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Counting Toroidal Binary Arrays, II

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Abstract

We derive formulas for (i) the number of distinct toroidal $n \times n$ binary arrays, allowing rotation of rows and/or columns as well as matrix transposition, and (ii) the number of distinct toroidal $n \times n$ binary arrays, allowing rotation and/or reflection of rows and/or columns as well as matrix transposition.

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1 Introduction

A previous paper [1] found the number of (distinct) toroidal $m \times n$ binary arrays, allowing rotation of rows and/or columns, to be

$$a(m,n) := \frac{1}{mn} \sum_{c \mid m} \sum_{d \mid n} \varphi(c)\varphi(d)\, 2^{mn/\operatorname{lcm}(c,d)}, \tag{1}$$

where φ is Euler's phi function and lcm stands for least common multiple. This is A184271 in the *On-Line Encyclopedia of Integer Sequences* [2]. The main diagonal is A179043. It was also shown that, allowing rotation and/or reflection of rows and/or columns, the number becomes

$$b(m,n) := b_1(m,n) + b_2(m,n) + b_3(m,n) + b_4(m,n), \tag{2}$$

where

$$b_1(m,n) := \frac{1}{4mn} \sum_{c \mid m} \sum_{d \mid n} \varphi(c)\varphi(d)\, 2^{mn/\operatorname{lcm}(c,d)},$$

$$b_2(m,n) := \frac{1}{4n} \sum_{d \mid n} \varphi(d)\, 2^{mn/d} + \begin{cases} (4n)^{-1} \sum' \varphi(d)(2^{(m+1)n/(2d)} - 2^{mn/d}), & \text{if } m \text{ is odd;} \\ (8n)^{-1} \sum' \varphi(d)(2^{mn/(2d)} + 2^{(m+2)n/(2d)} - 2 \cdot 2^{mn/d}), & \text{if } m \text{ is even,} \end{cases}$$

with $\sum' := \sum_{d \mid n:\, d \text{ is odd}}$,

$$b_3(m,n) := b_2(n,m),$$

and

$$b_4(m,n) := \begin{cases} 2^{(mn-3)/2}, & \text{if } m \text{ and } n \text{ are odd;} \\ 3 \cdot 2^{mn/2-3}, & \text{if } m \text{ and } n \text{ have opposite parity;} \\ 7 \cdot 2^{mn/2-4}, & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

(The formula for $b_2(m,n)$ given in [1] is simplified here.) This is A222188 in the *OEIS* [2]. The main diagonal is A209251.

Our aim here is to derive the corresponding formulas when $m = n$ and we allow matrix transposition as well. More precisely, we show that the number of (distinct) toroidal $n \times n$ binary arrays, allowing rotation of rows and/or columns as well as matrix transposition, is

$$\alpha(n) = \frac{1}{2}\, a(n,n) + \frac{1}{2n} \sum_{d \mid n} \varphi(d)\, 2^{n(n+d-2\lfloor d/2 \rfloor)/(2d)}, \tag{3}$$

where $a(n,n)$ is from (1). When we allow rotation and/or reflection of rows and/or columns as well as matrix transposition, the number becomes

$$\beta(n) = \frac{1}{2}\, b(n,n) + \frac{1}{4n} \sum_{d \mid n} \varphi(d)\, 2^{n(n+d-2\lfloor d/2 \rfloor)/(2d)} + \begin{cases} 2^{(n^2-5)/4}, & \text{if } n \text{ is odd;} \\ 5 \cdot 2^{n^2/4-3}, & \text{if } n \text{ is even,} \end{cases} \tag{4}$$

where $b(n,n)$ is from (2). These are the sequences A255015 and A255016, respectively, recently added to the *OEIS* [2].

