A Multidimensional Cesáro Type Identity and Applications

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Abstract

We generalize a well-known identity due to Cesáro and get some asymptotics on some multivariate gcd-sum functions. Several examples when the functions are completely multiplicative are given.

1 Introduction

In 1885, Cesáro [2] proved the following identity.

Lemma 1. *Let f be an arithmetic function and φ denote Euler's totient function. For every positive integer n, we have*

$$\sum_{i=1}^{n} f(\gcd(i,n)) = (f \star \varphi)(n).$$

This follows from

$$\sum_{i=1}^{n} f(\gcd(i,n)) = \sum_{d|n} f(d) \sum_{\substack{k \leqslant n/d \\ \gcd(k,n/d)=1}} 1 = \sum_{d|n} f(d) \varphi\left(\frac{n}{d}\right) = (f \star \varphi)(n)$$

where $(f \star g)(n) = \sum_{d|n} f(d)g(n/d)$ is the usual Dirichlet convolution product. It should be mentioned that such an identity also occurs with some other convolution products where the

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summation is over some subset of the set of the divisors of n. For instance, Tóth [3] showed that

$$\sum_{i \in \mathrm{Reg}(n)} f(\gcd(i,n)) = (f \odot \varphi)(n)$$

where $f \odot g$ is the unitary convolution product and where the notation $i \in \mathrm{Reg}(n)$ means that $1 \leqslant i \leqslant n$ and there exists an integer x such that $i^2 x \equiv i \pmod{n}$.

In this work, we intend to generalize Lemma 1 to higher dimensions and present some applications to various common number-theoretic functions.

The following result is easily deduced in a similar fashion to Lemma 1.

Proposition 2. *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $f : (\mathbb{Z}_{\geqslant 1})^r \longrightarrow \mathbb{C}$ be an arithmetic function of r variables. For any $n \in \mathbb{Z}_{\geqslant 1}$*

$$\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f\left(\gcd(i_1,n),\ldots,\gcd(i_r,n)\right) = \sum_{d_1 \mid n} \cdots \sum_{d_r \mid n} f\left(d_1,\ldots,d_r\right) \varphi(n/d_1) \cdots \varphi(n/d_r).$$

Proof. Similarly as in Lemma 1, we have

$$\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f\left(\gcd(i_1,n),\ldots,\gcd(i_r,n)\right)$$

$$= \sum_{d_1 \mid n} \cdots \sum_{d_r \mid n} f\left(d_1,\ldots,d_r\right) \sum_{\substack{k_1 \leq n/d_1 \\ \gcd(k_1,n/d_1)=1}} \cdots \sum_{\substack{k_r \leq n/d_r \\ \gcd(k_r,n/d_r)=1}} 1$$

$$= \sum_{d_1 \mid n} \cdots \sum_{d_r \mid n} f\left(d_1,\ldots,d_r\right) \varphi(n/d_1) \cdots \varphi(n/d_r)$$

as announced. $\square$

2 Applications

2.1 Main result

Theorem 3. *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $f : \mathbb{Z}_{\geqslant 1} \longrightarrow \mathbb{C}$ be a completely multiplicative function such that, for any prime number p, $|f(p)| \leqslant 1$. Set $\theta := \max_{p \,\mathrm{prime}} |1 - f(p)|$ and assume $\theta r \geqslant 1$.*

Then, for any x sufficiently large

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} f\left(\gcd(i_1,n)\right) \cdots f\left(\gcd(i_r,n)\right) = C_r(f)\, x^{r+1} + O\left(\left(x \log^{\theta} x\right)^r\right)$$

where

$$C_r(f) := \frac{1}{r+1} \prod_{p} \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha}}\left(1 - \left(1 - \left(\frac{f(p)}{p}\right)^{\alpha}\right)\frac{1 - f(p)}{p - f(p)}\right)^{r}\right). \quad (1)$$

This result immediately yields the following examples.

