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The Numbers $a^2 + b^2 - dc^2$

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Abstract

We say that a positive integer d is *special* if for every integer m there exist nonzero integers a, b, c such that $m = a^2 + b^2 - dc^2$. In this note we present examples and some properties of special numbers. Moreover, we present an infinite sequence of special numbers.

1 Introduction

Let d be a positive integer. If a, b, c are integers, then let $[a, b, c]_d$ denote the number $a^2 + b^2 - dc^2$. We say that d is *special* if for every integer m there exist nonzero integers a, b, c such that $m = [a, b, c]_d$.

We present examples and some properties of special numbers. Moreover, we present an infinite sequence of special numbers.

2 The numbers $a^2 + b^2 - c^2$

Observe that $0 = [3, 4, 5]_1$ and

$$\begin{array}{llll}
-1 = [2,2,3]_1, & 1 = [1,1,1]_1, & -6 = [3,1,4]_1, & 6 = [3,1,2]_1, \\
-2 = [1,1,2]_1, & 2 = [3,3,4]_1, & -7 = [1,1,3]_1, & 7 = [2,2,1]_1, \\
-3 = [3,2,4]_1, & 3 = [6,4,7]_1, & -8 = [2,2,4]_1, & 8 = [4,1,3]_1, \\
-4 = [2,1,3]_1, & 4 = [2,1,1]_1, & -9 = [6,2,7]_1, & 9 = [3,1,1]_1, \\
-5 = [4,2,5]_1, & 5 = [5,4,6]_1, & -10 = [5,1,6]_1, & 10 = [5,1,4]_1.
\end{array}$$

One of the problems presented in [3, Problem L25] states that every integer is of the form $[a,b,c]_1$, where a,b,c are integers. We will show that every integer is of the form $[a,b,c]_1$ where a,b,c are nonzero integers.

Proposition 1. *The number* 1 *is special, that is, for every integer* m *there exist nonzero integers* a,b,c *such that* $m = a^2 + b^2 - c^2$.

Proof. It follows from the following equalities:

$$2k - 1 = [2,\ k-2,\ k-3]_1, \quad 2k = [k,\ 1,\ k-1]_1$$

for $k \in \mathbb{Z}$, and $3 = [6,4,7]_1$, $5 = [5,4,6]_1$, $2 = [3,3,4]_1$. □

It is known [1, p. 38] that the equation $x^2 + y^2 - z^2 = 3$ has infinitely many solutions in positive integers. The equation $x^2 + y^2 - z^2 = 1997$ has also infinitely many solutions in positive integers [7, p. 9]. In the next proposition we show that the same is true for every integer.

Proposition 2. *For every integer* m *there are infinitely many triples* (a,b,c) *of nonzero integers such that* $m = a^2 + b^2 - c^2$.

Proof. This is a consequence of the following two equalities.

$$\begin{aligned} 2k-1 &= (2t)^2 + (2t^2-k)^2 - (2t^2-k+1)^2, \\ 2k &= (2t^2-2t-k)^2 + (2t-1)^2 - (2t^2-2t-k+1)^2, \end{aligned}$$

where k, t are integers. □

3 Properties of special numbers

In this section we present some elementary properties of special numbers. The following, well known lemma (see, for example, [5]), will play an important role.

Lemma 3. *A positive integer* m *is a sum of two integer squares if and only if all prime factors of* m *of the form* $4k+3$ *have even exponent in the prime factorization of* m.

Now we prove

Proposition 4. *Every special number is a sum of two integer squares. If a non-square positive integer* d *is special, then* d *is a sum of two nonzero integer squares.*

Proof. Let d be a special number. There exist nonzero integers a,b,c such that $[a,b,c]_d = d$. Thus, we have the equality

$$a^2 + b^2 = d(c^2 + 1),$$

which says that $d(c^2+1)$ ia a sum of two squares. Hence, by Lemma 3, all prime factors of $d(c^2+1)$ of the form $4k+3$ have even exponent in the prime factorization of $d(c^2+1)$. Since c^2+1 is also a sum of two squares, all prime factors of d of the form $4k+3$ have even exponent in the prime factorization of d. Hence, again by Lemma 3, d is a sum of two integer squares. Now it is also clear that if additionally d is non-square, then d is a sum of two nonzero integer squares. □

Note that $4 = 2^2 + 0^2$ is a sum of two integer squares and the number 4 is not special. The number $8 = 2^2 + 2^2$ is a sum of two nonzero squares and 8 is not special. In general we have

