	10	35	126	462	1716	6435	24309	92370	352671	1351854

$$[t^8]S^{[01]}(t) = \big|\{11111111, 111111110, 1111111100, 11111111000, 111111110000, 1111111100000, 11111111000000, 111111110000000, 1111111100000000\}\big| = 9$$

Table 8: Some expansions for $S^{[0^j1^j]}(t)$ (or, equivalently, $S^{[1^j0^j]}(t)$) and the set of words with $n = 8$ 1-bits, avoiding pattern $\mathfrak{p} = 01$ (or, equivalently, $\mathfrak{p} = 10$), so $j = 1$ in the family; moreover, for $j = 0$ the sequence corresponds to A001700, for $j = 1$ the sequence corresponds to A001477.

j/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	3	7	18	48	131	363	1017	2873	8169	23349	67024
2	1	3	10	32	109	377	1324	4697	16795	60425	218485	793259
3	1	3	10	35	123	445	1631	6036	22511	84460	318438	1205457
4	1	3	10	35	126	459	1699	6350	23911	90572	344737	1317397
5	1	3	10	35	126	462	1713	6418	24225	91979	350910	1344092
6	1	3	10	35	126	462	1716	6432	24293	92293	352317	1350272
7	1	3	10	35	126	462	1716	6435	24307	92361	352631	1351679
8	1	3	10	35	126	462	1716	6435	24310	92375	352699	1351993

$$\begin{aligned}[t^3]S^{[101]}(t) = \big|\{&111, 0111, 1110, 00111, 01110, 10011, 11001, 11100, 000111, 001110,\\ &010011, 011001, 011100, 100011, 100110, 110001, 110010, 111000\}\big| = 18\end{aligned}$$

Table 9: Some expansions for $S^{[(10)^j 1]}(t)$ and the set of words with $n = 3$ 1-bits, avoiding pattern $\mathfrak{p} = 101$, so $j = 1$ in the family; moreover, for $j = 1$ the sequence corresponds to A225034.

j/n	0	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	3	8	22	61	171	483	1373	3923	11257	32418	93644
2	1	3	10	33	113	393	1384	4920	17618	63456	229642	834342
3	1	3	10	35	124	449	1647	6099	22754	85394	322022	1219205
4	1	3	10	35	126	460	1703	6366	23974	90818	345691	1321092
5	1	3	10	35	126	462	1714	6422	24241	92042	351156	1345049
6	1	3	10	35	126	462	1716	6433	24297	92309	352380	1350518
7	1	3	10	35	126	462	1716	6435	24308	92365	352647	1351742
8	1	3	10	35	126	462	1716	6435	24310	92376	352703	1352009

$$\begin{aligned}[t^3]S^{[010]}(t) = \big|\{&111, 0111, 1011, 1101, 1110, 00111, 01101, 01110,\\ &10011, 10110, 11001, 11100, 000111, 001101, 001110, 011001,\\ &011100, 100011, 100110, 101100, 110001, 111000\}\big| = 22\end{aligned}$$

Table 10: Some expansions for $S^{[(01)^j 0]}(t)$ and the set of words with $n = 3$ 1-bits, avoiding pattern $\mathfrak{p} = 010$, so $j = 1$ in the family; moreover, for $j = 1$ the sequence corresponds to A025566.

Finally, by applying Theorem 6 to the pattern families already examined, we find the following result.

Corollary 9. *Let $L^{[\mathfrak{p}]}(t) = \sum_{n\geq 0} L_n^{[\mathfrak{p}]} t^n$ the generating function enumerating the set of binary words $\{w \in \{0,1\}^* : |w|_0 \leq |w|_1\}$ avoiding a Riordan pattern $\mathfrak{p}$ according to the length. We have the following particular cases:*

- *for $\mathfrak{p} = 1^{j+1}0^j$ we have*

$$L^{[\mathfrak{p}]}(t) = \frac{2t}{\sqrt{1-4t^2+4t^{2(j+1)}}\left(2t-1+\sqrt{1-4t^2+4t^{2(j+1)}}\right)};$$

- *for $\mathfrak{p} = 0^{j+1}1^j$ we have*

$$L^{[\mathfrak{p}]}(t) = \frac{2t(t^{2j}-1)}{\sqrt{1-4t^2+4t^{2(j+1)}}\left(1-2t+2t^{2j+1}-\sqrt{1-4t^2+4t^{2(j+1)}}\right)};$$

- *for $\mathfrak{p} = 1^j0^j$ and $\mathfrak{p} = 0^j1^j$ we have:*

$$L^{[\mathfrak{p}]}(t) = \frac{2t}{\sqrt{1-4t^2+2t^{2j}+t^{4j}}\left(-1+2t-t^{2j}+\sqrt{1-4t^2+2t^{2j}+t^{4j}}\right)};$$

- *for $\mathfrak{p} = (10)^j1$ we have*

$$L^{[\mathfrak{p}]}(t) = \frac{2t(t^{2j+2}-1)}{1-4t^2+3t^{2j+2}+(2t-1)\sqrt{Q(t)}};$$

- *for $\mathfrak{p} = (01)^j0$ we have*

$$L^{[\mathfrak{p}]}(t) = \frac{2t(t^{2j+2}-1)(t^{2j}-1)}{\sqrt{Q(t)}(t^{2j+2}-2t^{2j+1}+2t-1+\sqrt{Q(t)})},$$

where $Q(t) = 1-4t^2+2t^{2j+2}+4t^{2j+4}-3t^{4j+4}$.

