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Sums of Products of Generalized Ramanujan Sums

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Abstract

We consider weighted averages for the products $t_{k_1}^{(1)}(j)\cdots t_{k_n}^{(n)}(j)$ of generalized Ramanujan sums $t_{k_i}^{(i)}(j)=\sum_{d|\gcd(k_i,j)} f_i(d)g_i(k_i/d)h_i(j/d)$ with any arithmetical functions f_i, g_i and h_i $(i=1,\ldots,n)$, and derive formulas for several weighted averages with weights concerning completely multiplicative functions, completely additive functions, and others.

1 Introduction

Let $\gcd(k, j)$ be the greatest common divisor of the positive integer k and the integer j, and let $K = \operatorname{lcm}(k_1, \ldots, k_n)$ be the least common multiple of n-tuple positive integers $k_1, \ldots, k_n$. The function $c_k(j)$, as usual, denotes the Ramanujan sum defined as the sum of the m-th powers of the primitive k-th roots of unity, namely,

$$c_k(j) = \sum_{\substack{1 \le m \le k \\ \gcd(m,k)=1}} \exp\left(2\pi i \frac{mj}{k}\right),$$

which can be expressed as the well-known identity

$$\sum_{d \mid \gcd(k,j)} d\mu\left(\frac{k}{d}\right)$$

with the Möbius function μ. Anderson and Apostol [5] (also see [6, 7, 14, 17]) first introduced the sum

$$s_k(j) = \sum_{d \mid \gcd(k,j)} f(d) g\left(\frac{k}{d}\right)$$

with any arithmetical functions f and g, which is a generalization of Ramanujan's sum. This function is said to be the Anderson–Apostol sum. Now, let f be a completely multiplicative function, and let $g(k) = \mu(k)u(k)$ with u a multiplicative function. Assume that $f(p) \neq 0$ and $f(p) \neq u(p)$ for all primes p, and let $F(k) = (f * g)(k)$. Anderson and Apostol [5] derived the identity $s_k(j) = F(k)\mu(m)\, u(m)/F(m)$ with $m = k/\gcd(k, j)$. This is said to be *Hölder's identity* for $s_k(j)$, which has been considered by many mathematicians and physicists. For any fixed positive integer r, using the properties of the sum

$$\sum_{j=1}^{k} j^r c_k(j), \tag{1}$$

Alkan [1, 3] derived exact formulas involving Euler's and Jordan's functions for averages of the special values of L-functions. Tóth [18] showed a simpler proof of (1) and established some identities for other weighted averages of Ramanujan's sum with weights concerning logarithms, values of arithmetic functions for gcd's, the gamma function, the Bernoulli polynomials, and binomial coefficients. In a recent paper, Kiuchi, Minamide and Ueda [11] gave a generalization of some identities due to Tóth [18]; they showed that

$$\frac{1}{k^r} \sum_{j=1}^{k} j^r s_k(j) = \frac{1}{2}(f * g)(k) + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m}(f * \mathrm{id}_{1-2m} \cdot g)(k) \tag{2}$$

for any fixed positive integer r,

$$\sum_{j=1}^{k} s_k(j)\log j = (f\cdot\log * g\cdot \mathrm{id})(k) + (f * g\cdot \mathrm{Log})(k), \tag{3}$$

$$\sum_{j=1}^{k} s_k(j)\log\Gamma\left(\frac{j}{k}\right) = \log\sqrt{2\pi}\,\{(f*g\cdot\mathrm{id})(k) - (f*g)(k)\} - \frac{1}{2}(f*g\cdot\log)(k), \tag{4}$$

$$\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}s_k(j) = \sum_{d\mid k}\frac{f(d)}{d}g\left(\frac{k}{d}\right)\sum_{l=1}^{d}(-1)^{\frac{lk}{d}}\cos^k\frac{l\pi}{d} \tag{5}$$

and

$$\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right)s_k(j) = \frac{B_m}{k^{k-1}}(\mathrm{id}_{m-1}\cdot f * g)(k), \tag{6}$$

where the function $\mathrm{Log}\, d$ is given by $\log(d!)$ and Γ denotes the gamma function. Here, $B_m(x)$ [7] (also see [8, 9]) denotes the Bernoulli polynomials defined by the expansion

$$\frac{ze^{xz}}{e^z-1} = \sum_{m=0}^{\infty} B_m(x)\frac{z^m}{m!}$$

where $|z| < 2\pi$. The number B_m is the Bernoulli number given by $B_m(0)$. The expressions in (2)–(6) give a generalization of some interesting identities by Tóth [12] and Alkan [1, 2, 3, 4]. The function

$$E(k_1,\ldots,k_n) = \frac{1}{K}\sum_{j=1}^{K} c_{k_1}(j)\cdots c_{k_n}(j) \tag{7}$$

of the product $c_{k_1}(j)\cdots c_{k_n}(j)$ of Ramanujan's sums for n variables was investigated in the studies of Liskovets [12] and Tóth [20], which has some interesting formulas for combinatorial and topological applications. The expression (7) has been introduced by Mednykh and Nedela [15] to handle certain problems of enumerative combinatorics. They showed that all the values $E(k_1,\ldots,k_n)$ are nonnegative integers. Furthermore, Tóth derived that two interesting representations [20, Propositions 3 and 9], [19, Corollary 4] for E hold:

$$E(k_1,\ldots,k_n) = \sum_{d_1\mid k_1,\ldots,d_n\mid k_n}\frac{d_1\cdots d_n}{\mathrm{lcm}(d_1,\ldots,d_n)}\mu\left(\frac{k_1}{d_1}\right)\cdots\mu\left(\frac{k_n}{d_n}\right) \tag{8}$$

and

$$E(k_1,\ldots,k_n) = \frac{1}{K}\sum_{d\mid K} c_{k_1}(d)\cdots c_{k_n}(d)\phi\left(\frac{K}{d}\right), \tag{9}$$

where ϕ is the Euler totient function. He also noted that all values for E are nonnegative integers and that E is multiplicative as a function of several variables. Let $\widetilde{E}$ denote a generalization of E defined by

$$\widetilde{E}(k_1, \ldots, k_n) = \frac{1}{K} \sum_{j=1}^{K} s_{k_1}(j) \cdots s_{k_n}(j). \tag{10}$$

