Journal of Integer Sequences, Vol. 21 (2018),
Article 18.6.1

Poly-Bernoulli Numbers and Eulerian Numbers

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Abstract

In this note we prove, using combinatorial arguments, some new formulas connecting poly-Bernoulli numbers with negative indices to Eulerian numbers.

1 Introduction

Kaneko [10] introduced the poly-Bernoulli numbers A099594 during his investigations on multiple zeta values. He defined these numbers by their generating function:

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}, \tag{1}$$

where

$$\mathrm{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k}$$

is the classical polylogarithmic function. As the name indicates, poly-Bernoulli numbers are generalizations of the Bernoulli numbers. For $k=1$ $B_n^{(1)}$ are the classical Bernoulli numbers with $B_1 = \frac{1}{2}$. For negative k-indices poly-Bernoulli numbers are integers (see the values for small n, k in Table 1) and have interesting combinatorial properties.

$n \backslash k$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

Table 1: The poly-Bernoulli numbers $B_n^{(-k)}$

Poly-Bernoulli numbers enumerate several combinatorial objects arisen in different research areas, such as lonesum matrices, Γ-free matrices, acyclic orientations of complete bipartite graphs, alternative tableaux with rectangular shape, permutations with restriction on the distance between positions and values, permutations with excedance set $[k]$, etc. In [3, 4] the authors summarize the known interpretations, present connecting bijections and give further references.

In this note we are concerned only with poly-Bernoulli numbers with negative indices. For convenience, we let $B_{n,k}$ denote the poly-Bernoulli numbers $B_n^{(-k)}$.

Kaneko derived two formulas for the poly-Bernoulli numbers with negative indices: a formula that we call the basic formula, and an inclusion-exclusion type formula. The basic formula is

$$B_{n,k} = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\}, \tag{2}$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling number of the second kind A008277 that counts the number of partitions of an n-element set into k non-empty blocks [9]. The inclusion-exclusion type formula is

$$B_{n,k} = \sum_{n=0}^{\infty} (-1)^{n+m} m! \left\{ {n \atop m} \right\} (m+1)^k. \tag{3}$$

Kaneko's proofs were algebraic, based on manipulations of generating functions. The first combinatorial investigation of poly-Bernoulli numbers was done by Brewbaker [7]. He defined $B_{n,k}$ as the number of lonesum matrices of size $n \times k$. He proved combinatorially both formulas; hence, he proved the equivalence of the algebraic definition and the combinatorial one.

Bayad and Hamahata [2] introduced poly-Bernoulli polynomials by the following generating function:

$$\sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt}.$$

For negative indices the polylogarithmic function converges for $|z| < 1$ and equals to

$$\mathrm{Li}_{-k}(z) = \frac{\sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle z^{k-j}}{(1-z)^{k+1}}, \tag{4}$$

where $\left\langle {k \atop j} \right\rangle$ is the Eulerian number [9] A008282 given, for instance, by

$$\left\langle {k \atop j} \right\rangle = \sum_{i=0}^{j} (-1)^i \binom{k+1}{i} (j-i)^k. \tag{5}$$

In [2] the authors used analytical methods to show that for $k \leq 0$

$$B_n^{(k)}(x) = \sum_{j=0}^{|k|} \left\langle {|k| \atop j} \right\rangle \sum_{m=0}^{|k|-j} \binom{|k|-j}{m} (-1)^m (x+m-|k|-1)^n. \tag{6}$$

The evaluation of (6) at $x = 0$ leads to a new explicit formula of the poly-Bernoulli numbers involving Eulerian numbers.

Theorem 1. *[2] For all $k > 0$ and $n > 0$ we have*

$$B_{n,k} = \sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle \sum_{m=0}^{k-j} (-1)^m \binom{k-j}{m} (k+1-m)^n. \tag{7}$$

We see that the Eulerian numbers and the defining generating function of poly-Bernoulli numbers for negative k are strongly related.

In this note we prove this formula purely combinatorially. Moreover, we show four further new formulas for poly-Bernoulli numbers involving Eulerian numbers.

2 Main results

In our proofs a special class of permutations plays the key role. We call this permutation class *Callan permutations* because Callan introduced this class as a combinatorial interpretation of the poly-Bernoulli numbers [8]. We use the well-known notation $[N]$ for $\{1, 2, \ldots, N\}$.