In Table 11, Table 12, Table 13, Table 14 and Table 15 we report some expansions related to the $L^{[\mathfrak{p}]}(t)$ functions just defined, respectively.

j/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	3	3	7	7	15	15	31	31	63	63	127	127	255
2	1	1	3	4	11	15	38	55	135	201	483	736	1742	2699	6313
3	1	1	3	4	11	16	42	63	159	247	610	969	2354	3802	9117
4	1	1	3	4	11	16	42	64	163	255	634	1015	2482	4041	9752
5	1	1	3	4	11	16	42	64	163	256	638	1023	2506	4087	9880
6	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4095	9904
7	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9908

Table 11: Some expansions for $L^{[1^{j+1}0^j]}(t)$; moreover, for $j = 1$ the sequence corresponds to A052551.

j/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	3	4	9	13	26	39	73	112	201	313	546	859	1469
2	1	1	3	4	11	16	40	61	147	231	542	870	2004	3269	7423
3	1	1	3	4	11	16	42	64	161	253	622	999	2414	3942	9396
4	1	1	3	4	11	16	42	64	163	256	636	1021	2494	4071	9812
5	1	1	3	4	11	16	42	64	163	256	638	1024	2508	4093	9892
6	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9906
7	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9908

Table 12: Some expansions for $L^{[0^{j+1}1^j]}(t)$; moreover, for $j = 1$ the sequence corresponds to A079284.

j/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9908
1	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
2	1	1	3	4	10	14	33	48	109	163	362	552	1207	1868	4036
3	1	1	3	4	11	16	41	62	154	240	583	928	2217	3587	8459
4	1	1	3	4	11	16	42	64	162	254	629	1008	2455	4000	9614
5	1	1	3	4	11	16	42	64	163	256	637	1022	2501	4080	9853
6	1	1	3	4	11	16	42	64	163	256	638	1024	2509	4094	9899
7	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9907

Table 13: Some expansions for $L^{[0^j1^j]}(t)$ (or, equivalently, $L^{[1^j0^j]}(t)$); moreover, for $j = 0$ the sequence corresponds to A027306, for $j = 1$ the sequence corresponds to A008619.

j/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	3	3	7	8	19	23	53	66	150	191	429	555	1235
2	1	1	3	4	11	15	38	56	139	210	511	790	1892	2973	7034
3	1	1	3	4	11	16	42	63	159	248	614	978	2382	3857	9273
4	1	1	3	4	11	16	42	64	163	255	634	1016	2486	4050	9780
5	1	1	3	4	11	16	42	64	163	256	638	1023	2506	4088	9884
6	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4095	9904
7	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9908

Table 14: Some expansions for $L^{[(10)^j 1]}(t)$; moreover, no sequence is known in the literature, except for $j = 0$.

j/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	3	4	9	13	28	42	87	134	271	425	844	1342	2628
2	1	1	3	4	11	16	40	61	149	234	558	895	2098	3420	7909
3	1	1	3	4	11	16	42	64	161	253	624	1002	2430	3967	9492
4	1	1	3	4	11	16	42	64	163	256	636	1021	2496	4074	9828
5	1	1	3	4	11	16	42	64	163	256	638	1024	2508	4093	9894
6	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9906
7	1	1	3	4	11	16	42	64	163	256	638	1024	2510	4096	9908

Table 15: Some expansions for $L^{[(01)^j 0]}(t)$; moreover, no sequence is known in literature, except for $j = 0$.

3 Some combinatorial interpretations

In the previous section we proved results about the enumeration of words avoiding patterns from an algebraic point of view. The aim of this section is to analyze in more details some particular cases of the various pattern families. We approach these problems either combinatorially by providing an interpretation, or algebraically by computing the coefficients of the involved generating functions explicitly.

3.1 Enumeration with respect to the number of 1-bits

Corollary 10. *Consider pattern $\mathfrak{p} = 1^{j+1}0^j$. There is only one word in $\mathfrak{L}^{[\mathfrak{p}]}$ for $j = 0$; on the other hand, there are $S_n^{[\mathfrak{p}]} = 2^{n+1} - 1$ words for $j = 1$.*

Proof. When $j = 0$ the pattern to avoid is $\mathfrak{p} = 1$, therefore only words w in $\{\varepsilon\} \cup \{0\}^+$ are suitable choices; however, the constraint $|w|_0 \leq |w|_1$ makes $w = \varepsilon$ the only one.