Then Tóth [20, Proposition 19] deduced that two formulas for $\widetilde{E}$ hold:

$$\widetilde{E}(k_1, \ldots, k_n) = \sum_{d_1|k_1,\ldots,d_n|k_n} \frac{f(d_1)\cdots f(d_n)}{\operatorname{lcm}(d_1, \ldots, d_n)} g\left(\frac{k_1}{d_1}\right) \cdots g\left(\frac{k_n}{d_n}\right) \tag{11}$$

and

$$\widetilde{E}(k_1, \ldots, k_n) = \frac{1}{K} \sum_{d|K} s_{k_1}(d) \cdots s_{k_n}(d) \phi\left(\frac{K}{d}\right). \tag{12}$$

He also showed that if f and g are multiplicative functions, then (10) is multiplicative as a function of several variables. The weighted average of the products of Ramanujan's sum with weight concerning the function id_r for any fixed positive integer r defined by

$$S_r(k_1, \ldots, k_n) = \frac{1}{K^{r+1}} \sum_{j=1}^{K} j^r c_{k_1}(j) \cdots c_{k_n}(j) \tag{13}$$

was considered by Tóth [18, Proposition 7], who derived the following identity

$$\begin{aligned} S_r(k_1, \ldots, k_n) &= \frac{\phi(k_1)\cdots\phi(k_n)}{2K} \\ &+ \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{K^{2m}} \sum_{d_1|k_1,\ldots,d_n|k_n} \frac{d_1 \cdots d_n}{\operatorname{lcm}(d_1, \ldots, d_n)^{1-2m}} \mu\left(\frac{k_1}{d_1}\right) \cdots \mu\left(\frac{k_n}{d_n}\right). \end{aligned} \tag{14}$$

He presented a multivariable generalization of (1) connected to the orbicyclic arithmetic function, discussed by Liskovets [12] and Tóth [20]. Substituting $r = 1$ in (14), it follows that

$$S_1(k_1, \ldots, k_n) = \frac{\phi(k_1)\cdots\phi(k_n)}{2K} + \frac{E(k_1, \ldots, k_n)}{2},$$

which was given by Tóth [18, Corollary 2].

For any arithmetical functions f_i, g_i and h_i ($i = 1, 2, \ldots, n$), we define $s_{k_i}^{(i)}(j)$ and $t_{k_i}^{(i)}(j)$ by

$$s_{k_i}^{(i)}(j) = \sum_{d|\gcd(k_i,j)} f_i(d) g_i\left(\frac{k_i}{d}\right) \quad \text{and} \quad t_{k_i}^{(i)}(j) = \sum_{d|\gcd(k_i,j)} f_i(d) g_i\left(\frac{k_i}{d}\right) h_i\left(\frac{j}{d}\right),$$

respectively.

The first aim of this study is to derive some identities for weighted averages of the product $t^{(1)}_{k_1}(j)\cdots t^{(n)}_{k_n}(j)$ with weight function w_r, completely multiplicative, completely additive, and others, for any fixed positive integer r, namely

$$U_r(k_1,\ldots,k_n)=\sum_{j=1}^{K}w_r(j)t^{(1)}_{k_1}(j)\cdots t^{(n)}_{k_n}(j). \tag{15}$$

This sum is a generalization of some of the identities for weighted averages mentioned by Alkan [2], Liskovets [12], Tóth [18, 20] and Kiuchi, Minamide and Ueda [11]. To our knowledge, we derive some new and useful formulas in Theorems 1, 6 and 10. This study then aims to establish some identities for the weighted averages of the product $s^{(1)}_{k_1}(j)\cdots s^{(n)}_{k_n}(j)$ with weights concerning the gamma function, binomial coefficients, and Bernoulli polynomials, which provide a generalization of some interesting identities discussed by Tóth [18] and Kiuchi, Minamide and Ueda [11].

Notations. We use the following notation throughout this paper. For any positive integer n, the functions id and id_q as well as the unit function $\mathbf{1}$ are given by $\mathrm{id}(n)=n$, $\mathrm{id}_q(n)=n^q$ for any real number q and $\mathbf{1}(n)=1$, respectively. The symbols $*$ and $\cdot$ denote the Dirichlet convolution and the ordinary product of arithmetical functions, respectively. The function $\phi_s(n)$ defines the Jordan function by $(\mathrm{id}_s*\mu)(n)$ for any real number s.

2 Some formulas for $U_r(k_1,\ldots,k_n)$

We shall evaluate the function $U_r(k_1,\ldots,k_n)$. The weight function w_r of $U_r(k_1,\ldots,k_n)$ only deals with completely multiplicative functions and completely additive functions, and we introduce a simple proof of (15) and some useful formulas in Theorems 1 and 6. Moreover, we shall derive some identities of the weighted averages of the product $s^{(1)}_{k_1}(j)\cdots s^{(n)}_{k_n}(j)$ with weights concerning the gamma function, binomial coefficients, and the Bernoulli polynomials.

Theorem 1. *Let $f_1,\ldots,f_n$, $g_1,\ldots,g_n$, and $h_1,\ldots,h_n$ denote any arithmetical functions, and let*

$$\begin{aligned}&U_r(k_1,\ldots,k_n)\\&=\sum_{j=1}^{K}w_r(j)\sum_{d_1\mid\gcd(k_1,j)}f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)h_1\left(\frac{j}{d_1}\right)\cdots\sum_{d_n\mid\gcd(k_n,j)}f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)h_n\left(\frac{j}{d_n}\right).\end{aligned}$$

If w_r is a completely multiplicative function, we have

$$\begin{aligned} &U_r(k_1,\ldots,k_n) \\ &= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)w_r(\mathrm{lcm}(d_1,\ldots,d_n))\times \\ &\qquad \times \sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} w_r(l)h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_n}l\right), \end{aligned} \tag{16}$$

and if in addition, $h_1,\ldots,h_n$ are completely multiplicative functions, then

$$\begin{aligned} &U_r(k_1,\ldots,k_n) \\ &= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\times \\ &\times h_n\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_n}\right)w_r\left(\mathrm{lcm}(d_1,\ldots,d_n)\right)\sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} w_r(l)h_1\left(l\right)\cdots h_n\left(l\right). \end{aligned} \tag{17}$$