For an alternative description, we could define a group action on the set of $n \times n$ binary arrays, which has 2^{n^2} elements. If the group is generated by σ (row rotation) and τ (column rotation), then the number of orbits is given by $a(n,n)$; see [1]. If the group is generated by σ, τ, and ζ (matrix transposition), then the number of orbits is given by $\alpha(n)$; see Theorem 1 below. If the group is generated by σ, τ, ρ (row reflection), and θ (column reflection), then the number of orbits is given by $b(n,n)$; see [1]. If the group is generated by σ, τ, ρ, θ, and ζ, then the number of orbits is given by $\beta(n)$; see Theorem 2 below.

Both theorems are proved using Pólya's enumeration theorem (actually, the simplified unweighted version; see, e.g., van Lint and Wilson [3, Theorem 37.1, p. 524]).

To help clarify the distinction between the various group actions, we consider the case of 3×3 binary arrays as in [1]. When the group is generated by σ and τ (allowing rotation of rows and/or columns), there are 64 orbits, which were listed in [1]. When the group is generated by σ, τ, and ζ (allowing rotation of rows and/or columns as well as matrix transposition), there are 44 orbits, which are listed in Table 1 below. When the group is generated by σ, τ, ρ, and θ (allowing rotation and/or reflection of rows and/or columns), there are 36 orbits, which were listed in [1]. When the group is generated by σ, τ, ρ, θ, and ζ (allowing rotation and/or reflection of rows and/or columns as well as matrix transposition), there are 26 orbits, which are listed in Table 2 below.

Table 3 provides numerical values for $\alpha(n)$ and $\beta(n)$ for small n.

We take this opportunity to correct a small gap in the proof of Theorem 2 in [1]. The proof assumed implicitly that $m, n \geq 3$. The theorem is correct as stated for $m, n \geq 1$, so the proof is incomplete if m or n is 1 or 2. Following the proof of Theorem 2 below, we supply the missing steps.

2 Rotation of rows and columns, and matrix transposition

Let $X_n := \{0,1\}^{\{0,1,\ldots,n-1\}^2}$ be the set of $n \times n$ matrices of 0s and 1s, which has 2^{n^2} elements. Let $\alpha(n)$ denote the number of orbits of the group action on X_n by the group of order $2n^2$ generated by σ (row rotation), τ (column rotation), and ζ (matrix transposition). (Exception: If $n = 1$, the group is of order 1.)

Informally, $\alpha(n)$ is the number of (distinct) toroidal $n \times n$ binary arrays, allowing rotation of rows and/or columns as well as matrix transposition.

Theorem 1. *With $a(n,n)$ defined using (1), $\alpha(n)$ is given by (3).*

Proof. Let us assume that $n \geq 2$. By Pólya's enumeration theorem,

$$\alpha(n) = \frac{1}{2n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{A_{ij}} + 2^{E_{ij}}), \tag{5}$$

Table 1: A list of the 44 orbits of the group action in which the group generated by σ, τ, and ζ acts on the set of 3×3 binary arrays. (Rows and/or columns can be rotated and matrices can be transposed.) Each orbit is represented by its minimal element in 9-bit binary form. Subscripts indicate orbit size. Bars separate different numbers of 1s.

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}_1 \Big| \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}_9 \Big| \begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}_9 \Big|$$

$$\begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}_6 \begin{pmatrix}0&0&0\\0&0&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\1&0&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&0&1\\1&1&0\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\1&0&0\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}_3$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}_3 \Big| \begin{pmatrix}0&0&0\\0&0&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\1&0&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&1&1\\1&1&0\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\1&1&1\\0&0&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}_9$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}_{18} \Big| \begin{pmatrix}0&0&0\\0&1&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\1&1&1\\0&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}_9$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&0\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\1&0&1\\1&1&0\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\1&1&0\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&0\\1&1&0\end{pmatrix}_9 \Big|$$

$$\begin{pmatrix}0&0&0\\1&1&1\\1&1&1\end{pmatrix}_6 \begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\1&1&0\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\1&1&1\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&1\\1&1&0\end{pmatrix}_{18} \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}_3$$