When $j = 1$ the pattern to avoid is $\mathfrak{p} = 110$ and we observe that the generic binomial $\binom{r}{k}$ can be interpreted as the number of binary words with r 0-bits containing k occurrences of the substring 10, which we call an *inversion* with respect to pattern $\mathfrak{p} = 110$. In order to build a word in the language we start from the substring 0^r for $r \in \{0, \ldots, n\}$ and select $k \in \{0, \ldots, r\}$ 0-bits, transforming each one using the mapping $0 \mapsto 10$, while preventing the transformation of the 0-bit in the 10 just introduced. This maneuver introduces k inversions and the selection can be done in $\binom{r}{k}$ ways; finally, we pad on the right with a strip 1^{n-k}, because it is mandatory for a word to have n 1-bits. Hence there is one padding for each set of inversions and there is no other way to avoid $\mathfrak{p}$. Therefore

$$\sum_{r=0}^{n} \sum_{k=0}^{r} \binom{r}{k} = 2^{n+1} - 1 = S_n^{[\mathfrak{p}]},$$

as can be verified algebraically by extracting the coefficient of the generating function

$$S^{[\mathfrak{p}]}(t) = \frac{1}{1 - 3t + 2t^2} = \frac{2}{1 - 2t} - \frac{1}{1 - t},$$

as required. □

The same argument can be rewritten in term of sets, which allows us to give a constructive approach. Let $\mathcal{S}_{n,k,i}$ be the set of binary words containing n and k occurrences of 1-bits and 0-bits, respectively, with i inversions, namely an occurrence of the subsequence 10. By union with respect to i and k, we get the sets $\mathcal{S}_{n,k}^{[\mathfrak{p}]}$ and $\mathcal{S}_n^{[\mathfrak{p}]}$, formally

$$\mathcal{S}_n^{[\mathfrak{p}]} = \bigcup_{k \in \{0,\ldots,n\}} \mathcal{S}_{n,k}^{[\mathfrak{p}]} = \bigcup_{i \in \{0,\ldots,k\}} \mathcal{S}_{n,k,i}^{[\mathfrak{p}]} = \left(\bigcup_{i \in \{0,\ldots,k\}} \mathcal{S}_{k,k,i}^{[\mathfrak{p}]} \right) \times \left\{ 1^{n-k} \right\}.$$

For the sake of clarity, we enumerate all binary words avoiding $\mathfrak{p} = 110$ containing $n = 3$ 1-bits, formally we partition $\mathcal{S}_3^{[\mathfrak{p}]}$ as follows:

$$\begin{aligned} \mathcal{S}_3^{[\mathfrak{p}]} = \mathcal{S}_{0,0,0}^{[\mathfrak{p}]} &\times \{111\} \\ \cup \left(\mathcal{S}_{1,1,0}^{[\mathfrak{p}]} \cup \mathcal{S}_{1,1,1}^{[\mathfrak{p}]} \right) &\times \{11\} \\ \cup \left(\mathcal{S}_{2,2,0}^{[\mathfrak{p}]} \cup \mathcal{S}_{2,2,1}^{[\mathfrak{p}]} \cup \mathcal{S}_{2,2,2}^{[\mathfrak{p}]} \right) &\times \{1\} \\ \cup \left(\mathcal{S}_{3,3,0}^{[\mathfrak{p}]} \cup \mathcal{S}_{3,3,1}^{[\mathfrak{p}]} \cup \mathcal{S}_{3,3,2}^{[\mathfrak{p}]} \cup \mathcal{S}_{3,3,3}^{[\mathfrak{p}]} \right) &\times \{\varepsilon\} \end{aligned}$$

where

$$\mathcal{S}^{[\mathfrak{p}]}_{3,0} = \mathcal{S}^{[\mathfrak{p}]}_{0,0,0} \times \{111\} = \{\varepsilon\} \times \{111\} = \{111\}$$

$$\mathcal{S}^{[\mathfrak{p}]}_{3,1} = \left(\mathcal{S}^{[\mathfrak{p}]}_{1,1,0} \cup \mathcal{S}^{[\mathfrak{p}]}_{1,1,1}\right) \times \{11\} = \{0111\} \cup \{1011\}$$

$$\mathcal{S}^{[\mathfrak{p}]}_{3,2} = \left(\mathcal{S}^{[\mathfrak{p}]}_{2,2,0} \cup \mathcal{S}^{[\mathfrak{p}]}_{2,2,1} \cup \mathcal{S}^{[\mathfrak{p}]}_{2,2,2}\right) \times \{1\} = \{00111\} \cup \{10011, 01011\} \cup \{10101\}$$

$$\begin{aligned}\mathcal{S}^{[\mathfrak{p}]}_{3,3} = \left(\mathcal{S}^{[\mathfrak{p}]}_{3,3,0} \cup \mathcal{S}^{[\mathfrak{p}]}_{3,3,1} \cup \mathcal{S}^{[\mathfrak{p}]}_{3,3,2} \cup \mathcal{S}^{[\mathfrak{p}]}_{3,3,3}\right) \times \{\varepsilon\} &= \{000111\} \cup \{001011, 100011, 010011\} \\ &\cup \{101001, 100101, 010101\} \cup \{101010\},\end{aligned}$$

the same set of words shown in Table 6.