Proposition 5. *If a positive integer d is divisible by 4, then d is not special.*

Proof. Let $d = 4k$ where k is a positive integer, and assume that d is special. Then $a^2+b^2-dc^2 = 3$ for some nonzero integers a, b, c. This implies that the number a^2+b^2 is of the form $4k+3$. But integers of the form $4k+3$ are not sums of two squares. Thus the assumption that d is special leads to a contradiction. □

Proposition 6. *If a positive integer d is divisible by a prime number of the form $4k+3$, then d is not special.*

Proof. Let p be a prime number of the form $4k+3$. Assume that $p \mid d$ and d is special. Then d is a sum of two squares (by Proposition 4) and this implies (by Lemma 3) that $p^2 \mid d$. Moreover, there exist nonzero integers a, b, c such that $a^2+b^2-dc^2 = p$. In this case p divides the sum of two squares a^2+b^2 and so, again by Lemma 3, the integer a^2+b^2 is divisible by p^2. Hence, p^2 divides p. Thus the assumption that d is special leads to a contradiction. □

As a consequence of the above propositions we obtain the following theorem.

Theorem 7. *Every special number is of the form q or $2q$, where either $q = 1$ or q is a product of prime numbers of the form $4k+1$.*

Question 8. Let $d = q$ or $d = 2q$, where q is a product of prime numbers of the form $4k+1$. Is it true that d is a special number?

We do not know the answer to the above question.

Proposition 9. *Let d be a non-square positive integer and let m be an integer. Assume that there exists a triple (a, b, c) of positive integers such that $[a, b, c]_d = m$. Then such triples (a, b, c) are infinitely many.*

Proof. Let $[a, b, c]_d = m$ for some positive integers a, b, c. Then the Pell equation

$$x^2 - dz^2 = m - b^2$$

has a solution in positive integers $(x, z) = (a, c)$. It follows from the theory of Pell equations [5, 2, 4] that then this equation has infinitely many positive solutions. Let (u, v) be such a solution. Then the triple (u, b, v) is a solution in positive integers of the equation $x^2+y^2-dz^2 = m$. □

4 Examples

We already know that the number 1 is special. In this section we present the all special numbers smaller than 50.

Consider the case $d = 2$. Let us recall that $[a, b, c]_2 = a^2 + b^2 - 2c^2$. Observe that $0 = [1, 1, 1]_2$ and we have

$$\begin{array}{rrrr}
-1 = [4,1,3]_2, & 1 = [8,3,6]_2, & -6 = [1,1,2]_2, & 6 = [2,2,1]_2, \\
-2 = [12,4,9]_2, & 2 = [3,1,2]_2, & -7 = [4,3,4]_2, & 7 = [4,3,3]_2, \\
-3 = [2,1,2]_2, & 3 = [2,1,1]_2, & -8 = [3,1,3]_2, & 8 = [3,1,1]_2, \\
-4 = [8,2,6]_2, & 4 = [16,6,12]_2, & -9 = [5,4,5]_2, & 9 = [4,1,2]_2, \\
-5 = [3,2,3]_2, & 5 = [3,2,2]_2, & -10 = [2,2,3]_2, & 10 = [3,3,2]_2.
\end{array}$$

Proposition 10. *The number 2 is special, that is, for every integer m there exist nonzero integers a, b, c such that $m = a^2 + b^2 - 2c^2$.*

Proof. This is a consequence of the equalities $2k - 1 = [k - 1,\ k,\ k - 1]_2$, $4k = [k - 1,\ k + 1,\ k - 1]_2$, $4k + 2 = [k - 3,\ k + 1,\ k - 2]_2$ (where k is an integer), and $1 = [8, 3, 6]_2$, $-1 = [4, 1, 3]_2$, $-4 = [9, 2, 6]_2$. $4 = [16, 6, 12]$, $-2 = [12, 4, 9]_2$, $10 = [3, 3, 2]_2$, $14 = [4, 4, 3]_2$. □

Note the following consequence of Propositions 10 and 9.

Proposition 11. *For every integer m there are infinitely many triples (a, b, c), of nonzero integers such that $m = a^2 + b^2 - 2c^2$.*

Example 12. Some solutions (x, y, z) of the equation $x^2 + y^2 - 2z^2 = 1$:

$$\begin{array}{rrrrr}
(8,3,6), & (15,8,12), & (24,15,20), & (33,8,24), & (35,24,30), \\
(48,3,34), & (48,17,36), & (48,35,42), & (63,48,56), & (72,33,56), \\
(72,15,52), & (80,63,72), & (93,8,66), & (93,48,74), & (99,80,90).
\end{array}$$

Example 13. For every integer a we have $[a + 2, a, a + 1]_2 = 2$.

