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Note on Total Positivity for a Class of Recursive Matrices

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Abstract

In this note, we study the total positivity of a class of infinite recursive matrices that depend on three infinite sets of independent variables and on an integer parameter. We give a simple algebraic proof and provide a few examples.

1 Introduction

A (finite or infinite) matrix M is said to be *totally positive* (TP) if its minors of all orders are nonnegative. Total positivity is a powerful concept that arises frequently in various branches of mathematics, statistics, probability, mechanics, economics, and computer science [4, 10]. The totally positive matrices play an important role in the theory of total positivity [1, 5, 8, 9, 10]. Recently, Chen et al. [6, 7] presented some sufficient conditions for the total positivity of Riordan arrays and recursive matrices. Brenti [4] introduced a class of recursive matrices that depend on three infinite sets of independent variables and on an integer parameter. More precisely, let the infinite matrix $M = [M_{n,k}]_{n,k\geq 0}$ be defined by

$$M_{0,0} = 1, \qquad M_{n,k} = z_n M_{n-t,k-1} + y_n M_{n-1-t,k-1} + x_n M_{n-1,k} \tag{1}$$

for $n + k \geq 1$, where $t \in \mathbb{N}$, $M_{n,k} = 0$ if either $n < 0$ or $k < 0$, and $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, and $(z_n)_{n\geq 0}$ are nonnegative sequences. Brenti [4] associated a planar network with the matrix and used its planarity to prove the total positivity of M.

Theorem 1 ([4, Theorem 4.3])**.** *Let* $M = [M_{n,k}]_{n,k\geq 0}$ *be defined by* (1)*. Then* M *is* TP*.*

In this note we give a simple algebraic proof of Theorem 1. With the method of our proof, the total positivity of many recursive matrices can be easily proved, and a few examples are provided in Section 3.

2 Proof of Theorem 1

We first review some basic facts about TP matrices.

Lemma 2 ([11, Proposition 1.6])**.** *Assume* A *is a nonsingular totally positive matrix. Then so is* $JA^{-1}J$*, where* J *is the diagonal matrix with diagonal entries alternately* 1 *and* -1*.*

An operation that preserves total positivity is the following form of iteration. Let $A = [a_{ij}]_{i,j=1}^{n}$ be an $n\times n$ matrix. Define the matrix $B = [b_{ij}]_{i,j=1}^{n}$ by $b_{1j} = a_{1j}, j = 1,\ldots,n$, and for $i\geq 2, b_{ij} = \sum_{k=1}^{n} b_{i-1,k}a_{kj}, j = 1,\ldots,n$. Pinkus [11] showed that if A is TP, then B is also TP. Actually, this result can be generalized to infinite matrices when $A = [a_{i,j}]_{i,j\geq 0}$ is an upper triangular matrix. It is clear that $B = [b_{i,j}]_{i,j\geq 0}$ is TP if and only if its leading principal submatrices $[b_{i,j}]_{0\leq i,j\leq n}$ are all TP [6, Lemma 2.1], and $[b_{i,j}]_{0\leq i,j\leq n}$ is obtained directly from $[a_{i,j}]_{0\leq i,j\leq n}$. Then we have the following.

Lemma 3. *Let* $A^T = [\alpha_1,\alpha_2,\ldots]$ *be an infinite upper triangular matrix, where for* $j\geq 1$, α_j *denotes the infinite column vector. Define the matrix* $B^T = [\beta_1,\beta_2,\ldots]$ *by* $\beta_1 = \alpha_1$*, and for* $i\geq 2$, ${\beta_i}^T = {\beta_{i-1}}^T A$*. If* A *is* TP*, then so is* B*.*

To prove Theorem 1, we construct two matrices

$$A = [a_{ij}]_{i\geq 0,j\geq 0} = \begin{bmatrix} 0 & 1 & x_1 & x_1x_2 & x_1x_2x_3 & \cdots \\ & z_0 & y_1+x_1z_0 & x_2(y_1+x_1z_0) & x_2x_3(y_1+x_1z_0) & \\ & & z_1 & y_2+x_2z_1 & x_3(y_2+x_2z_1) & \\ & & & z_2 & y_3+x_3z_2 & \\ & & & & z_3 & \\ & & & & & \ddots \end{bmatrix}$$

and

$$I_t = \begin{bmatrix} 1 & O \\ O & U \end{bmatrix},$$

where $U = [\delta_{i+t,j}]_{i,j\geq 0}$ and $\delta_{i,j}$ is the Kronecker delta, i.e.,

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i\neq j. \end{cases}$$

Clearly, $I_0 = I$ and I_tA is the matrix obtained from A by removing rows of A from the second row to the $(t+1)$th row. Applying Lemma 3 to the matrix I_tA, we get $B = [O,\widetilde{B}]$, where $\widetilde{B} = [b_{i,j}]_{i\geq 0,j\geq 0}$.