Corollary 11. *Consider pattern $\mathfrak{p} = 0^{j+1}1^j$. There is one word $S^{[\mathfrak{p}]}_n = 1$ for each $n \in \mathbb{N}$ in $\mathfrak{L}^{[\mathfrak{p}]}$ when $j = 0$; on the other hand, there are $(n+2)2^{n-1}$ words for $j = 1$.*

Proof. When $j = 0$ the pattern to avoid is $\mathfrak{p} = 0$, therefore only words $w = 1^n$ are suitable. Hence there is one of them for each $n \in \mathbb{N}$.

When $j = 1$ the pattern to avoid is $\mathfrak{p} = 001$, therefore we extract the n-th coefficient after instantiation of the corresponding generating function:

$$[t^n]S^{[\mathfrak{p}]}_n(t) = [t^n]\frac{1-t}{(1-2t)^2} = (n+2)2^{n-1},$$

as required.

We also provide a combinatorial interpretation of the theorem; first of all, we observe that sequence $S^{[\mathfrak{p}]}_n$ is the binomial transform of the sequence of the positive integers $(n+1)_{n\in\mathbb{N}}$, formally

$$S^{[\mathfrak{p}]}_n = (n+2)2^{n-1} = \sum_{k=0}^{n} \binom{n}{k}(k+1),$$

where the generic summand $\binom{n}{k}(k+1)$ can be interpreted as the number of binary words with n 1-bits containing $n-k$ occurrences of the substring 01, which we call an *inversion* respect to the pattern $\mathfrak{p} = 001$. We construct the set of words avoiding $\mathfrak{p}$ to show the bijection with the previous assertion as follows: if in a word w there are $n-k$ occurrences of the substring 01 then w contains $2n-2k$ bits in total, $n-k$ of both kinds. Since it is mandatory that the number of 1 is n, we add k 1-bits to it, resulting in a new word w' of length $2n-k$, which can be augmented with at most k additional 0-bits, according to the constraint $|w'|_0 \leq |w'|_1$. In order to build a word with the structure of w', we start from the substring 1^n and select $n-k$ 1-bits, transforming each one using the mapping $1 \mapsto 01$, simultaneously to prevent transforming 1-bit in 01 just introduced. This maneuver introduces $n-k$ inversions and the selection can be done in $\binom{n}{k}$ ways; moreover, we are free to pad on the right with 0^i strips, for $i \in \{0, \ldots, k\}$, hence there are $k+1$ paddings for each set of inversions. Therefore, since there can be up to n inversions,

$$\sum_{k=0}^{n} \binom{n}{n-k}(k+1) = (n+2)2^{n-1}$$

concludes the proof by symmetry of binomial coefficients. $\square$

Corollary 12. *Consider pattern $\mathfrak{p} = 0^j1^j$ (or, equivalently, $\mathfrak{p} = 1^j0^j$). There are*

$$S_n^{[\mathfrak{p}]} = \sum_{k=0}^{n} \binom{n+k}{k} = \binom{2n+1}{n}$$

words in $\mathfrak{L}^{[\mathfrak{p}]}$ for $j = 0$; on the other hand, there are $S_n^{[\mathfrak{p}]} = n+1$ words for $j = 1$.

Proof. When $j = 0$ there is no pattern to avoid and this situation corresponds to the enumeration of binary words $\{w \in \{0,1\}^* : |w|_0 \le |w|_1 = n\}$. After instantiating the generating function $S^{[\mathfrak{p}]}(t)$, we extract the n-th coefficient, as follows:

$$[t^n]S_n^{[\mathfrak{p}]}(t) = [t^n]\frac{1-\sqrt{1-4t}}{2t\sqrt{1-4t}} = \frac{1}{2}\binom{2(n+1)}{n+1} = \binom{2n+1}{n+1} = \binom{2n+1}{n},$$

which can be simplified by using the identity

$$\binom{r+s+1}{s} = \sum_{q=0}^{s} \binom{r+q}{q},$$

as desired. It is possible to state the following combinatorial interpretation: since the maximum number of 0-bits is n, we reserve n boxes for them. To the left of each box reserve one more box and, finally, another one to the right of the very last box. In this way we have reserved $2n+1$ boxes where we can put n 1-bits in $\binom{2n+1}{n}$ ways, as required.