$$\begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix}_3 \Big| \begin{pmatrix}0&0&1\\1&1&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\1&1&0\\1&1&1\end{pmatrix}_9 \Big| \begin{pmatrix}0&1&1\\1&1&1\\1&1&1\end{pmatrix}_9 \Big| \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}_1$$

where A_{ij} (resp., E_{ij}) is the number of cycles in the permutation $\sigma^i\tau^j$ (resp., $\sigma^i\tau^j\zeta$); here σ rotates the rows (row 0 becomes row 1, row 1 becomes row 2, $\ldots$, row $n-1$ becomes row 0), τ rotates the columns, and ζ transposes the matrix. We know from [1] that

$$a(n,n) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{A_{ij}}, \tag{6}$$

so it remains to find E_{ij}. The permutation ζ has n fixed points and $\binom{n}{2}$ transpositions, so $E_{00} = n(n+1)/2$.

Notice that σ and τ commute, whereas $\sigma\zeta = \zeta\tau$ and $\tau\zeta = \zeta\sigma$. Let $(i,j) \in \{0,1,\ldots,n-$

Table 2: A list of the 26 orbits of the group action in which the group generated by σ, τ, ρ, θ, and ζ acts on the set of 3×3 binary arrays. (Rows and/or columns can be rotated and/or reflected and matrices can be transposed.) Each orbit is represented by its minimal element in 9-bit binary form. Subscripts indicate orbit size. Bars separate different numbers of 1s.

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}_1 \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}_{18} \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}_6 \begin{pmatrix}0&0&0\\0&0&1\\0&1&1\end{pmatrix}_{36}$$

$$\begin{pmatrix}0&0&0\\0&0&1\\1&1&0\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}_6 \Bigg| \begin{pmatrix}0&0&0\\0&0&1\\1&1&1\end{pmatrix}_{36} \begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\1&0&1\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}_{36} \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&1&1\\1&1&1\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\1&1&0\\1&1&0\end{pmatrix}_9 \Bigg| \begin{pmatrix}0&0&0\\1&1&1\\1&1&1\end{pmatrix}_6 \begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}_{36}$$

$$\begin{pmatrix}0&0&1\\1&1&0\\1&1&1\end{pmatrix}_{36} \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}_6 \Bigg| \begin{pmatrix}0&0&1\\1&1&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix}_{18} \Bigg| \begin{pmatrix}0&1&1\\1&1&1\\1&1&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}_1$$

Table 3: The values of $\alpha(n)$ and $\beta(n)$ for $n = 1, 2, \ldots, 12$.

n	$\alpha(n)$	$\beta(n)$
1	2	2
2	6	6
3	44	26
4	2209	805
5	674384	172112
6	954623404	239123150
7	5744406453840	1436120190288
8	144115192471496836	36028817512382026
9	14925010120653819583840	3731252531904348833632
10	6338253001142965335834871200	1584563250300891724601560272
11	10985355337065423791175013899922368	2746338834266358751489231123956672
12	77433143050453552587418968170813573149024	19358285762613388352671214587818634041520

$1\}^2 - \{(0,0)\}$ be arbitrary. Then

$$(\sigma^i \tau^j \zeta)^2 = (\sigma^i \tau^j \zeta)(\zeta \tau^i \sigma^j) = \sigma^{i+j} \tau^{i+j},$$

hence

$$(\sigma^i\tau^j\zeta)^{2d} = \sigma^{(i+j)d}\tau^{(i+j)d} = ((\sigma\tau)^{i+j})^d,$$
$$(\sigma^i\tau^j\zeta)^{2d+1} = \sigma^{(i+j)d+i}\tau^{(i+j)d+j}\zeta.$$

Clearly, $(\sigma^i\tau^j\zeta)^{2d+1}$ cannot be the identity permutation, so $\sigma^i\tau^j\zeta$ is of even order. Using the fact that, in the cyclic group $\{a, a^2, \ldots, a^{n-1}, a^n = e\}$ of order n, a^k is of order $n/\gcd(k,n)$, we find that the permutation $\sigma^i\tau^j\zeta$ is of order $2d$, where $d := n/\gcd(i+j,n)$. Therefore, every cycle of this permutation must have length that divides $2d$.