Proof of Theorem 1. We first show that $M = \widetilde{B}^T$. For $n = 0$, it is readily verified that $b_{0,k} = M_{k,0}$. For $n \geq 1$, we have

$$\begin{aligned} b_{n,k} =& \prod_{i=t+2}^{k} x_i(y_{t+1} + x_{t+1}z_t)b_{n-1,0} + \prod_{i=t+3}^{k} x_i(y_{t+2} + x_{t+2}z_{t+1})b_{n-1,1} \\ &+ \cdots + (y_k + x_k z_{k-1})b_{n-1,k-t-1} + z_k b_{n-1,k-t} \\ =& x_k b_{n,k-1} + y_k b_{n-1,k-t-1} + z_k b_{n-1,k-t}. \end{aligned}$$

Thus $M = \widetilde{B}^T$.

By the definition of TP and the fact that $I_t A$ is a submatrix of A, it suffices to show that A is TP. Let

$$X = \begin{bmatrix} 1 & 0 & & & & \\ & 1 & x_1 & & & \\ & & 1 & x_2 & & \\ & & & 1 & x_3 & \\ & & & & 1 & \ddots \\ & & & & & \ddots \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & & & & \\ & z_0 & y_1 & & & \\ & & z_1 & y_2 & & \\ & & & z_2 & y_3 & \\ & & & & z_3 & \ddots \\ & & & & & \ddots \end{bmatrix}.$$

It is obvious that X and Y are TP. Note that by column operations, we can turn A into Y. Thus A has a factorization of the form

$$A = YJX^{-1}J,$$

where J is defined in Lemma 2, which implies that $JX^{-1}J$ is TP. It is known that the product of two TP matrices is still TP by the classic Cauchy-Binet formula. Thus A is TP. This completes the proof. □

3 Applications

In this section we consider some examples for different t. A basic example for case $t = 0$ and $t = 1$ is the Delannoy square and the Delannoy triangle [12, A008288]. Many properties and applications of Delannoy numbers have been discussed [2, 3, 13, 15]. The Delannoy numbers $d_{n,k}$ are defined as the numbers of lattice paths from $(0, 0)$ to (n, k) with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$. It is well known that the Delannoy numbers satisfy the recursion

$$d_{n,k} = d_{n,k-1} + d_{n-1,k-1} + d_{n-1,k}.$$

Brenti [4] showed that the Delannoy square

$$D = [d_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 5 & 7 & \\ 1 & 5 & 13 & 25 & \\ 1 & 7 & 25 & 63 & \\ \vdots & & & & \ddots \end{bmatrix}$$

is TP. Let $d(n,k) = d_{n-k,k}$ denote the corresponding Delannoy triangle $\bar{D} = [d(n,k)]_{n,k\geq 0}$, i.e.,

$$\bar{D} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 3 & 1 & & \\ 1 & 5 & 5 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Then the entries in $\bar{D}$ satisfy the recurrence relation

$$d(n+1,k+1) = d(n,k) + d(n-1,k) + d(n,k+1).$$

Wang and Yang [14] showed that $\bar{D}$ is TP.

With the method of our proof, applying Lemma 3 to the matrices

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 2 & 2 & \\ & & 1 & 2 & 2 & \\ & & & 1 & 2 & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ & 0 & 1 & 2 & 2 & \\ & & 0 & 1 & 2 & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix},$$

we get

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 3 & 5 & 7 & \\ 0 & 1 & 5 & 13 & 25 & \\ 0 & 1 & 7 & 25 & 63 & \\ 0 & 1 & 9 & 41 & 129 & \\ \vdots & & & & & \ddots \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ & 0 & 1 & 3 & 5 & \\ & & 0 & 1 & 5 & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix}.$$

Then the total positivity of the Delannoy square and the Delannoy triangle immediately follows.

An example for case $t = 2$ is the Harer-Zagier numbers $g(n,k)$ [12, A035309], which count the number of ways to glue all edges of a $2n$-gon pairwise, so as to produce a surface of given genus k. The first three columns of $G = [g(n,k)]_{n,k\geq 0}$ (for $k = 0,1,2$) are respectively the Catalan numbers [12, A000108], A002802, and A006298. The entries of G satisfy the recursion

$$(n+2)g(n+1,k) = (4n+2)g(n,k) + (4n^3 - n)g(n-1,k-1), \tag{2}$$

where $g(n,k) = 0$ if either $n < 0$ or $k < 0$, $g(0,0) = 1$, and $g(0,k) = 0$. So,

$$G = \begin{bmatrix} 1 & & & & \\ 1 & & & & \\ 2 & 1 & & & \\ 5 & 10 & & & \\ 14 & 70 & 21 & & \\ 42 & 420 & 483 & & \\ 132 & 2310 & 6468 & 1485 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Using Lemma 3 with

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 5 & 14 & 42 & \cdots \\ & & & 1 & \frac{5}{2} & 7 & 21 & \\ & & & & \frac{15}{2} & 21 & 63 & \\ & & & & & 21 & 63 & \\ & & & & & & 42 & \\ & & & & & & & \ddots \end{bmatrix},$$

we get

$$B = \begin{bmatrix} 0 & 1 & 1 & 2 & 5 & 14 & 42 & \cdots \\ & & & 1 & 10 & 70 & 420 & \\ & & & & & 21 & 483 & \\ & & & & & & & \ddots \end{bmatrix}.$$

Then we have the following.

Corollary 4. *Let $G = [g(n,k)]_{n,k\geq 0}$ be defined by (2). Then G is TP.*

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