Consider now the case $d = 5$. Let us recall that $[a, b, c]_5 = a^2 + b^2 - 5c^2$. Observe that $0 = [1, 2, 1]_5$ and we have

$$\begin{array}{rrrr}
-1 = [12,10,7]_5, & 1 = [10,9,6]_5, & -2 = [3,3,2]_5, & 2 = [9,1,4]_5, \\
-3 = [1,1,1]_5, & 3 = [2,2,1]_5, & -4 = [5,4,3]_5, & 4 = [20,3,9]_5, \\
-5 = [6,2,3]_5, & 5 = [3,1,1]_5, & -6 = [7,5,4]_5, & 6 = [5,1,2]_5, \\
-7 = [3,2,2]_5, & 7 = [6,4,3]_5, & -8 = [6,1,3]_5, & 8 = [3,2,1]_5, \\
-9 = [10,4,5]_5, & 9 = [5,2,2]_5, & -10 = [7,11,6]_5, & 10 = [3,9,4]_5.
\end{array}$$

Proposition 14. *The number 5 is special, that is, for every integer m there exist nonzero integers a, b, c such that $m = a^2 + b^2 - 5c^2$.*

Proof. It follows from the equalities

$$k^2 + (2k-2)^2 - 5(k-1)^2 = 2k-1, \quad (k-2)^2 + (2k-1)^2 - 5(k-1)^2 = 2k,$$

and $= -1 = [12, 10, 7]_5$, $1 = [10, 9, 6]_5$, $2 = [9, 1, 4]_5$, $4 = [20, 3, 9]_5$. □

Note the following consequence of Propositions 14 and 9.

Proposition 15. *For every integer m there are infinitely many triples (a, b, c), of positive integers such that $m = a^2 + b^2 - 5c^2$.*

Proposition 16. *Let $d = q$ or $d = 2q$, where q is a product of prime numbers of the form $4k+1$. If $d \leqslant 50$, then d is special.*

Proof. If $d < 10$, then $d = 1, 2$ or 5, and we already know that in this case d is special. If $d \geqslant 10$, then we have the following equalities:

$$\begin{array}{rcccl}
[k, 3k-3, k-1]_{10} & = & [k-5, 3k-8, k-3]_{10} & = & 2k-1, \\
[k+1, 3k-3, k-1]_{10} & = & [k-9, 3k-13, k-5]_{10} & = & 4k, \\
[k-1, 3k+1, k]_{10} & = & [k-21, 3k-39, k-14]_{10} & = & 4k+2.
\end{array}$$

$$\begin{array}{rcccl}
[2k-4, 3k-10, k-3]_{13} & = & [2k-30, 3k-36, k-13]_{13} & = & 2k-1, \\
[2k-3, 3k-2, k-1]_{13} & = & [2k-29, 3k-54, k-17]_{13} & = & 2k.
\end{array}$$

$$\begin{array}{rcccl}
[k, 4k-4, k-1]_{17} & = & [k-34, 4k-106, k-27]_{17} & = & 2k-1, \\
[k-8, 4k-19, k-5]_{17} & = & [k-76, 4k-357, k-65]_{17} & = & 2k.
\end{array}$$

$$\begin{array}{rcccl}
[3k-18, 4k-30, k-7]_{25} & = & [3k-68, 4k-80, k-21]_{25} & = & 2k-1, \\
[3k-4, 4k-3, k-1]_{25} & = & [3k-104, 4k-153, k-37]_{25} & = & 2k.
\end{array}$$

$$\begin{array}{rcccl}
[k, 5k-5, k-1]_{26} & = & [k-13, 5k-44, k-9]_{26} & = & 2k-1. \\
[k+1, 5k-5, k-1]_{26} & = & [k-25, 5k-83, k-17]_{26} & = & 4k, \\
[k-5, 5k-9, k-2]_{26} & = & [k-57, 5k-217, k-44]_{26} & = & 4k+2.
\end{array}$$

$$\begin{array}{rcccl}
[2k-8, 5k-14, k-3]_{29} & = & [2k-66, 5k-188, k-37]_{29} & = & 2k-1, \\
[2k-7, 5k-26, k-5]_{29} & = & [2k-65, 5k-142, k-29]_{29} & = & 2k.
\end{array}$$