When $j = 1$ the pattern to avoid is $\mathfrak{p} = 01$ (or, equivalently, $\mathfrak{p} = 10$), therefore only words $w \in \{1^n\} \times \bigcup_{s\in\{0,\dots,n\}}\{0^s\}$ are suitable, which are $n+1$, one for each value that s can take. $\square$

Last two patterns $\mathfrak{p} = (10)^j1$ and $\mathfrak{p} = (01)^j0$ are harder to study: for $j = 0$ there are $S_n^{[\mathfrak{p}]} = [\![n = 0]\!]$ and $S_n^{[\mathfrak{p}]} = 1$ words, respectively. On the other hand, when $j = 1$ we report only the instantiated generating functions

$$S^{[101]}(t) = \frac{(1+t)\left(1-3t-\sqrt{1-2t-3t^2}\right)}{2t(3t-1)},$$

$$S^{[010]}(t) = \frac{1-2t-3t^2-(1-t)\sqrt{1-2t-3t^2}}{2t^2(3t-1)}.$$

As pointed out by an anonymous referee, the previous functions can be rewritten as

$$S^{[101]}(t) = \frac{(1+t)(1-tM(t))}{1-3t},$$

$$S^{[010]}(t) = \frac{(1+tM(t))(1-tM(t))}{1-3t},$$

where $M(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$ is the Motzkin numbers' generating function. Such rewriting shows a relation among Motzkin numbers and powers of 3, which is not easy to state bijectively, to the best of our knowledge. However, for their generating functions, we have the following identity

$$\frac{1}{1-3t} = \frac{M(t)}{(1-tM(t))^2}, \tag{3}$$

and thus, by using the fundamental rule of Riordan arrays (2), we can express the functions $S^{[101]}(t)$ and $S^{[010]}(t)$ in terms of the Motzkin triangle and the sequence $(1, 2, 2, 2, \ldots)$

$$S^{[101]}(t) = \frac{(1+t)M(t)}{1-tM(t)} = \frac{(1+tM(t))^2}{1-tM(t)} = (1+tM(t), tM(t)) * \frac{1+t}{1-t},$$
$$S^{[010]}(t) = \frac{(1+tM(t))M(t)}{1-tM(t)} = (M(t), tM(t)) * \frac{1+t}{1-t},$$

as illustrated in Table 16.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	2	1					
3	4	5	3	1				
4	9	12	9	4	1			
5	21	30	25	14	5	1		
6	51	76	69	44	20	6	1	
7	127	196	189	133	70	27	7	1

Table 16: The Motzkin triangle corresponding to the Riordan array $(M(t), tM(t))$ and to sequence A064189. Multiplying the matrix by the column vector $(1, 2, 2, 2, 2, \ldots)$ we get the sequence $S_n^{[010]} = (1, 3, 8, 22, 61, \cdots)$. Similarly, with matrix $(1 + tM(t), tM(t))$ we get the sequence $S_n^{[101]} = (1, 3, 7, 18, 48, \cdots)$.

Identity (3) has a combinatorial interpretation in terms of compact-rooted directed animals or domino towers (see Ardila [1, Example 21, pp. 21–22] and the references therein); Motzkin triangle corresponds to sequence A064189 and has several combinatorial interpretations.

3.2 Enumeration with respect to the length

Corollary 13. *Consider pattern $\mathfrak{p} = 1^{j+1}0^j$. There is one word in $\mathfrak{L}^{[\mathfrak{p}]}$ for $j = 0$; on the other hand, there are $2^{m+1} - 1$ words, where $n = 2m + [\![n \text{ is odd}]\!]$, for $j = 1$.*

Proof. When $j = 0$ the pattern to avoid is $\mathfrak{p} = 1$, therefore instantiating the generating function we have $L^{[\mathfrak{p}]}(t) = 1$, as required.

When $j = 1$ the pattern to avoid is $\mathfrak{p} = 110$, therefore we instantiate and extract the n-th coefficient

$$L_n^{[\mathfrak{p}]} = [t^n]\frac{2}{1-2t^2} + [t^{n-1}]\frac{2}{1-2t^2} - [t^n]\frac{1}{1-t}$$

and proceed by cases on the parity of n. If $n = 2m$ then the second term in the rhs disappears, otherwise if $n = 2m+1$ the first term disappears; in both cases it is required to perform $[u^m]\frac{2}{1-2u} = 2^{m+1}$ where $u = t^2$, as required.