If w_r is a completely additive function, we have

$$\begin{aligned} &U_r(k_1,\ldots,k_n) \\ &= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)w_r(\mathrm{lcm}(d_1,\ldots,d_n))\times \\ &\qquad \times \sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_n}l\right) \\ &+ \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\times \\ &\qquad \times \sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} w_r(l)h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_n}l\right), \end{aligned} \tag{18}$$

and if in addition, $h_1, \dots, h_n$ are completely multiplicative functions, then

$$
\begin{aligned}
&U_r(k_1, \dots, k_n) \qquad (19)\\
&= \sum_{d_1|k_1,\dots,d_n|k_n} f_1(d_1) g_1\left(\frac{k_1}{d_1}\right) h_1\left(\frac{\operatorname{lcm}(d_1,\dots,d_n)}{d_1}\right) \cdots f_n(d_n) g_n\left(\frac{k_n}{d_n}\right) h_n\left(\frac{\operatorname{lcm}(d_1,\dots,d_n)}{d_n}\right) \times \\
&\quad \times w_r(\operatorname{lcm}(d_1,\dots,d_n)) \sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\dots,d_n)}} h_1(l) \cdots h_n(l) \\
&+ \sum_{d_1|k_1,\dots,d_n|k_n} f_1(d_1) g_1\left(\frac{k_1}{d_1}\right) h_1\left(\frac{\operatorname{lcm}(d_1,\dots,d_n)}{d_1}\right) \cdots f_n(d_n) g_n\left(\frac{k_n}{d_n}\right) h_n\left(\frac{\operatorname{lcm}(d_1,\dots,d_n)}{d_n}\right) \times \\
&\quad \times \sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\dots,d_n)}} w_r(l) h_1(l) \cdots h_n(l).
\end{aligned}
$$

Remark 2. The formulas (16) and (17) immediately imply a generalization of the two formulas (8) and (11). We substitute $w_r = \mathbf{1}$, $f_1 = \cdots = f_n = f$, $g_1 = \cdots = g_n = g$, and $h_1 = \cdots = h_n = \mathbf{1}$ in (16) to obtain the formula (11).

The formulas (16)–(19) give an analogue and a generalization of result of Tóth [22, Proposition 1]. For any positive integer k, Kiuchi, Minamide and Ueda [11] recently showed that if w_r is a completely multiplicative function, then the identity

$$\sum_{j=1}^{k} w_r(j) t_k(j) = (f \cdot w_r * g \cdot W)(k) \qquad (20)$$

holds with $W(d) = \sum_{m=1}^{d} w_r(m) h(m)$, and if w_r is a completely additive function, then the identity

$$\sum_{j=1}^{k} w_r(j) t_k(j) = (f \cdot w_r * g \cdot H)(k) + (f * g \cdot W)(k) \qquad (21)$$

holds with $H(d) = \sum_{m=1}^{d} h(m)$. Thus, the formulas (16) and (18) give a generalization of (20) and (21), respectively. Substituting $w_r = \mathrm{id}_r$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (17), $w_r = \mathrm{id}_r$ and $g_1 = \cdots = g_n = \mathbf{1}$ in (16), and $w_r = \mathrm{id}_r$ and $f_1 = \cdots = f_1 = \mathbf{1}$ in (16), we derive the following formulas (22), (23), and (24), respectively.

Corollary 3. *Let the notation be as above. Then we have*

$$\frac{1}{K^{r+1}}\sum_{j=1}^{K} j^r s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j) \tag{22}$$
$$= \frac{1}{2K}(f_1 * g_1)(k_1)\cdots(f_n * g_n)(k_n)$$
$$+\frac{1}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}\frac{B_{2m}}{K^{2m}}\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)^{1-2m}}g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right),$$

$$\sum_{j=1}^{K} j^r \sum_{d_1|\gcd(k_1,j)} f_1(d_1)h_1\left(\frac{j}{d_1}\right)\cdots\sum_{d_n|\gcd(k_n,j)} f_n(d_n)h_n\left(\frac{j}{d_n}\right) \tag{23}$$
$$=\sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)\cdots f_n(d_n)\,\mathrm{lcm}(d_1,\ldots,d_n)^r\times$$
$$\times\sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} l^r h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_n}l\right)$$

and

$$\sum_{j=1}^{K} j^r \sum_{d_1|\gcd(k_1,j)} g_1\left(\frac{k_1}{d_1}\right)h_1\left(\frac{j}{d_1}\right)\cdots\sum_{d_n|\gcd(k_n,j)} g_n\left(\frac{k_n}{d_n}\right)h_n\left(\frac{j}{d_n}\right) \tag{24}$$
$$=\sum_{d_1|k_1,\ldots,d_n|k_n} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\mathrm{lcm}(d_1,\ldots,d_n)^r\times$$
$$\times\sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} l^r h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_n}l\right).$$

The formula (22) also gives a generalization of the formula (2). As an application of Corollary 3, we give some formulas for weighted averages of the products of the arithmetical functions of Anderson–Apostol sums.

Example 4. Let the notation be as above. Then we have

$$\frac{1}{K^{r+1}}\sum_{j=1}^{K} j^r \gcd(k_1,j)\cdots\gcd(k_n,j) \tag{25}$$
$$=\frac{k_1k_2\cdots k_n}{2K}+\frac{1}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}\frac{B_{2m}}{K^{2m}}\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{\phi(d_1)\cdots\phi(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)^{1-2m}},$$

(26)

$$\frac{1}{K^{2r}}\sum_{j=1}^{K} j^{2r}\sum_{d_1\,|\,\gcd(k_1,j)}\frac{\phi(d_1)}{d_1}\cdots\sum_{d_1|(k_r,j)}\frac{\phi(d_r)}{d_r} \tag{27}$$

$$= \frac{1}{2}\sum_{d_1|k_1,\ldots,d_r|k_r}\frac{\phi(d_1)\cdots\phi(d_r)}{d_1\cdots d_r}$$

$$+\frac{K}{2r+1}\sum_{m=0}^{r}\binom{2r+1}{2m}\frac{B_{2m}}{K^{2m}}\sum_{d_1|k_1,\ldots,d_r|k_r}\frac{\phi(d_1)}{d_1}\cdots\frac{\phi(d_r)}{d_r}\operatorname{lcm}(d_1,\ldots,d_r)^{2m-1},$$

$$\frac{1}{K^{r+n+1}}\sum_{j=1}^{K} j^{r+n}\sum_{d_1\,|\,\gcd(k_1,j)}\frac{f_1(d_1)}{d_1}\cdots\sum_{d_n|(k_n,j)}\frac{f_n(d_n)}{d_n} \tag{28}$$

$$= \frac{1}{2K}\sum_{d_1|k_1}\frac{f_1(d_1)}{d_1}\cdots\sum_{d_n|k_n}\frac{f_n(d_1)}{d_n}$$