Corollary 4. *Let $r \in \mathbb{Z}_{\geqslant 1}$.*

1. *For any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \frac{1}{\gcd(i_1, n) \cdots \gcd(i_r, n)} = C_r x^{r+1} + O\left((x \log x)^r\right),$$

where

$$C_r = \frac{1}{r+1} \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \left(1 - \frac{1 - p^{-2\alpha}}{p+1}\right)^r\right).$$

2. *Let λ be the Liouville function. Then for any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \lambda\left(\gcd(i_1, n)\right) \cdots \lambda\left(\gcd(i_r, n)\right) = C_r x^{r+1} + O\left(x^r (\log x)^{2r}\right),$$

where

$$C_r = \frac{1}{r+1} \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \left(1 - \frac{2\left(1 - (-1)^\alpha p^{-\alpha}\right)}{p+1}\right)^r\right).$$

3. *Let ℓ be an odd prime number, $(\cdot/\ell)$ be the Legendre symbol modulo ℓ and define [1]*

$$\Omega_\ell(n) := \sum_{\substack{p \mid n \\ (p/\ell)=-1}} 1 \quad \text{and} \quad \lambda_\ell(n) := (-1)^{\Omega_\ell(n)}.$$

Then for any x sufficiently large

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \lambda_\ell\left(\gcd(i_1, n)\right) \cdots \lambda_\ell\left(\gcd(i_r, n)\right) = C_r x^{r+1} + O\left(x^r (\log x)^{2r}\right),$$

where

$$C_r = \frac{1}{r+1} \prod_{(p/\ell)=-1} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \left(1 - \frac{2\left(1 - (-1)^\alpha p^{-\alpha}\right)}{p+1}\right)^r\right).$$

4. *Let $q > 1$ be an integer and χ be a real non-principal Dirichlet character modulo q. Then for any x sufficiently large*

$$\sum_{n \leqslant x} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \chi\left(\gcd(i_1, n)\right) \cdots \chi\left(\gcd(i_r, n)\right) = C_r x^{r+1} + O\left(x^r (\log x)^{2r}\right),$$

where

$$C_r = \frac{1}{r+1} \prod_{\chi(p)=-1} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha} \left(1 - \frac{2\left(1 - (-1)^\alpha p^{-\alpha}\right)}{p+1}\right)^r\right)$$
$$\times \prod_{\chi(p)=0} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{r-1}\right).$$

2.2 Average order of certain multiplicative functions

The proof of Theorem 3 uses the following tool, which is of independent interest.

Proposition 5. *Let F be a multiplicative function for which there exists $\kappa \geqslant 1$ such that, for all prime powers p^α*

$$|(F \star \mu)(p^\alpha)| \leqslant \kappa p^{-\alpha}.$$

Then

$$\sum_{n \leqslant x} F(n) = x \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{F(p^\alpha)}{p^\alpha}\right) + O\left((\log x)^\kappa\right).$$

Proof. Let $G := F \star \mu$. The function G is multiplicative and satisfies

$$|G(n)| \leqslant \frac{\kappa^{\omega(n)}}{n} \quad (n \in \mathbb{Z}_{\geqslant 1}).$$

For all $x \geqslant 1$, we then have

$$\sum_{d \leqslant x} \frac{|G(d)|}{d} \leqslant \sum_{d \leqslant x} \frac{\kappa^{\omega(d)}}{d^2} \ll 1,$$

so that the series $\sum_{d \geqslant 1} G(d)/d$ is absolutely convergent and one may check that

$$1 + \sum_{\alpha=1}^{\infty} \frac{G(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{F(p^\alpha)}{p^\alpha}\right) \quad (p \text{ prime}).$$

Hence

$$\begin{aligned}
\sum_{n \leqslant x} F(n) &= \sum_{n \leqslant x} (G \star \mathbf{1})(n) = \sum_{d \leqslant x} G(d) \left\lfloor \frac{x}{d} \right\rfloor \\
&= x \sum_{d \leqslant x} \frac{G(d)}{d} + O\left(\sum_{d \leqslant x} |G(d)|\right) \\
&= x \sum_{d=1}^{\infty} \frac{G(d)}{d} + O\left(\sum_{d \leqslant x} |G(d)| + x \sum_{d > x} \frac{|G(d)|}{d}\right) \\
&= x \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{p^\alpha}\right) + O\left(R(x)\right)
\end{aligned}$$