It is possible to state a combinatorial interpretation using an argument similar to the one given in the proof of Corollary 10. Let $n = 2m$, therefore a word w needs to have $m + j$ 1-bits, where $j \in \{0, \ldots, m\}$; conversely, w needs to have $n - m - j = m - j$ 0-bits. Fixing j in the given range, from the substring 0^{m-j} we select $i \in \{0, \ldots, m-j\}$ 0-bits to introduce i inversions, namely i occurrences of 10, applying the mapping $0 \mapsto 10$ simultaneously. This maneuver keeps the original 0-bits and introduces at most $m-j$ 1-bits, so we pad with 1-bits on the right in order to have the required $m + j$ 1-bits in the entire word; finally, selections can be done in

$$\sum_{j=0}^{m}\sum_{i=0}^{m-j}\binom{m-j}{i} = \sum_{j=0}^{m} 2^{m-j} = 2^{m+1} - 1$$

ways, because padding can be done in only one way, completing the case for n even.

Let $n = 2m+1$, therefore a word w needs to have $m+1+j$ 1-bits, where $j \in \{0, \ldots, m\}$; conversely, w needs to have $n - m - 1 - j = m - j$ 0-bits. Fixing j in the given range, from the substring 0^{m-j} we select $i \in \{0, \ldots, m-j\}$ 0-bits to introduce i inversions as done in the even case, introducing at most $m - j$ 1-bits, and padding as necessary to have $m+1+j$ 1-bits, the total number of selections equals the one given for the even case, completing the case for n odd. □

Corollary 14. *Consider pattern $\mathfrak{p} = 0^{j+1}1^j$. There is one word $L_n^{[\mathfrak{p}]} = 1$ for each $n \in \mathbb{N}$ in $\mathfrak{L}^{[\mathfrak{p}]}$ when $j = 0$; on the other hand, there are $L_n^{[\mathfrak{p}]} = F_{n+3} - 2^m$ words if $n = 2m$ else $L_n^{[\mathfrak{p}]} = F_{n+3} - 2^{m+1}$ words if $n = 2m+1$, for $j = 1$, where F_n is the n-th Fibonacci number.*

Proof. When $j = 0$ the pattern to avoid is $\mathfrak{p} = 0$, therefore suitable words of length n are of the form $w = 1^n$. Hence $L_n^{[\mathfrak{p}]} = 1$ for each $n \in \mathbb{N}$.

When $j = 1$ the pattern to avoid is $\mathfrak{p} = 001$, therefore we instantiate and extract the n-th coefficient

$$L_n^{[\mathfrak{p}]} = 2[t^{n+1}]\frac{t}{1-t-t^2} + [t^n]\frac{t}{1-t-t^2} - [t^n]\frac{1}{1-2t^2} - 2[t^{n-1}]\frac{1}{1-2t^2}$$

in order to have $L_n^{[\mathfrak{p}]} = 2F_{n+1} + F_n - a_n = F_{n+3} - a_n$, where $a_{2m} = 2^m$ and $a_{2m+1} = 2^{m+1}$.

It is possible to state a combinatorial interpretation using an argument similar to the one given in the proof of Corollary 11. Let $n = 2m$, therefore a word w needs to have $m + j$ 1-bits, where $j \in \{0, \ldots, m\}$; conversely, w needs to have $n - m - j = m - j$ 0-bits. Fixing j

in the given range, from the substring 1^{m+j} we select $i \in \{0, \dots, m-j\}$ 1-bits to introduce i inversions, namely i occurrences of 01, applying the mapping $1 \mapsto 01$ simultaneously. This maneuver keeps the original 1-bits and introduces at most $m-j$ 0-bits; finally, selections can be done in $\sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{m+j}{i}$ ways. In order to find a closed form for the double summation, we inspect the region of the Pascal triangle taken into account; marking with $\circ$ the involved binomials

$$
\begin{array}{c|ccccccccccc}
n/j & 0 & 1 & \dots & m-1 & m & m+1 & \dots & 2m-1 & 2m & \dots \\
\hline
0 & & & & & & & & & & \\
\vdots & & & & & & & & & & \\
m-1 & & & & & & & & & & \\
m & \circ & \circ & \dots & \circ & \circ & & & & & \\
m+1 & \circ & \circ & \dots & \circ & & & & & & \\
\vdots & \vdots & \vdots & \iddots & & & & & & & \\
2m-1 & \circ & \circ & & & & & & & & \\
2m & \circ & & & & & & & & & \\
2m+1 & & & & & & & & & & \\
\vdots & & & & & & & & & &
\end{array}
$$

and using the identity $\binom{r+1}{k+1} - \binom{s}{k+1} = \sum_{i=s}^{r} \binom{i}{k}$ for rearranging the summation and identities $2^n = \sum_{i=0}^{n} \binom{n}{i}$ and $F_{n+1} = \sum_{i=0}^{n} \binom{n-i}{i}$ to collect terms, we obtain

$$\sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{m+j}{i} = \sum_{k=0}^{m} \binom{2m+1-k}{k+1} - \binom{m}{k+1} = F_{2m+3} - 2^m = L_{2m}^{[\mathfrak{p}]},$$

completing the case for n even.