$$+\frac{1}{r+n+1}\sum_{m=0}^{\lfloor\frac{r+n}{2}\rfloor}\binom{r+n+1}{2m}\frac{B_{2m}}{K^{2m}}\sum_{d_1|k_1,\ldots,d_n|k_n} f_1\left(d_1\right)\cdots f_n\left(d_n\right)\frac{\operatorname{lcm}(d_1,\ldots,d_n)^{2m-1}}{d_1\cdots d_n},$$

and

$$\frac{1}{K^{r+n+1}}\sum_{j=1}^{K} j^{r+n}\sum_{d_1\,|\,\gcd(k_1,j)} g_1\left(\frac{k_1}{d_1}\right)\frac{1}{d_1}\cdots\sum_{d_n\,|\,\gcd(k_n,j)} g_n\left(\frac{k_n}{d_n}\right)\frac{1}{d_n} \tag{29}$$

$$= \frac{1}{2K}\left(g_1 * \frac{1}{\mathrm{id}}\right)(k_1)\cdots\left(g_n * \frac{1}{\mathrm{id}}\right)(k_n)$$

$$+\frac{1}{r+n+1}\sum_{m=0}^{\lfloor\frac{r+n}{2}\rfloor}\binom{r+n+1}{2m}\frac{B_{2m}}{K^{2m}}\sum_{d_1|k_1,\ldots,d_n|k_n} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\frac{\operatorname{lcm}(d_1,\ldots,d_n)^{2m-1}}{d_1\cdots d_n}.$$

We substitute $f_1 = \cdots = f_n = \phi$ and $g_1 = \cdots = g_n = \mathbf{1}$ in (22) to obtain (25) using $(\phi * \mathbf{1})(\gcd(k_i, j)) = \gcd(k_i, j)$ $(i = 1, \ldots, n)$. The formula (25) is an analogue of the identity

$$\frac{1}{K}\sum_{j=1}^{K}\gcd(k_1,j)\cdots\gcd(k_n,j) = \sum_{d_1|k_1,\ldots,d_n|k_n}\frac{\phi(d_1)\cdots\phi(d_n)}{\operatorname{lcm}(d_1,\ldots,d_n)} \tag{30}$$

[20, Proposition 12], [21, Corollary 2]. The formula (30) was mentioned by Liskovets [12], and it was considered by Deitmar, Koyama and Kurokawa [10] in a special case, investigating analytic properties of Igusa-type zeta-functions. Using the arguments of elementary probability theory, the explicit formula for the values of (30) derived in [10] was re-proved by Minami [13] for the general case $m_1, \ldots, m_n$. We substitute $n = r$, $f_1 = \cdots = f_n = \phi$

and $h_1 = \cdots = h_n = \mathrm{id}$ in (23) and use (51) below to obtain (27). The formulas (23) and (24) imply a generalization of two formulas derived in [11], namely

$$\frac{1}{k^r}\sum_{j=1}^{k} j^r \sum_{d|\gcd(k,j)} f(d)h\left(\frac{j}{d}\right) = \sum_{d|k} f\left(\frac{k}{d}\right)\frac{1}{d^r}\sum_{l=1}^{d} h(l)l^r$$

and

$$\frac{1}{k^r}\sum_{j=1}^{k} j^r \sum_{d|\gcd(k,j)} g\left(\frac{k}{d}\right) h\left(\frac{j}{d}\right) = \sum_{d|k} \frac{g(d)}{d^r}\sum_{l=1}^{d} h(l)l^r,$$

which follow from (20). Substituting $h_1 = \cdots = h_n = \mathrm{id}$ in (23) and (24), respectively, and using (51) below we easily obtain the two formulas (28) and (29).

We shall evaluate some identities of weighted averages for the product $t_{k_1}^{(1)} \cdots t_{k_n}^{(n)}$ of the weight w_r with the completely additive function. We set $w_r = \log$ and substitute $h_1 = \cdots = h_n = \mathbf{1}$ in (18), $g_1 = \cdots = g_n = \mathbf{1}$ in (18), and $f_1 = \cdots = f_n = \mathbf{1}$ in (18) to obtain the following formulas (31), (32), and (33), respectively.

Corollary 5. *Let the notation be as above. Then we have*

$$\begin{aligned}&\sum_{j=1}^{K} s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\log j \\ &= K \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\frac{\log \operatorname{lcm}(d_1,\ldots,d_n)}{\operatorname{lcm}(d_1,\ldots,d_n)} \\ &\quad + \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\log\left(\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}\right)!\end{aligned} \tag{31}$$

$$\begin{aligned}&\sum_{j=1}^{K}\log j\sum_{d_1|\gcd(k_1,j)} f_1(d_1)h_1\left(\frac{j}{d_1}\right)\cdots\sum_{d_n|\gcd(k_n,j)} f_n(d_n)h_n\left(\frac{j}{d_n}\right) \\ &= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)\cdots f_n(d_n)\log\operatorname{lcm}(d_1,\ldots,d_n)\times \\ &\quad\times\sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}} h_1\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_n}l\right) \\ &+ \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)\cdots f_n(d_n)\sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}} h_1\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_1\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_n}l\right)\log l\end{aligned} \tag{32}$$

and

$$\sum_{j=1}^{K}\log j \sum_{d_1|\gcd(k_1,j)} g_1\left(\frac{k_1}{d_1}\right) h_1\left(\frac{j}{d_1}\right)\cdots \sum_{d_n|\gcd(k_n,j)} g_n\left(\frac{k_n}{d_n}\right) h_n\left(\frac{j}{d_n}\right) \tag{33}$$

$$= \sum_{d_1|k_1,\ldots,d_n|k_n} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\log \operatorname{lcm}(d_1,\ldots,d_n)\times$$

$$\times \sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}} h_1\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_n}l\right)$$

$$+ \sum_{d_1|k_1,\ldots,d_n|k_n} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\times$$

$$\times \sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}} h_1\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\operatorname{lcm}(d_1,\ldots,d_n)}{d_n}l\right)\log l.$$

(31) implies a generalization of (3). (32) and (33) give a generalization of the formulas

$$\sum_{j=1}^{k}\log j \sum_{d|\gcd(k,j)} f(d)h\left(\frac{j}{d}\right) = (f\cdot\log * H)(k) + (f * I)(k),$$

and

$$\sum_{j=1}^{k}\log j \sum_{d|\gcd(k,j)} g\left(\frac{k}{d}\right) h\left(\frac{j}{d}\right) = (\log * g\cdot H)(k) + (\mathbf{1} * g\cdot W)(k)$$

with $I(d) = \sum_{l=1}^{d} h(l)l$, $H(d) = \sum_{m=1}^{d} h(m)$ and $W(d) = \sum_{m=1}^{d} w_r(m)h(m)$. These two identities immediately follow from (21).