Definition 2. *Callan permutation* of $[n+k]$ is a permutation such that each substring whose support belongs to $N = \{1, 2, \ldots, n\}$ or $K = \{n+1, n+2, \ldots, n+k\}$ is increasing.

Let $\mathcal{C}_n^k$ denote the set of Callan permutations of $[n+k]$. We call the elements in N the *left-value elements* and the elements in K the *right-value elements*. For instance, for $n = 2$ and $k = 2$, the Callan permutations are (writing the left-value elements in red, right-value elements in blue) as follows:

$$1234, 1324, 1423, 1342, 2314, 2413, 2341,$$
$$3124, 3142, 3241, 3412, 4123, 4132, 4231.$$

It is easy to see that Callan permutations are enumerated by the poly-Bernoulli numbers, but for the sake of completeness, we recall a sketch of the proof.

Theorem 3. *[8]*

$$|\mathcal{C}_n^k| = \sum_{m=0}^{\min(n,k)} (m!)^2 \left\{ {n+1 \atop m+1} \right\} \left\{ {k+1 \atop m+1} \right\} = B_{n,k}.$$

Proof. (Sketch) We extend our universe with 0, a special left-value element, and $n+k+1$, a special right-value element. Define $\widehat{N} = N \cup \{0\}$ and $\widehat{K} = K \cup \{n+k+1\}$. Let $\pi \in \mathcal{C}_n^k$. Let $\widetilde{\pi} = 0\pi(n+k+1)$. Divide $\widetilde{\pi}$ into maximal blocks of consecutive elements in such a way that each block is a subset of $\widehat{N}$ (*left blocks*) or a subset of $\widehat{K}$ (*right blocks*). The partition starts with a left block (the block of 0) and ends with a right block (the block of $(n+k+1)$). So the left and right blocks alternate, and their number is the same, say $m+1$.

Describing a Callan permutation is equivalent to specifying m, a partition $\Pi_{\widehat{N}}$ of $\widehat{N}$ into $m+1$ classes (one class is the class of 0, the other m classes are called *ordinary classes*), a partition $\Pi_{\widehat{K}}$ of $\widehat{K}$ into $m+1$ classes (m of them not containing $(n+k+1)$, these are the ordinary classes), and two orderings of the ordinary classes. After specifying the components, we need to merge the two ordered set of classes (starting with the nonordinary class of $\widehat{N}$ and ending with the nonordinary class of $\widehat{K}$), and list the elements of classes in increasing order. The classes of our partitions will form the blocks of the Callan permutations. We will refer to the blocks coming from ordinary classes as *ordinary blocks*.

This proves the claim. $\square$

The main results of this note are the next five formulas for the number of Callan permutations and hence, for the poly-Bernoulli numbers. We present elementary combinatorial proofs of the theorems in the next section. Theorem 8 is equivalent to Theorem 1; we recall the theorem in the combinatorial setting.

Theorem 4. *For all $k > 0$ and $n > 0$ the identity*

$$|\mathcal{C}_n^k| = \sum_{m=0}^{\min(n,k)} \sum_{i=0}^{n} \sum_{j=0}^{k} \left\langle {n \atop i} \right\rangle \left\langle {k \atop j} \right\rangle \binom{n+1-i}{m+1-i} \binom{k+1-j}{m+1-j} = B_{n,k} \tag{8}$$

holds.

Theorem 5. *For all $k > 0$ and $n > 0$ the identity*

$$|\mathcal{C}_n^k| = \sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle \sum_{m=0}^{k+2-j} \binom{k+2-j}{m} (m+j-1)! \left\{ {n \atop m+j-1} \right\} = B_{n,k} \tag{9}$$

holds.

Theorem 6. *For all $k > 0$ and $n > 0$ the identity*

$$|\mathcal{C}_n^k| = \sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle \sum_{m=0}^{j-1} (-1)^m \binom{j-1}{m} (k+1-m)^n = B_{n,k} \tag{10}$$

holds.

Theorem 7. *For all $k > 0$ and $n > 0$ the identity*

$$|\mathcal{C}_n^k| = \sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle \sum_{m=0}^{j+1} \binom{j+1}{m} (m+k-j)! \left\{ {n \atop m+k-j} \right\} = B_{n,k} \tag{11}$$

holds.