Let $n = 2m+1$, therefore a word w needs to have $m+1+j$ 1-bits, where $j \in \{0, \dots, m\}$; conversely, w needs to have $n-m-1-j = m-j$ 0-bits. Fixing j in the given range, from the substring 1^{m+1+j} select $i \in \{0, \dots, m-j\}$ 1-bits to introduce i inversions as done for the even case; in parallel, selections can be done in $\sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{m+1+j}{i}$ ways. The involved region in the Pascal triangle has the same shape as the one shown for the even case translated one row to the bottom, so binomials lying on row m are excluded and binomials $\binom{2m+1-k}{k}$ are included, for $k \in \{0, \dots, m\}$. Therefore we rewrite

$$\sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{m+1+j}{i} = \sum_{k=0}^{m} \binom{2m+2-k}{k+1} - \binom{m+1}{k+1} = F_{2m+4} - 2^{m+1} = L_{2m+1}^{[\mathfrak{p}]},$$

completing the case for n odd. $\square$

Corollary 15. *Consider pattern $\mathfrak{p} = 0^j 1^j$ (or, equivalently, $\mathfrak{p} = 1^j 0^j$). There are 2^{n-1} words in $\mathfrak{L}^{[\mathfrak{p}]}$ if n is odd else $2^{n-1} + \frac{1}{2}\binom{2m}{m}$ where $n = 2m$, for $j = 0$; on the other hand, there are $L_n^{[\mathfrak{p}]} = m+1$ words, where $n = 2m + [\![n \text{ is odd}]\!]$, for $j = 1$.*

Proof. When $j = 0$ the pattern to avoid is $\mathfrak{p} = \varepsilon$, namely the empty word, therefore instantiating the generating function we have

$$L^{[\mathfrak{p}]}(t) = \frac{1}{2(1-2t)} + \frac{1}{2\sqrt{1-4t^2}}$$

we extract the coefficient $L_n^{[\mathfrak{p}]} = 2^{n-1} + \frac{a_n}{2}$, where $a_{2m+1} = 0$ and $a_{2m} = \binom{2m}{m}$, as required. We observe that these values correspond to the summation $\sum_{i=0}^{m} \binom{n}{i}$ for $n = 2m, 2m+1, \ldots$, where the binomial coefficient computes the number of ways to choose i 0-bits among n bits, and this gives the combinatorial interpretation.

When $j = 1$ the pattern to avoid is $\mathfrak{p} = 01$ (or, equivalently, $\mathfrak{p} = 10$), therefore after instantiation

$$L^{[\mathfrak{p}]}(t) = \frac{1}{4(1-t)} + \frac{1}{2(1-t)^2} + \frac{1}{4(1+t)}$$

we extract the n-th coefficient $L_n^{[\mathfrak{p}]} = \frac{1}{4} + \frac{(-1)^n}{4} + \frac{n+1}{2}$, so either $n = 2m$ or $n = 2m+1$ entails $L_n^{[\mathfrak{p}]} = m + 1$, as required.

A combinatorial interpretation can be given as follows: if $n = 2m$ then suitable words have structure $1^m 1^j 0^{m-j}$ for $j \in \{0, \ldots, m\}$, and there are $m + 1$ of them. On the contrary, if $n = 2m + 1$ holds then suitable words have structure $1^{m+1} 1^j 0^{m-j}$ for $j \in \{0, \ldots, m\}$, they are $m + 1$ in number again, as required. $\square$

As before, last two patterns $\mathfrak{p} = (01)^j 0$ and $\mathfrak{p} = (10)^j 1$ are harder to study and we avoid to report formulas about $L^{[\mathfrak{p}]}(t)$ functions because we have not a meaningful combinatorial interpretation: we only point out that these functions can be expressed in terms of $M(t^2)$, where $M(t)$ is the generating function of Motzkin numbers, similarly to the corresponding $S^{[\mathfrak{p}]}(t)$ functions.

4 Conclusions

As a final remark, we observe a structural properties of matrices $\mathcal{R}^{[\mathfrak{p}]}$ against the studied families of patterns. As it is well-known (see, e.g., Shapiro, Getu, Woan, and Woodson [12]), Riordan arrays constitute a group with respect to the usual row-by-column product between matrices, and the product of two Riordan arrays D_1 and D_2 is defined as follows:

$$D_1 * D_2 = (d_1(t),\ h_1(t)) * (d_2(t),\ h_2(t)) = (d_1(t)d_2(h_1(t)),\ h_2(h_1(t))).$$

Moreover, the Riordan array $I = (1,\ t)$ acts as the identity and the inverse of $D = (d(t), h(t))$ is the Riordan array:

$$D^{-1} = \left(\frac{1}{d(\overline{h}(t))}, \overline{h}(t)\right)$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$.