Next, we shall evaluate the function $U_r(k_1,\ldots,k_n)$, which is another representation of (16) and (18). Using the method of Tóth [20], we have the following formulas.

Theorem 6. *Let $k_1,\ldots,k_n$ be any positive integers and let $K = \operatorname{lcm}(k_1,\ldots,k_n)$ be the least common multiple of n-tuple integers $k_1,\ldots,k_n$. If w_r is completely multiplicative function, then*

$$U_r(k_1,\ldots,k_n) = (w_r\cdot t_{k_1}^{(1)}\cdots t_{k_n}^{(n)} * W_r)(K) \tag{34}$$

with

$$W_r(d) = \sum_{\substack{l=1\\(l,d)=1}}^{d} w_r(l),$$

and if w_r is a completely additive function, then

$$U_r(k_1,\ldots,k_n) = (w_r\cdot t_{k_1}^{(1)}\cdots t_{k_n}^{(n)} * \phi)(K) + (t_{k_1}^{(1)}\cdots t_{k_n}^{(n)} * W_r)(K). \tag{35}$$

Remark 7. We substitute $w_r = \mathbf{1}$, $f_1 = \cdots = f_n = \mathrm{id}$, $g_1 = \cdots = g_n = \mu$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (34) to obtain (9), and $w_r = \mathbf{1}, f_1 = \cdots = f_n = f, g_1 = \cdots = g_n = g$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (34) to obtain the formula (12). The formulas (34) and (35) are an analogue of (20) and (21), respectively.

We substitute $w_r = \mathrm{id}$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (34) and use (54) below to obtain the formula (36), which is a generalization of (14). We also substitute $w_r = \log$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (35) and use (52) below to obtain the formula (37).

Corollary 8. *Let the notation be as above. Then we have*

$$\begin{aligned}&\frac{1}{K^r}\sum_{j=1}^{K} j^r s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j) \\ &= \frac{1}{2}s_{k_1}^{(1)}(K)\cdots s_{k_n}^{(n)}(K) + \frac{1}{r+1}\sum_{d\mid K} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}\phi_{1-2m}(d)\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\sum_{j=1}^{K} s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\log j \\ &= (s_{k_1}^{(1)}\cdots s_{k_n}^{(n)}\cdot\log *\phi)(K) + \sum_{d\mid K} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\sum_{e\mid d}\mu(e)\,\mathrm{Log}\,\frac{d}{e} \\ &\quad - \sum_{d\mid K} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\phi(d)\sum_{p\mid d}\frac{\log p}{p-1}\end{aligned} \tag{37}$$

where $\mathrm{Log}\, d$ *is given by* $\log(d!)$.

As an application of Corollary 8, we provide two formulas for the weighted averages of the product $\gcd(k_1, j)\gcd(k_2, j)\cdots\gcd(k_n, j)$ of the gcd's.

Example 9. Let the notation be as above. Then we have

$$\begin{aligned}&\frac{1}{K^r}\sum_{j=1}^{K} j^r \gcd(k_1, j)\cdots\gcd(k_n, j) \\ &= \frac{1}{2}\gcd(k_1, K)\cdots\gcd(k_n, K) \\ &+ \frac{1}{r+1}\sum_{d\mid K}\gcd\left(k_1, \frac{K}{d}\right)\cdots\gcd\left(k_n, \frac{K}{d}\right)\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}\phi_{1-2m}(d),\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\sum_{j=1}^{K} \gcd(k_1, j) \cdots \gcd(k_n, j) \log j \qquad (39)\\
&= \sum_{d|K} \gcd(k_1, d) \cdots \gcd(k_n, d) (\log d) \phi\left(\frac{K}{d}\right)\\
&+ \sum_{d|K} \gcd\left(k_1, \frac{K}{d}\right) \cdots \gcd\left(k_n, \frac{K}{d}\right) \sum_{e|d} \mu(e) \mathrm{Log} \frac{d}{e}\\
&- \sum_{d|K} \gcd\left(k_1, \frac{K}{d}\right) \cdots \gcd\left(k_n, \frac{K}{d}\right) \phi(d) \sum_{p|d} \frac{\log p}{p-1}.
\end{aligned}$$

We substitute $w_r = \mathrm{id}_r$, $f_1 = \cdots = f_n = \phi$ and $g_1 = \cdots = g_n = \mathbf{1}$ in (36) to obtain (38), which is a generalization of Tóth's result [20, Proposition 14]:

$$\frac{1}{K} \sum_{j=1}^{K} \gcd(k_1, j) \cdots \gcd(k_n, j) = \frac{1}{K} \sum_{d|K} \gcd(d, k_1) \cdots \gcd(d, k_n) \phi\left(\frac{K}{d}\right).$$

We also substitute $w_r = \log$, $f_1 = \cdots = f_n = \phi$ and $g_1 = \cdots = g_n = \mathbf{1}$ in (37) to obtain (39).

Lastly, we shall consider some weighted averages of the product $s_{k_1}^{(1)}(j) \cdots s_{k_n}^{(n)}(j)$ with weights concerning the gamma function, binomial coefficients, and Bernoulli polynomials. To state Theorem 10, we use the well-known multiplication formula of Gauss–Legendre [8, Proposition 9.6.33] for the gamma function

$$\prod_{j=1}^{n} \Gamma\left(\frac{j}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \qquad (40)$$

for any positive integer n, and the binomial formula [22, (27)]

$$\sum_{k=0}^{[n/r]} \binom{n}{kr} = \frac{1}{r} \sum_{j=1}^{r} \left(1 + \exp\left(2\pi i \frac{j}{r}\right)\right)^n = \frac{2^n}{r} \sum_{j=1}^{r} \cos^n \frac{j\pi}{r} \cos \frac{nj\pi}{r} \qquad (41)$$

for any positive integers n and r. Furthermore, we use the well-known formula for the Bernoulli polynomial [8, Proposition 9.1.3]

$$\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) = \frac{B_m}{k^{m-1}} \qquad (42)$$

for any positive integer k.