Theorem 8. *[2] For all $k > 0$ and $n > 0$ the identity*

$$|\mathcal{C}_n^k| = \sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle \sum_{m=0}^{k-j} (-1)^m \binom{k-j}{m} (k+1-m)^n = B_{n,k}. \tag{12}$$

holds.

3 Proofs of the main results

Eulerian numbers play the crucial role in these formulas. Though Eulerian numbers are well-known, we think it could be helpful for readers who are not so familiar with this topic to recall some basic combinatorial properties.

3.1 Eulerian numbers

First, we need some definitions and notation. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$. We call $i \in [n-1]$ a *descent* (resp., *ascent*) of π if $\pi_i > \pi_{i+1}$ (resp., $\pi_i < \pi_{i+1}$). Let $D(\pi)$ (resp., $A(\pi)$) denote the set of descents (resp., the set of ascents) of the permutation π. For instance, $\pi = 361487925$ has 3 descents and $D(\pi) = \{2, 5, 7\}$, while it has 5 ascents and $A(\pi) = \{1, 3, 4, 6, 8\}$.

Eulerian numbers $\left\langle {k \atop j} \right\rangle$ counts the permutations of $[k]$ with $j-1$ descents. A permutation $\pi \in S_n$ with $j-1$ descents is the union of j increasing subsequences of consecutive entries, so called *ascending runs*. So, in other words $\left\langle {k \atop j} \right\rangle$ is the number of permutations of $[k]$ with j ascending runs. In our example, π is the union of 4 ascending runs: 36, 148, 79, and 25.

There are several identities involving Eulerian numbers, see, for instance, [6, 9]. We will use a strong connection between the surjections/ordered partitions and Eulerian numbers:

$$r!\left\{ {k \atop r} \right\} = \sum_{j=0}^{r} \left\langle {k \atop j} \right\rangle \binom{k-j}{r-j}. \tag{13}$$

Proof. We take all the partitions of $[k]$ into r classes. Order the classes, and list the elements in increasing order. This way we obtain permutations of $[k]$. Counting by multiplicity we get $r!\left\{ {k \atop r} \right\}$ permutations. All of these have at most r ascending runs.

Take a permutation with $j(\leq r)$ ascending runs. How many times did we list it in the previous paragraph? We split the ascending runs by choosing $r-j$ places out of the $k-j$ ascents to obtain all the initial r blocks. The multiplicity is $\binom{k-j}{r-j}$. This proves our claim. $\square$

Inverting (13) immediately gives

$$\left\langle {k \atop j} \right\rangle = \sum_{r=1}^{j} (-1)^{j-r} r! \left\{ {k \atop r} \right\} \binom{k-r}{j-r}.$$

In the previous section we mentioned the close relation between Eulerian numbers and the polylogarithmic function $\mathrm{Li}_k(x)$. Here we recall one possible derivation of the identity (4) following [6].

$$\begin{aligned}
\sum_{j=0}^{k} \left\langle {k \atop j} \right\rangle x^j &= \sum_{j=0}^{\infty} \left\langle {k \atop j} \right\rangle x^j = \sum_{j=0}^{\infty}\sum_{i=0}^{j} (-1)^i \binom{k+1}{i} (j-i)^k x^j = \\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} (-1)^i \binom{k+1}{i} l^k x^{i+l} = \sum_{i=0}^{\infty} (-1)^i \binom{k+1}{i} x^i \sum_{l=0}^{\infty} l^k x^l \\
&= (1-x)^{k+1} \mathrm{Li}_{-k}(x).
\end{aligned}$$

Substituting in (5) for $\left\langle {k \atop j} \right\rangle$, changing the order of the summation, using the transformation $l = j - i$, and finally applying the binomial theorem, we get the result.

3.2 Combinatorial proofs of the theorems

We turn our attention now to the proofs of our theorems. For the sake of convenience, thanks to our color coding (left-value elements are red, and right-value elements are blue), we rewrite the set of right-value elements as $K = \{1, 2, \ldots, k\}$, and $\widehat{K} = K \cup \{k+1\}$. We can do this without essentially changing Callan permutations, since we just need the distinction between the elements N and K and an order in N and K. If we separately consider the left-value elements and right-value elements in the permutation π, the elements of N form a permutation of $[n]$, and the elements of K form a permutation of $[k]$. We let π^r denote the permutation restricted to the right-value elements, and we let π^ℓ denote the permutation restricted to the left-value elements. Further, let $\widetilde{\pi}^r$ and $\widetilde{\pi}^\ell$ denote the extended versions of the permutations. For instance, for $\widetilde{\pi} = 0231454728183569679,$ we have $\widetilde{\pi}^r = 145283679$, while $\widetilde{\pi}^\ell = 0234718569$.