We claim that all cycles have length d or $2d$. Accepting that for now, let us determine how many cycles have length d. A cycle that includes entry (k,l) has length d if (k,l) is a fixed point of $(\sigma^i\tau^j\zeta)^d$. For this to hold we must have d odd (otherwise there would be no fixed points because we have excluded the case $i = j = 0$ and $(i+j)d/2 = \operatorname{lcm}(i+j,n)/2$ is not a multiple of n). Since

$$(\sigma^i\tau^j\zeta)^d = \sigma^{(i+j)(d-1)/2+i}\tau^{(i+j)(d-1)/2+j}\zeta,$$

we must also have

$$(k,l) = ([l + (i+j)(d-1)/2 + j], [k + (i+j)(d-1)/2 + i]), \tag{7}$$

where $d := n/\gcd(i+j,n)$ and, for simplicity, $[r] := (r \bmod n) \in \{0,1,\ldots,n-1\}$. For each $k \in \{0,1,\ldots,n-1\}$, there is a unique l (namely, $l := [k + (i+j)(d-1)/2 + i]$) such that (7) holds; indeed,

$$\begin{aligned}
[l + (i+j)(d-1)/2 + j] &= [[k + (i+j)(d-1)/2 + i] + (i+j)(d-1)/2 + j] \\
&= [k + (i+j)(d-1)/2 + i + (i+j)(d-1)/2 + j] \\
&= [k + (i+j)d] \\
&= [k + (i+j)(n/\gcd(i+j,n))] \\
&= [k + \operatorname{lcm}(i+j,n)] \\
&= k.
\end{aligned}$$

This shows that there are n fixed points of $(\sigma^i\tau^j\zeta)^d$. Each cycle of length d of $\sigma^i\tau^j\zeta$ will account for d such fixed points, hence there are n/d such cycles. All remaining cycles will have length $2d$, and so there are $n(n-1)/(2d)$ of these. The total number of cycles is therefore $n(n+1)/(2d)$.

The other possibility is that d is even and all cycles have the same length, $2d$, so there are $n^2/(2d)$ of them. Notice that d is a divisor of n, so the contribution to

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{E_{ij}}$$

from odd d is

$$\sum_{d \mid n\colon d \text{ is odd}} n\varphi(d) 2^{n(n+1)/(2d)} \tag{8}$$

and from even d is

$$\sum_{d \mid n\colon d \text{ is even}} n\varphi(d) 2^{n^2/(2d)}. \tag{9}$$

The reason for the coefficient $n\varphi(d)$ is that, if $d \mid n$, then the number of elements of the cyclic group $\{e, \sigma\tau, (\sigma\tau)^2, \ldots, (\sigma\tau)^{n-1}\}$ that are of order d is $\varphi(d)$. And for a given $(i, j) \in \{0, 1, \ldots, n-1\}^2$, there are n pairs $(k, l) \in \{0, 1, \ldots, n-1\}^2$ such that $[k + l] = [i + j]$. Putting (8) and (9) together, we obtain

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{E_{ij}} = \sum_{d \mid n} n\varphi(d) 2^{n(n+d-2\lfloor d/2 \rfloor)/(2d)}, \tag{10}$$

which, together with (5) and (6), yields (3).