Theorem 10. *Let $k_1, \ldots, k_n$ be any positive integers and let $K = \mathrm{lcm}(k_1, \ldots, k_n)$ be the least common multiple of n-tuple integers $k_1, \ldots, k_n$. Then we have*

$$\begin{aligned}
&\sum_{j=1}^{K} s_{k_1}^{(1)}(j) \cdots s_{k_n}^{(n)}(j) \log \Gamma\left(\frac{j}{K}\right) \qquad (43)\\
&= K \log \sqrt{2\pi} \sum_{d_1|k_1,\ldots,d_n|k_n} \frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\\
&\quad - (f_1 * g_1)(k_1)\cdots(f_n * g_n)(k_n) \log \sqrt{2\pi K}\\
&\quad + \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1) g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n) g_n\left(\frac{k_n}{d_n}\right) \log \sqrt{\mathrm{lcm}(d_1,\ldots,d_n)},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=0}^{K} \binom{K}{j} s_{k_1}^{(1)}(j) \cdots s_{k_n}^{(n)}(j) \qquad (44)\\
&= 2^K \sum_{d_1|k_1,\ldots,d_n|k_n} \frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right) \times\\
&\quad \times \sum_{l=1}^{\mathrm{lcm}(d_1,\ldots,d_n)} (-1)^{\frac{Kl}{\mathrm{lcm}(d_1,\ldots,d_n)}} \cos^K \frac{l\pi}{\mathrm{lcm}(d_1,\ldots,d_n)},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{K-1} B_m\left(\frac{j}{K}\right) s_{k_1}^{(1)}(j) \cdots s_{k_n}^{(n)}(j) \qquad (45)\\
&= \frac{B_m}{K^{m-1}} \sum_{d_1|k_1,\ldots,d_n|k_n} \frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)^{1-m}} g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right).
\end{aligned}$$

As an application of Theorem 10, we give three formulas for weighted averages of the product $\gcd(k_1, j)\gcd(k_2, j)\cdots\gcd(k_n, j)$ of the gcd's.

Example 11. Let the notation be as above. Then we have

$$\begin{aligned}
&\sum_{j=1}^{K} \gcd(k_1, j)\cdots\gcd(k_n, j) \log \Gamma\left(\frac{j}{K}\right) \qquad (46)\\
&= K \log \sqrt{2\pi} \sum_{d_1|k_1,\ldots,d_n|k_n} \frac{\phi(d_1)\cdots\phi(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)} + \frac{1}{2} \sum_{d_1|k_1,\ldots,d_n|k_n} \phi(d_1)\cdots\phi(d_n) \log \mathrm{lcm}(d_1,\ldots,d_n)\\
&\quad - k_1 k_2 \cdots k_n \log \sqrt{2\pi K},
\end{aligned}$$

$$\frac{1}{2^K}\sum_{j=0}^{K}\binom{K}{j}\gcd(k_1,j)\cdots\gcd(k_n,j) \tag{47}$$

$$= \sum_{d_1|k_1,\ldots,d_n|k_n}\frac{\phi(d_1)\cdots\phi(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)}\sum_{l=1}^{\mathrm{lcm}(d_1,\ldots,d_n)}(-1)^{\frac{Kl}{\mathrm{lcm}(d_1,\ldots,d_n)}}\cos^K\frac{l\pi}{\mathrm{lcm}(d_1,\ldots,d_n)}$$

and

$$\sum_{j=0}^{K-1}B_m\left(\frac{j}{K}\right)\gcd(k_1,j)\cdots\gcd(k_n,j) = \frac{B_m}{K^{m-1}}\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{\phi(d_1)\cdots\phi(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)^{1-m}}. \tag{48}$$

We substitute $f_1=\cdots=f_n=\phi$ and $g_1=\cdots=g_n=\mathbf{1}$ in (43) and $f_1=\cdots=f_n=\phi$ and $g_1=\cdots=g_n=\mathbf{1}$ in (44) to obtain (46) and (47), respectively. We also substitute $f_1=\cdots=f_n=\phi$ and $g_1=\cdots=g_n=\mathbf{1}$ in (45) to obtain (48), which is an analogue of (30).

3 Proofs of Theorems 1, 6, 10 and Corollaries 3, 5, 8

Proof of Theorem 1. Since

$$t_{k_i}^{(i)}(j) = \sum_{d_i|\gcd(k_i,j)} f_i(d_i)g_i\left(\frac{k_i}{d_i}\right)h_i\left(\frac{j}{d_i}\right) \quad (i=1,2,\ldots,n),$$

we have

$$U_r(k_1,\ldots,k_n) \tag{49}$$

$$= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\sum_{\substack{j=1\\ d_1|j,\ldots,d_n|j}}^{K} w_r(j)h_1\left(\frac{j}{d_1}\right)\cdots h_n\left(\frac{j}{d_n}\right)$$

$$= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\times$$

$$\times\sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} w_r\left(l\,\mathrm{lcm}(d_1,\ldots,d_n)\right)h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_n,\ldots,d_n)}{d_n}l\right).$$

We use the completely multiplicative function w_r in (49) to obtain

$$U_r(k_1,\ldots,k_n) \tag{50}$$

$$= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)w_r\left(\mathrm{lcm}(d_1,\ldots,d_n)\right)\times$$

$$\times\sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}} w_r(l)\,h_1\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{d_1}l\right)\cdots h_n\left(\frac{\mathrm{lcm}(d_n,\ldots,d_n)}{d_n}l\right),$$

and if $h_1, \dots, h_n$ are completely multiplicative functions, (50) gives

$$\begin{aligned} &U_r(k_1, \dots, k_n) \\ &= \sum_{d_1|k_1,\dots,d_n|k_n} f_1(d_1) g_1\left(\frac{k_1}{d_1}\right) h_1\left(\frac{\operatorname{lcm}(d_1,\dots,d_n)}{d_1}\right) \cdots f_n(d_n) g_n\left(\frac{k_n}{d_n}\right) \times \\ &\quad \times h_n\left(\frac{\operatorname{lcm}(d_1,\dots,d_n)}{d_n}\right) w_r\left(\operatorname{lcm}(d_1,\dots,d_n)\right) \sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\dots,d_n)}} w_r(l) h_1(l) \cdots h_n(l). \end{aligned}$$

Similarly, as in the proof of the above, using the completely additive function w_r in (49), we have the identity (18), and if $h_1, \dots, h_n$ are completely multiplicative functions, we establish the identity (19). This completes the proof of Theorem 1. □