Proof. We consider the last entries of the blocks in the restricted permutations $\widetilde{\pi}^\ell$ (resp., $\widetilde{\pi}^r$). Some of the blocks end with a descent and some of the blocks do not. (The special elements 0 and $k+1$ are neither descents nor ascents of the permutations.) Let i be the number of ascending runs in $\widetilde{\pi}^\ell$ and j be the number of ascending runs in $\widetilde{\pi}^r$. Further, let m be the number of ordinary blocks of both types. The $i-1$ descents of $\widetilde{\pi}^\ell$ determine i last elements of blocks; hence, we are missing $m-(i-1)$ blocks with an ascent as last element. Similarly, the $j-1$ descents of $\widetilde{\pi}^r$ determine $j-1$ blocks and there are $m-(j-1)$ additional blocks with an ascent as last element.

Given a pair $(\widetilde{\pi}^\ell, \widetilde{\pi}^r)$ with $|D(\widetilde{\pi}^\ell)| = i-1$ and $|D(\widetilde{\pi}^r)| = j-1$, we can construct a Callan permutation, according to the above arguments. In our running example, $\widetilde{\pi}^\ell = 0234718569$, we need to choose $3-2=1$ from $9-2$ possibilities. In general, we need to choose $m-(i-1)$ (as last elements of blocks) from $n-(i-1)$ possibilities. And analogously for $\widetilde{\pi}^r$, we need to choose $m-(j-1)$ from $k-(j-1)$ possibilities. Hence, for a given pair $(\widetilde{\pi}^\ell, \widetilde{\pi}^r)$ with $|D(\widetilde{\pi}^\ell)| = i-1$ and $|D(\widetilde{\pi}^r)| = j-1$ we have

$$\binom{n+1-i}{m+1-i}\binom{k+1-j}{m+1-j}$$

different corresponding Callan permutations. Since the number of pairs $(\widetilde{\pi}^\ell, \widetilde{\pi}^r)$ with $|D(\widetilde{\pi}^\ell)| = i-1$ and $|D(\widetilde{\pi}^r)| = j-1$ is $\left\langle{n \atop i}\right\rangle\left\langle{k \atop j}\right\rangle$ The identity (8) is proven. $\square$

Note that (8) is actually a rewriting of the basic combinatorial formula (2) in terms of Eulerian numbers using the relation (13) between the number of ordered partitions and Eulerian numbers.

Now we enumerate Callan permutations according to the number of descents in π^r. Given a permutation π^r with $j-1$ descents, we determine the number of ways to merge π^r with left-value elements to obtain a valid Callan permutation. Let $D = \{d_1, d_2, \ldots, d_{j-1}\}$ be the set of descents of π^r. In our running example, $j=3$ and $D = \{3, 5\}$. We code the positions of the left-values comparing to the right-values by a word w. We let w_i be the number of right-values that are to the left of the left-value i. In our example, $w_1 = 5$, since there are 5 right-value elements preceding the left-value 1, $w_2 = 0$, because there are no right-value elements preceding the left-value 2, etc. Hence, $w = 500366356$. Note that the blocks of the left-value elements can be recognized from the word: The positions i, for which the values w_i are the same, are the elements of the same block. We call a word *valid with respect to* π^r if the augmentation of π^r according to the word w leads to a valid Callan permutation.

Observation 9. A word w is valid with respect to a permutation π^r if and only if it contains every value d_i of the descent set of π^r.

Proof. In a Callan permutation the substrings restricted to K or N are increasing subsequences. Given π^r with descent set D, each $d_i \in D$ has to be the last element of a block in the ordered partition of the set K, the set of right-value elements. Hence, each right-value with position d_i in π^r has to be followed by a left-value element in the Callan permutation. In our word w at the position of this left-value element there is a d_i.