$$
\begin{aligned}
[3k-7,5k-16,k-3]_{34} &= [3k-24,5k-33,k-7]_{34} &&= 2k-1,\\
[3k-11,5k-27,k-5]_{34} &= [3k-45,5k-61,k-13]_{34} &&= 4k,\\
[3k-1,5k+1,k]_{34} &= [3k-69,5k-135,k-26]_{34} &&= 4k+2.
\end{aligned}
$$

$$
\begin{aligned}
[k,6k-6,k-1]_{37} &= [k-74,6k-376,k-63]_{37} &&= 2k-1,\\
[k-18,6k-77,k-13]_{37} &= [k-166,6k-891,k-149]_{37} &&= 2k.
\end{aligned}
$$

$$
\begin{aligned}
[4k-48,5k-68,k-13]_{41} &= [4k-130,5k-150,k-31]_{41} &&= 2k-1,\\
[4k-5,5k-4,k-1]_{41} &= [4k-251,5k-332,k-65]_{41} &&= 2k.
\end{aligned}
$$

$$
\begin{aligned}
[k,7k-7,k-1]_{50} &= [k-25,7k-132,k-19]_{50} &&= 2k-1,\\
[k+1,7k-7,k-1]_{50} &= [k-49,7k-257,k-37]_{50} &&= 4k,\\
[k-11,7k-41,k-6]_{50} &= [k-111,7k-641,k-92]_{50} &&= 4k+2.
\end{aligned}
$$

□

By similar methods we are ready to prove, using a computer, that the same is true for $d < 1000$. Hence, we know that if $d < 1000$, then the answer to Question 8 is affirmative.

5 An infinite sequence of special numbers

In this section we prove that the set of special numbers is infinite. In our proof we use the following well known lemma [5, 2, 4] concerned with the sequence [6, A001110]. Let us recall that every number of the form $t_n = \frac{n(n+1)}{2} = 1 + 2 + \cdots + n$ is called *triangular* .

Lemma 17. *There are infinitely many square triangular numbers. Examples:*

$$t_1 = 1^2, \quad t_8 = 6^2, \quad t_{49} = 35^2, \quad t_{288} = 204^2, \quad t_{1681} = 1189^2.$$

Proof. The Pell equation $x^2 - 8y^2 = 1$ has infinitely many solutions in positive integers. Let (x, y) be one of such solutions. Then x is odd. Let $x = 2n + 1$ where n is a positive integer. Then we have $t_n = \frac{n(n+1)}{2} = y^2$. □

Theorem 18. *There are infinitely many special numbers.*

Proof. We know from the previous lemma that there are infinitely many positive integers u such that $u^2 = \frac{k(k+1)}{2}$ for some positive integer k. Let $d = (2u)^2 + 1$ with $u \geqslant 2$. Observe that $d = k^2 + (k+1)^2$. We will show that the number d is special. Let m be an integer.

First assume that m is even. Let $m = 2s$, where s is an integer. We have the equality

$$\Big((k+1)(s-1)+1\Big)^2 + \Big(k(s-1)-1\Big)^2 - d\big(s-1\big)^2 = 2s.$$

Thus, if $m = 2s$ with $s \neq 1$, then there exist nonzero integers a, b, c such that $[a, b, c]_d = m$. Consider the case $s = 1$, that is, $m = 2$. Since d is non-square, the Pell equation $x^2 - dz^2 = 1$ has a solution (x, z) such that x, z are positive integers. Then we have $[x, 1, z]_d = 2$. Therefore, every even integer m is of the form $[a, b, c]_d$ with nonzero integers a, b, c.

Now assume that m is odd. Let $m = 2s - 1$ where s is an integer. We have the equality

$$s^2 + (2us - 2u)^2 - d(s-1)^2 = 2s - 1.$$

Thus, if $m = 2s-1$ with $s \neq 1$, then there exist positive integers a, b, c such that $[a, b, c]_d = m$. Consider the case $s = 1$, that is, $m = 1$. Since $d - 4$ is non-square (because $d = 4u^2 + 1$ with $u \geqslant 2$), the Pell equation $x^2 - (d-4)z^2 = 1$ has a solution (x, z) such that x, z are positive integers. Then we have $[x, 2z, z]_d = 1$. Therefore, every odd integer m is also of the form $[a, b, c]_d$ with nonzero integers a, b, c. □

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