Proof of Corollary 3. We substitute $w_r = \mathrm{id}_r$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (17) and use

$$\sum_{m=1}^{N} m^r = \frac{N^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} N^{r+1-2m} \tag{51}$$

for any positive integer $N > 1$ [8, Proposition 9.2.12], [9, Section 3.9] to obtain

$$\begin{aligned} &\sum_{j=1}^{K} j^r s_{k_1}^{(1)}(j) \cdots s_{k_n}^{(n)}(j) \\ &= \sum_{d_1|k_1,\dots,d_n|k_n} f_1(d_1) g_1\left(\frac{k_1}{d_1}\right) \cdots f_n(d_n) g_n\left(\frac{k_n}{d_n}\right) \operatorname{lcm}(d_1,\dots,d_n)^r \sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\dots,d_n)}} l^r \\ &= \sum_{d_1|k_1,\dots,d_n|k_n} f_1(d_1) g_1\left(\frac{k_1}{d_1}\right) \cdots f_n(d_n) g_n\left(\frac{k_n}{d_n}\right) \operatorname{lcm}(d_1,\dots,d_n)^r \times \\ &\times \left\{ \frac{1}{2}\left(\frac{K}{\operatorname{lcm}(d_1,\dots,d_n)}\right)^r + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} \left(\frac{K}{\operatorname{lcm}(d_1,\dots,d_n)}\right)^{r+1-2m} \right\} \\ &= \frac{(f_1 * g_1)(k_1) \cdots (f_n * g_n)(k_n)}{2} K^r \\ &+ \frac{K^{r+1}}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{K^{2m}} \sum_{d_1|k_1,\dots,d_n|k_n} \frac{f_1(d_1) \cdots f_n(d_n)}{\operatorname{lcm}(d_1,\dots,d_n)^{1-2m}} g_1\left(\frac{k_1}{d_1}\right) \cdots g_n\left(\frac{k_n}{d_n}\right), \end{aligned}$$

which proves (22). We substitute $g_1 = \cdots = g_n = \mathbf{1}$ and $w_r = \mathrm{id}_r$ in (16) and $f_1 = \cdots = f_n = \mathbf{1}$ and $w_r = \mathrm{id}_r$ in (16) to obtain the formulas for (23) and (24), respectively. □

Proof of Corollary 5. We substitute $w_r = \log$ and $h_1 = \cdots = h_n = \mathbf{1}$ in (18) to obtain

$$\begin{aligned}
&\sum_{j=1}^{K} s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\log j \\
&= K \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\frac{\log \operatorname{lcm}(d_1,\ldots,d_n)}{\operatorname{lcm}(d_1,\ldots,d_n)} \\
&\quad + \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\sum_{l=1}^{\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}}\log l \\
&= K \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\frac{\log \operatorname{lcm}(d_1,\ldots,d_n)}{\operatorname{lcm}(d_1,\ldots,d_n)} \\
&\quad + \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\log\left(\frac{K}{\operatorname{lcm}(d_1,\ldots,d_n)}\right)!.
\end{aligned}$$

Furthermore, we substitute $g_1 = \cdots = g_n = \mathbf{1}$ and $w_r = \log$ in (18) and $f_1 = \cdots = f_n = \mathbf{1}$ and $w_r = \log$ in (18) to get the formulas (32) and (33), respectively. □

Proof of Theorem 6. Let $K = \operatorname{lcm}(k_1,\ldots,k_n)$ be the least common multiple of n-tuple positive integers $k_1,\ldots,k_n$. Note that $t_k(j) = t_k(\gcd(k,j))$ for any positive integers k and j. Since $\gcd(\gcd(j,K),k_i) = \gcd(j,\gcd(K,k_i)) = \gcd(j,k_i)$ for any $i \in \{1,2,\ldots,n\}$, we observe that $t_{k_i}^{(i)}(j)$ is equal to $t_{k_i}^{(i)}(\gcd(j,K))$. If w_r is a completely multiplicative function, we have

$$\begin{aligned}
U_r(k_1,\ldots,k_n) &= \sum_{j=1}^{K} w_r(j)t_{k_1}^{(1)}(\gcd(j,K))\cdots t_{k_n}^{(n)}(\gcd(j,K)) \\
&= \sum_{d|K} w_r(d)t_{k_1}^{(1)}(d)\cdots t_{k_n}^{(n)}(d)\sum_{\substack{l=1 \\ \gcd\left(l,\frac{K}{d}\right)=1}}^{\frac{K}{d}} w_r(l) \\
&= \sum_{d|K} w_r\left(\frac{K}{d}\right)t_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots t_{k_n}^{(n)}\left(\frac{K}{d}\right)\sum_{\substack{l=1 \\ \gcd(l,d)=1}}^{d} w_r(l),
\end{aligned}$$

which completes the proof of the formula (34). Similarly, as in the proof of (34), we have (35). □

To prove Corollary 8, we need the following formula.

Lemma 12. *For any positive integers $N > 1$ and r, we have*

$$\sum_{\substack{l=1 \\ \gcd(l,N)=1}}^{N} \log l = \sum_{d|N}\mu\left(\frac{N}{d}\right)\log(d!) - \phi(N)\sum_{p|N}\frac{\log p}{p-1} \tag{52}$$

and

$$\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}=\frac{r+1}{2}. \tag{53}$$

Proof. Using the well-known identity

$$\sum_{d|N}\frac{\mu(d)}{d}\log d=-\frac{\phi(N)}{N}\sum_{p|N}\frac{\log p}{p-1}$$

[8, Exercise 10.8.45 (c)], we have

$$\begin{aligned}\sum_{\substack{l=1\\ \gcd(l,N)=1}}^{N}\log l&=\sum_{d|N}\mu(d)\sum_{j=1}^{N/d}\log(dj)\\ &=\sum_{d|N}\mu(d)\log\left(\frac{N}{d}\right)!+N\sum_{d|N}\frac{\mu(d)}{d}\log d\\ &=\sum_{d|N}\mu(d)\log\left(\frac{N}{d}\right)!-\phi(N)\sum_{p|N}\frac{\log p}{p-1},\end{aligned}$$

which completes the proof of (52). From Theorem 12.15 in [7], we have

$$\sum_{m=0}^{r}\binom{r+1}{m}B_m=0.$$

It follows that

$$\binom{r+1}{0}B_0+\binom{r+1}{1}B_1+\binom{r+1}{2}B_2+\ldots+\binom{r+1}{2[\frac{r}{2}]}B_{2[\frac{r}{2}]}=0.$$