It remains to prove our claim that, for $(i, j) \in \{0, 1, \ldots, n-1\}^2 - \{(0, 0)\}$, the permutation $\sigma^i\tau^j\zeta$ cannot have any cycles whose length is a proper divisor of $d := n/\gcd(i + j, n)$. Let $c \mid d$ with $1 \leq c < d$. We must show that $(\sigma^i\tau^j\zeta)^c$ has no fixed points. We can argue as above with c in place of d. For (k, l) to be a fixed point of $(\sigma^i\tau^j\zeta)^c$ we must have $(i+j)c$ a multiple of n. But $d := n/\gcd(i + j, n)$ is the smallest integer c such that $(i + j)c$ is a multiple of n because $(i + j)n/\gcd(i + j, n) = \mathrm{lcm}(i + j, n)$.

Finally, we excluded the case $n = 1$ at the beginning of the proof, but we notice that the formula (3) gives $\alpha(1) = 2$, which is correct. □

3 Rotation and reflection of rows and columns, and matrix transposition

Let $X_n := \{0, 1\}^{\{0,1,\ldots,n-1\}^2}$ be the set of $n \times n$ matrices of 0s and 1s, which has 2^{n^2} elements. Let $\beta(n)$ denote the number of orbits of the group action on X_n by the group of order $8n^2$ generated by σ (row rotation), τ (column rotation), ρ (row reflection), θ (column reflection), and ζ (matrix transposition). (Exceptions: If $n = 2$, the group is of order 8; if $n = 1$, the group is of order 1.)

Informally, $\beta(n)$ is the number of (distinct) toroidal $n \times n$ binary arrays, allowing rotation and/or reflection of rows and/or columns as well as matrix transposition.

Theorem 2. *With $b(n, n)$ defined using (2), $\beta(n)$ is given by (4).*

Proof. Let us assume that $n \geq 3$. (We will treat the cases $n = 1$ and $n = 2$ later.) By Pólya's enumeration theorem,

$$\beta(n) = \frac{1}{8n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{A_{ij}} + 2^{B_{ij}} + 2^{C_{ij}} + 2^{D_{ij}} + 2^{E_{ij}} + 2^{F_{ij}} + 2^{G_{ij}} + 2^{H_{ij}}),$$

where A_{ij} (resp., B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) is the number of cycles in the permutation $\sigma^i\tau^j$ (resp., $\sigma^i\tau^j\rho$, $\sigma^i\tau^j\theta$, $\sigma^i\tau^j\rho\theta$, $\sigma^i\tau^j\zeta$, $\sigma^i\tau^j\rho\zeta$, $\sigma^i\tau^j\theta\zeta$, $\sigma^i\tau^j\rho\theta\zeta$); here σ rotates the rows (row 0 becomes row 1, row 1 becomes row 2, $\ldots$, row $n-1$ becomes row 0), τ rotates the columns, ρ reflects the rows (rows 0 and $n-1$ are interchanged, rows 1 and $n-2$ are interchanged, $\ldots$, rows $\lfloor n/2\rfloor - 1$ and $n - \lfloor n/2\rfloor$ are interchanged), θ reflects the columns, and ζ transposes the matrix. The order of the group generated by σ, τ, ρ, θ, and ζ is $8n^2$, using the assumption that $n \geq 3$.

We have already evaluated

$$a(n,n) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{A_{ij}},$$

$$\alpha(n) = \frac{1}{2n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(2^{A_{ij}} + 2^{E_{ij}}),$$

and

$$b(n,n) = \frac{1}{4n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(2^{A_{ij}} + 2^{B_{ij}} + 2^{C_{ij}} + 2^{D_{ij}}),$$

so

$$\beta(n) = \frac{1}{2}\,b(n,n) + \frac{1}{4}\left(\alpha(n) - \frac{1}{2}a(n,n)\right) + \frac{1}{8n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(2^{F_{ij}} + 2^{G_{ij}} + 2^{H_{ij}}). \tag{11}$$