The Pascal triangle and its inverse correspond to the Riordan arrays

$$P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right) \qquad P^{-1} = \left(\frac{1}{1+t}, \frac{t}{1+t}\right)$$

respectively. Therefore, for every Riordan array $\mathcal{R}^{[\mathfrak{p}]}$ we can compute $B^{[\mathfrak{p}]} = P^{-1} * \mathcal{R}^{[\mathfrak{p}]}$,, which is equivalent to saying that $\mathcal{R}^{[\mathfrak{p}]}$ is the binomial transform of $B^{[\mathfrak{p}]}$, or $\mathcal{R}^{[\mathfrak{p}]} = P * B^{[\mathfrak{p}]}$.

For the sake of clarity, consider the pattern family $\mathfrak{p} = 1^{j+1}0^j$, so for $j = 1$ we have the minor

$$\mathcal{R}^{[110]} = \begin{bmatrix} 1 &&&&&&&&&& \\ 2 & 1 &&&&&&&&& \\ 4 & 2 & 1 &&&&&&&& \\ 8 & 4 & 2 & 1 &&&&&&& \\ 16 & 8 & 4 & 2 & 1 &&&&&& \\ 32 & 16 & 8 & 4 & 2 & 1 &&&&& \\ 64 & 32 & 16 & 8 & 4 & 2 & 1 &&&& \\ 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 &&& \\ 256 & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 && \\ 512 & 256 & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 & \\ 1024 & 512 & 256 & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 \end{bmatrix},$$

which corresponds to

$$B^{[110]} = \begin{bmatrix} 1 &&&&&&&&&& \\ 1 & 1 &&&&&&&&& \\ 1 & 0 & 1 &&&&&&&& \\ 1 & 1 & -1 & 1 &&&&&&& \\ 1 & 0 & 2 & -2 & 1 &&&&&& \\ 1 & 1 & -2 & 4 & -3 & 1 &&&&& \\ 1 & 0 & 3 & -6 & 7 & -4 & 1 &&&& \\ 1 & 1 & -3 & 9 & -13 & 11 & -5 & 1 &&& \\ 1 & 0 & 4 & -12 & 22 & -24 & 16 & -6 & 1 && \\ 1 & 1 & -4 & 16 & -34 & 46 & -40 & 22 & -7 & 1 & \\ 1 & 0 & 5 & -20 & 50 & -80 & 86 & -62 & 29 & -8 & 1 \end{bmatrix}$$

defined by functions $d^{[110]}(t) = \frac{1}{1-t}$ and $h^{[110]}(t) = \frac{t}{1+t}$, which expands to $h^{[110]}(t) = t - t^2 + t^3 - t^4 + t^5 - t^6 + t^7 - t^8 + t^9 - t^{10} + \mathcal{O}(t^{11})$.

On the other hand, the Riordan array $\mathcal{R}^{[11100]}$, that is $j = 2$ in the family, is the binomial transform of

$$B^{[11100]} = \begin{bmatrix} 1 &&&&&&&&&& \\ 1 & 1 &&&&&&&&& \\ 3 & 1 & 1 &&&&&&&& \\ 5 & 3 & 1 & 1 &&&&&&& \\ 15 & 7 & 3 & 1 & 1 &&&&&& \\ 31 & 16 & 9 & 3 & 1 & 1 &&&&& \\ 87 & 43 & 17 & 11 & 3 & 1 & 1 &&&& \\ 201 & 101 & 55 & 18 & 13 & 3 & 1 & 1 &&& \\ 543 & 271 & 119 & 67 & 19 & 15 & 3 & 1 & 1 && \\ 1331 & 666 & 341 & 141 & 79 & 20 & 17 & 3 & 1 & 1 & \\ 3533 & 1766 & 826 & 411 & 167 & 91 & 21 & 19 & 3 & 1 & 1 \end{bmatrix}$$

defined by functions $d^{[11100]}(t) = \sqrt{\frac{1+t}{1-t-5t^2+t^3}}$ and $h^{[11100]}(t) = \frac{1+2t+t^2-\sqrt{(1-t-5t^2+t^3)(1+t)}}{2(1+t)^2}$, which expands to $h^{[11100]}(t) = t + 2t^4 - t^5 + 7t^6 + 24t^8 + 17t^9 + 98t^{10} + \mathcal{O}\left(t^{11}\right)$.

Riordan arrays $B^{[\mathfrak{p}]}$ can be completely defined by using the results of Theorem 4 and the product rule of the Riordan group. Doing so, for each pattern family studied in this work with $j > 1$, the Riordan array $\mathcal{R}^{[\mathfrak{p}]}$ appears to be the binomial transform of another Riordan array $B^{[\mathfrak{p}]}$ with non-negative integer coefficients, although it is not easy to spot this property looking at the corresponding h functions because their series expansions might contain negative coefficients, as shown for matrices $B^{[110]}$ and $B^{[11100]}$. This fact could be further investigated from an algebraic and combinatorial point of view and possibly yield interesting combinatorial interpretations also in the case $j > 1$.

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