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say, and by partial summation, we finally get

$$
\begin{aligned}
R(x) \;\ll\; & \sum_{d \leqslant x} |G(d)| + x \int_x^\infty \left(\sum_{d \leqslant t} |G(d)| \right) \frac{\mathrm{d}t}{t^2} \\
\ll\; & \sum_{d \leqslant x} \frac{\kappa^{\omega(d)}}{d} + x \int_x^\infty \left(\sum_{d \leqslant t} \frac{\kappa^{\omega(d)}}{d} \right) \frac{\mathrm{d}t}{t^2} \\
\ll\; & (\log x)^\kappa + x \int_x^\infty (\log t)^\kappa \, \frac{\mathrm{d}t}{t^2} \ll (\log x)^\kappa
\end{aligned}
$$

as required. $\qquad\square$

2.3 Proof of Theorem 3

Let f satisfy the hypotheses of Theorem 3 and define

$$
F_r(n) := \left(\frac{(\varphi \star f)(n)}{n} \right)^r \quad (n, r \in \mathbb{Z}_{\geqslant 1}).
$$

We start with the following lemma.

Lemma 6. *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $f : \mathbb{Z}_{\geqslant 1} \longrightarrow \mathbb{C}$ be a completely multiplicative function such that, for any prime p, $|f(p)| \leqslant 1$. Set $\theta := \max_{p \text{ prime}} |1 - f(p)|$. For any prime powers p^α and any $r \in \mathbb{Z}_{\geqslant 1}$*

$$
\left| (F_r \star \mu)(p^\alpha) \right| \leqslant \frac{\theta r}{p^\alpha}.
$$

Proof. We first note that

$$
\left| 1 - \left(1 - \left(\frac{f(p)}{p} \right)^\alpha \right) \frac{1 - f(p)}{p - f(p)} \right| \leqslant 1 \quad (p \text{ prime}, \ \alpha \in \mathbb{Z}_{\geqslant 0}). \tag{2}
$$

Now for any prime powers p^α, straightforward computations show that

$$
F_r(p^\alpha) = \left(1 - \left(1 - \left(\frac{f(p)}{p} \right)^\alpha \right) \frac{1 - f(p)}{p - f(p)} \right)^r
$$

so that

$$
\begin{aligned}
& \left| (F_r \star \mu)(p^\alpha) \right| \\
=\; & \left| \left(1 - \left(1 - \left(\frac{f(p)}{p} \right)^\alpha \right) \frac{1 - f(p)}{p - f(p)} \right)^r - \left(1 - \left(1 - \left(\frac{f(p)}{p} \right)^{\alpha-1} \right) \frac{1 - f(p)}{p - f(p)} \right)^r \right|.
\end{aligned}
$$

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Note that

$$|(1-a)^r - (1-b)^r| = \left|(b-a)\sum_{k=0}^{r-1}(1-a)^k(1-b)^{r-k-1}\right| \leqslant r\rho^{r-1}|b-a|$$

whenever $|1-a|$, $|1-b| \leqslant \rho$ $(\rho > 0)$ so that, using (2)

$$|(F_r \star \mu)(p^\alpha)| \leqslant r\left(\frac{|f(p)|}{p}\right)^\alpha \left|1 - \frac{1}{f(p)}\right| \leqslant \frac{\theta r}{p^\alpha}|f(p)|^{\alpha-1} \leqslant \frac{\theta r}{p^\alpha}$$

as asserted. $\qquad\square$

We now are in a position to prove Theorem 3.

Proof of Theorem 3. From Lemma 6, we may apply Proposition 5 to the function F_r and get

$$\sum_{n\leqslant x} F_r(n) = (r+1)\,C_r(f)\,x + O\left((\log x)^{\theta r}\right)$$

where $C_r(f)$ is defined in (1). Now from Proposition 2, we get

$$\sum_{n\leqslant x}\sum_{i_1=1}^{n}\cdots\sum_{i_r=1}^{n} f\left(\gcd(i_1,n)\right)\cdots f\left(\gcd(i_r,n)\right) = \sum_{n\leqslant x}(\varphi \star f)^r(n) = \sum_{n\leqslant x}n^r F_r(n)$$

and we conclude the proof using partial summation. $\qquad\square$

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References

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