Let us begin with

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{H_{ij}}.$$

Here we are concerned with the permutations $\sigma^i\tau^j\rho\theta\zeta$ for $(i,j) \in \{0,1,\ldots,n-1\}^2$. We will need some multiplication rules for the permutations σ, τ, ρ, θ, and ζ, specifically

$$\sigma\tau = \tau\sigma, \quad \sigma\theta = \theta\sigma, \quad \tau\rho = \rho\tau, \quad \rho\theta = \theta\rho, \quad \sigma\rho = \rho\sigma^{-1}, \quad \tau\theta = \theta\tau^{-1},$$

and

$$\sigma\zeta = \zeta\tau, \quad \tau\zeta = \zeta\sigma, \quad \rho\zeta = \zeta\theta, \quad \theta\zeta = \zeta\rho.$$

It follows that (with $\tau^{-i} := (\tau^{-1})^i$)

$$\sigma^i\tau^j\rho\theta\zeta = \sigma^i\tau^j\zeta\theta\rho = \zeta\tau^i\sigma^j\theta\rho = \zeta\theta\tau^{-i}\sigma^j\rho = \zeta\theta\rho\tau^{-i}\sigma^{-j},$$

and hence

$$(\sigma^i\tau^j\rho\theta\zeta)^2 = (\sigma^i\tau^j\rho\theta\zeta)(\zeta\theta\rho\tau^{-i}\sigma^{-j}) = \sigma^{i-j}\tau^{-i+j} = (\sigma\tau^{-1})^{i-j} = (\sigma^{-1}\tau)^{-i+j}. \tag{12}$$

In particular, if $i \in \{0,1,\ldots,n-1\}$, then the permutation $\sigma^i\tau^i\rho\theta\zeta$ is of order 2. Furthermore, under this permutation, the entry in position (k,l) moves to position $(n-1-[l+i], n-1-[k+i])$, where, as before, $[r] := (r \bmod n) \in \{0,1,\ldots,n-1\}$. Thus, (k,l) is a fixed point if and only if

$$(k,l) = (n-1-[l+i], n-1-[k+i]). \tag{13}$$

For each $k \in \{0,1,\ldots,n-1\}$ there is a unique $l \in \{0,1,\ldots,n-1\}$ (namely $l := n-1-[k+i]$) such that (13) holds; indeed,

$$\begin{aligned} n-1-[l+i] &= n-1-[n-1-[k+i]+i] = n-1-[n-1-(k+i)+i] \\ &= n-1-[n-1-k] = n-1-(n-1-k) = k. \end{aligned}$$

Thus, $\sigma^i\tau^i\rho\theta\zeta$ with $i \in \{0,1,\ldots,n-1\}$ is of order 2 and has exactly n fixed points, hence $\binom{n}{2}$ transpositions. This implies that $H_{ii} = n(n+1)/2$ for such i.

Now we let $(i,j) \in \{0,1,\ldots,n-1\}^2$ be arbitrary but with $i \neq j$. Let us generalize (12) to

$$\begin{aligned} (\sigma^i\tau^j\rho\theta\zeta)^{2d} &= \sigma^{(i-j)d}\tau^{(-i+j)d} = ((\sigma\tau^{-1})^{i-j})^d = ((\sigma^{-1}\tau)^{-i+j})^d, \\ (\sigma^i\tau^j\rho\theta\zeta)^{2d+1} &= \sigma^{(i-j)d+i}\tau^{(-i+j)d+j}\rho\theta\zeta. \end{aligned}$$

The proof proceeds much like the proof of Theorem 1. Specifically, $\sigma^i\tau^j\rho\theta\zeta$ is of order $2d$, where $d := n/\gcd(|i-j|,n)$. All cycles have length d or $2d$. In fact, if d is odd, there are n/d cycles of length d and $n(n-1)/(2d)$ cycles of length $2d$. If d is even, there are $n^2/(2d)$ cycles, all of length $2d$. And for a given $(i,j) \in \{0,1,\ldots,n-1\}^2$, there are n pairs $(k,l) \in \{0,1,\ldots,n-1\}^2$ such that $[k-l] = [|i-j|]$. We arrive at the conclusion that

$$\frac{1}{8n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{H_{ij}} = \frac{1}{8n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{E_{ij}} = \frac{1}{4}\left(\alpha(n) - \frac{1}{2}a(n,n)\right). \tag{14}$$