Since $B_0=1$, $B_1=-\frac{1}{2}$ and $B_{2m+1}=0$ for any positive integer m, we obtain the formula (53). $\square$

Proof of Corollary 8. We substitute $w_r=\mathrm{id}_r$ and $h_1=\cdots=h_n=\mathbf{1}$ in (34) and use the formula [16, Corollary 4]

$$\sum_{\substack{m=1\\ \gcd(m,N)=1}}^{N}m^r=\frac{N^{r+1}}{r+1}\sum_{m=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r+1}{2m}B_{2m}\phi_{1-2m}(N) \tag{54}$$

for any positive integers $N>1$, and r to obtain

$$\begin{aligned}
&\sum_{j=1}^{K} j^r s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\\
&=\sum_{d\mid K} d^r s_{k_1}^{(1)}(d)\cdots s_{k_n}^{(n)}(d) \sum_{\substack{l=1\\ \gcd\left(l,\frac{K}{d}\right)=1}}^{K/d} l^r\\
&=K^r s_{k_1}^{(1)}(K)\cdots s_{k_n}^{(n)}(K)+\frac{K^r}{r+1}\sum_{\substack{d\mid K\\ d<K}} s_{k_1}^{(1)}(d)\cdots s_{k_n}^{(n)}(d)\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}\phi_{1-2m}\left(\frac{K}{d}\right)\\
&=K^r s_{k_1}^{(1)}(K)\cdots s_{k_n}^{(n)}(K)+\frac{K^r}{r+1}\sum_{\substack{d\mid K\\ d>1}} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}\phi_{1-2m}(d)\\
&=K^r s_{k_1}^{(1)}(K)\cdots s_{k_n}^{(n)}(K)-\frac{K^r}{r+1}s_{k_1}^{(1)}(K)\cdots s_{k_n}^{(n)}(K)\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}\\
&+\frac{K^r}{r+1}\sum_{d\mid K} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m}B_{2m}\phi_{1-2m}(d).
\end{aligned}$$

Using the well-known identity (53), we obtain the formula (36). We also substitute $w_r=\log$ and $h_1=\cdots=h_n=\mathbf{1}$ in (35), and use (52) to obtain

$$\begin{aligned}
&\sum_{j=1}^{K} s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\log j\\
&\quad=(s_{k_1}^{(1)}\cdots s_{k_n}^{(n)}\cdot\log*\phi)(K)+\sum_{d\mid K} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\sum_{e\mid d}\mu(e)\log\left(\frac{d}{e}\right)!\\
&\qquad-\sum_{d\mid K} s_{k_1}^{(1)}\left(\frac{K}{d}\right)\cdots s_{k_n}^{(n)}\left(\frac{K}{d}\right)\phi(d)\sum_{p\mid d}\frac{\log p}{p-1}.
\end{aligned}$$

□

Proof of Theorem 10. Using (40) and substituting $w_r(d)=\log\Gamma\left(\frac{d}{K}\right)$ and $h_1=\cdots=h_1=\mathbf{1}$

in (49), we have

$$\begin{aligned}
&\sum_{j=1}^{K} s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\log\Gamma\left(\frac{j}{K}\right)\\
&=\sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\sum_{l=1}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}}\log\Gamma\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{K}l\right)\\
&=\sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\times\\
&\quad\times\left\{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}\log\sqrt{2\pi}-\log\sqrt{2\pi K}+\log\sqrt{\mathrm{lcm}(d_1,\ldots,d_n)}\right\}\\
&=K\log\sqrt{2\pi}\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)}g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\\
&\quad-\log\sqrt{2\pi K}\sum_{d_1|k_1} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots\sum_{d_n|k_n} f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\\
&\quad+\sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\log\sqrt{\mathrm{lcm}(d_1,\ldots,d_n)},
\end{aligned}$$

which completes the proof of (43). We set $w_r(j)=\binom{K}{j}$ and $h_1=\cdots=h_1=\mathbf{1}$. Using (41), we have

$$\begin{aligned}
&\sum_{j=0}^{K}\binom{K}{j}s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)\\
&=\sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\sum_{m=0}^{\frac{K}{\mathrm{lcm}(d_1,\cdots,d_n)}}\binom{K}{\mathrm{lcm}(d_1,\ldots,d_n)m}\\
&=2^K\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)}\times\\
&\qquad\times g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\sum_{j=1}^{\mathrm{lcm}(d_1,\ldots,d_n)}\cos^K\frac{j\pi}{\mathrm{lcm}(d_1,\ldots,d_n)}\cos\frac{Kj\pi}{\mathrm{lcm}(d_1,\ldots,d_n)}\\
&=2^K\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)}\times\\
&\qquad\times g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right)\sum_{j=1}^{\mathrm{lcm}(d_1,\ldots,d_n)}(-1)^{\frac{Kj}{\mathrm{lcm}(d_1,\ldots,d_n)}}\cos^K\frac{j\pi}{\mathrm{lcm}(d_1,\ldots,d_n)},
\end{aligned}$$

hence, we obtain (44). Setting $w_r(j) = B_m\left(\frac{j}{K}\right)$ and $h_1 = \cdots = h_1 = \mathbf{1}$ and using (42), we have

$$\sum_{j=0}^{K-1} B_m\left(\frac{j}{K}\right) s_{k_1}^{(1)}(j)\cdots s_{k_n}^{(n)}(j)$$

$$= \sum_{d_1|k_1,\ldots,d_n|k_n} f_1(d_1)g_1\left(\frac{k_1}{d_1}\right)\cdots f_n(d_n)g_n\left(\frac{k_n}{d_n}\right)\sum_{l=0}^{\frac{K}{\mathrm{lcm}(d_1,\ldots,d_n)}-1} B_m\left(\frac{\mathrm{lcm}(d_1,\ldots,d_n)}{K}l\right)$$

$$= \frac{B_m}{K^{m-1}}\sum_{d_1|k_1,\ldots,d_n|k_n}\frac{f_1(d_1)\cdots f_n(d_n)}{\mathrm{lcm}(d_1,\ldots,d_n)^{1-m}}g_1\left(\frac{k_1}{d_1}\right)\cdots g_n\left(\frac{k_n}{d_n}\right).$$

Hence, we prove (45). □

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