For the converse, assume that our word w contains at least one d_i, for any $d_i \in D(\pi^r)$. There is at least one left-value element with d_i in w at its position. This implies that if we combine π^r and π^ℓ then in π^r the position of the descent will be interrupted by a left-value element. The combined permutation will be a Callan permutation. $\square$

Corollary 10. *The number of valid words with respect to π^r depends only on the number of descents in π^r.*

We let w^{j-1} denote a word that is valid to a π^r with $j-1$ descents and $W(\pi^r)$ denote the set of words w^{j-1}. The number of Callan permutations of size $n+k$ is the number of pairs (π^r, w^{j-1}), where π^r is a permutation of $[k]$ with $j-1$ descents and $w^{j-1} \in W(\pi^r)$. We denote $|W(\pi^r)|$ by $w(j-1)$. Hence,

$$|\mathcal{C}_n^k| = \sum_{j=1}^{k} \left\langle {k \atop j} \right\rangle w(j-1).$$

The next two proofs are based on two different ways of determining $w(j-1)$, i.e., enumerating those w^{j-1}'s that are valid to a π^r with $j-1$ descents.

Proof. Fix π^r and take a $w^{j-1} \in W(\pi^r)$. Then w^{j-1} corresponds to an ordered partition of $[n]$ into at least $j-1$ blocks. Let $j-1+m$ be the number of the blocks.

First, we take an ordered partition of $\{1, 2, \ldots, n\}$ into $m+j-1$ non-empty blocks in $(m+j-1)!\left\{{n \atop m+j-1}\right\}$ ways. Then we refine the partition of π^r, defined by the descents. For the refinement we need to choose additional places for the m blocks. These places can be before the first element of π^r, or at an ascent. We have $\binom{k+2-j}{m}$ choices. This proves (9). $\square$

Proof. Now we calculate $w(j-1)$ using the inclusion-exclusion principle. The total number of words of length n with entries $\{0, 1, \ldots, k\}$ ($w_i = 0$ if the left-value i is in the first block of the Callan permutation) is $(k+1)^n$. We have to reduce this number with the number of invalid words with respect to π^r, with the words that do not contain at least one of the $d_i \in D$. Let A_s be the set of words that do not contain the value s. The quantity $|\overline{\cup_{s\in D} A_s}|$ is to be determined. Clearly, $|A_s| = (k+1-1)^n$ and this number does not depend on the choice of s; hence, we have $\sum_{s\in D} |A_s| = k^n(j-1)$. Then $|A_s \cap A_t| = (k+1-2)^n$ and $\sum_{s,t\in D} |A_s \cap A_t| = (k-1)^n \binom{j-1}{2}$. Analogously, $|\cap_{l=1}^m A_{s_l}| = (k+1-m)^n \binom{j-1}{m}$. The inclusion-exclusion principle gives

$$w(j-1) = \sum_{m=0}^{j-1} (-1)^m \binom{j-1}{m} (k+1-m)^n,$$

and this implies (10). □

Proof. Finally, the identities (11) and (12) follow by the symmetry of Eulerian numbers. If we reverse a permutation of $[k]$ with $j-1$ descents we obtain a permutation with $k-(j-1)-1$ descents. According to our previous arguments a pair (π^r, w^{k-j}), where π^r is a permutation with $k-j$ descents and w^{k-j} is a valid word with respect to π^r determines a Callan permutation. Hence,

$$|\mathcal{C}_n^k| = \sum_{j=1}^{k} \left\langle {k \atop k-j+1} \right\rangle w(k-j) = \sum_{j=1}^{k} \left\langle {k \atop j} \right\rangle w(k-j).$$

We have two formulas for $w(k-j)$.

$$w(k-j) = \sum_{m=0}^{j+1} \binom{j+1}{m} (m+k-j)! \left\{ {n \atop m+k-j} \right\},$$
$$w(k-j) = \sum_{m=0}^{k-j} (-1)^m \binom{k-j}{m} (k+1-m)^n.$$

This implies (11) and (12). □

4 Acknowledgments

The authors thank the referee for carefully reading the manuscript and providing helpful suggestions.

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2010 *Mathematics Subject Classification*: Primary 05A05; Secondary: 05A15, 05A19, 11B83.
Keywords: combinatorial identity, Eulerian number, poly-Bernoulli number.

(Concerned with sequences A008277, A008282, and A099594.)

Received April 6 2018; revised versions received April 10 2018; July 6 2018. Published in *Journal of Integer Sequences*, July 11 2018.

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