Next we evaluate

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{F_{ij}} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{G_{ij}}, \tag{15}$$

where the equality holds by symmetry. We consider the permutations $\sigma^i\tau^j\rho\zeta$ for $(i,j) \in \{0,1,\ldots,n-1\}^2$. From the multiplication rules, it follows that

$$\sigma^i\tau^j\rho\zeta = \zeta\theta\tau^{-i}\sigma^j$$

and hence

$$(\sigma^i\tau^j\rho\zeta)^2 = (\sigma^i\tau^j\rho\zeta)(\zeta\theta\tau^{-i}\sigma^j) = \sigma^i\tau^j\rho\theta\tau^{-i}\sigma^j = \sigma^{i-j}\tau^{i+j}\rho\theta = \theta\rho\sigma^{-i+j}\tau^{-i-j}, \tag{16}$$

which implies

$$(\sigma^i\tau^j\rho\zeta)^4 = (\sigma^{i-j}\tau^{i+j}\rho\theta)(\theta\rho\sigma^{-i+j}\tau^{-i-j}) = e.$$

So the permutation $\sigma^i\tau^j\rho\zeta$ is of order 4. The entry in position (k,l) moves to position $([l+j], n-1-[k+i])$ under this permutation. Thus, $(k,l) \in \{0,1,\ldots,n-1\}^2$ is a fixed point of $\sigma^i\tau^j\rho\zeta$ if and only if

$$(k,l) = ([l+j], n-1-[k+i]).$$

There is a solution (k,l) if and only if there exists $l \in \{0,1,\ldots,n-1\}$ such that, with $k := [l+j]$, we have $n-1-[k+i] = l$ or, equivalently,

$$[l+i+j] = n-1-l. \tag{17}$$

When $i+j \leq n-1$, (17) is equivalent to

$$l+i+j = n-1-l \quad \text{or} \quad l+i+j-n = n-1-l$$

or to

$$l = (n-1-i-j)/2 \quad \text{or} \quad l = (2n-1-i-j)/2.$$

If n is odd and $i+j$ is odd, then there is one fixed point, $(k,l) = ([(2n-1-i+j)/2], [(2n-1-i-j)/2])$. If n is odd and $i+j$ is even, then there is one fixed point, $(k,l) = ([(n-1-i+j)/2], [(n-1-i-j)/2])$. If n is even and $i+j$ is odd, then there are two fixed points, namely

$$(k,l) = ([(n-1-i+j)/2], [(n-1-i-j)/2]),$$
$$(k,l) = ([(2n-1-i+j)/2], [(2n-1-i-j)/2]).$$

Finally, if n is even and $i+j$ is even, then there is no fixed point.

When $i+j \geq n$, (17) is equivalent to

$$l+i+j-n = n-1-l \quad \text{or} \quad l+i+j-2n = n-1-l$$

or to

$$l = (2n-1-i-j)/2 \quad \text{or} \quad l = (3n-1-i-j)/2.$$

If n is odd and $i+j$ is odd, then there is one fixed point, $(k,l) = ([(2n-1-i+j)/2], [(2n-1-i-j)/2])$. If n is odd and $i+j$ is even, then there is one fixed point, $(k,l) = ([(3n-1-i+j)/2], [(3n-1-i-j)/2]) = ([(n-1-i+j)/2], [(n-1-i-j)/2])$. If n is even and $i+j$ is odd, then there are two fixed points, namely

$$(k,l) = ([(2n-1-i+j)/2], [(2n-1-i-j)/2]),$$
$$(k,l) = ([(n-1-i+j)/2], [(n-1-i-j)/2]).$$

Finally, if n is even and $i+j$ is even, then there is no fixed point. Notice that the results are the same for $i+j \geq n$ as for $i+j \leq n-1$.

Using (16), under the permutation $(\sigma^i\tau^j\rho\zeta)^2$, the entry in position (k,l) moves to position $(n-1-[k+i-j], n-1-[l+i+j])$. Thus, $(k,l) \in \{0,1,\dots,n-1\}^2$ is a fixed point of $(\sigma^i\tau^j\rho\zeta)^2$ if and only if

$$(k,l) = (n-1-[k+i-j], n-1-[l+i+j]).$$

A necessary and sufficient condition on (k,l) is (17) together with $[k+i-j] = n-1-k$. Solutions have l as before. On the other hand, k must satisfy

$$k+i-j-n = n-1-k, \quad k+i-j = n-1-k, \quad \text{or} \quad k+i-j+n = n-1-k,$$

or equivalently,

$$k = [(n-1-i+j)/2] \quad \text{or} \quad k = [(2n-1-i+j)/2].$$

If n is odd, the only fixed points of $(\sigma^i\tau^j\rho\zeta)^2$ are those already shown to be fixed points of $\sigma^i\tau^j\rho\zeta$. If n is even and $i+j$ is odd, there are two fixed points of $(\sigma^i\tau^j\rho\zeta)^2$ that are not fixed points of $\sigma^i\tau^j\rho\zeta$, namely

$$(k,l) = ([(n-1-i+j)/2], [(2n-1-i-j)/2]),$$
$$(k,l) = ([(2n-1-i+j)/2], [(n-1-i-j)/2]).$$

Finally, there are no fixed points when n is even and $i+j$ is even.

Consequently, if n is odd, then the permutation $\sigma^i\tau^j\rho\zeta$, which is of order 4, has only one fixed point. Therefore, it has one cycle of length 1 and $(n^2-1)/4$ cycles of length 4. Thus,

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{F_{ij}} = n^2 2^{(n^2+3)/4}.$$

For even n, if $i+j$ is odd, then the permutation $\sigma^i\tau^j\rho\zeta$ has two cycles of length 1 and one cycle of length 2, and the remaining cycles are of length 4. If $i+j$ is even, then all cycles of the permutation $\sigma^i\tau^j\rho\zeta$ are of length 4, hence there are $n^2/4$ of them. Thus,

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} 2^{F_{ij}} = \frac{1}{2}n^2 2^{(n^2-4)/4+3} + \frac{1}{2}n^2 2^{n^2/4} = 5n^2 2^{n^2/4-1}.$$

These results, together with (3), (10), (11), (14), and (15), yield (4).

Finally, recall that we have assumed that $n \geq 3$. We notice that the formula (4) gives $\beta(1) = 2$ and $\beta(2) = 6$, which are correct, as we can see by direct enumeration. □

In the derivation of (2) in [1], the proof requires $m, n \geq 3$ because the group $D_m \times D_n$ used in the application of Pólya's enumeration theorem (D_m being the dihedral group of order $2m$), is incorrect if m or n is 1 or 2. If $m = 2$, row rotation and row reflection are the same, so the latter is redundant. Thus, D_2 should be replaced by C_2, the cyclic group of order 2. The reason (2) is still valid is that $b_1(2,n) = b_2(2,n)$ and $b_3(2,n) = b_4(2,n)$, as is easily verified. If $m = 1$, again row reflection is redundant, so D_1 should be replaced by C_1. Here (2) remains valid because $b_1(1,n) = b_2(1,n)$ and $b_3(1,n) = b_4(1,n)$. A similar remark applies to $n = 2$ and $n = 1$, except that here $b_1(m,2) = b_3(m,2)$, $b_2(m,2) = b_4(m,2)$, $b_1(m,1) = b_3(m,1)$, and $b_2(m,1) = b_4(m,1)$.

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