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A Short Proof of Carlitz's Bernoulli Number Identity

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Abstract

For an identity related to Bernoulli numbers, stated by Carlitz, rediscovered and reproved by various researchers, an extremely short and direct proof is provided which uses a bivariate exponential function.

1 Introduction and result

The very recent paper [2] deals with the remarkable identity

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k}$$

involving the Bernoulli numbers (B_n). We learn that it originated as a problem of Carlitz [1], with a solution by Shannon [3], which uses induction. It was rediscovered by Vassilev and Vassilev-Missana [4]. The paper by Gould and Quaintance [2] introduces and uses a binomial transform to prove it.

In this quick note, I would like to present a proof by (exponential) generating functions, which is perhaps the most direct argument. It goes like this:

$$\begin{aligned}
F(z,x) &:= \sum_{m\ge 0}\frac{z^m}{m!}\sum_{n\ge 0}\frac{x^n}{n!}(-1)^m\sum_{k=0}^{m}\binom{m}{k}B_{n+k}\\
&= \sum_{n\ge 0}\frac{x^n}{n!}\sum_{k\ge 0}\frac{B_{n+k}}{k!}\sum_{m\ge k}(-z)^m\frac{1}{(m-k)!}\\
&= \sum_{n\ge 0}\frac{x^n}{n!}\sum_{k\ge 0}\frac{B_{n+k}}{k!}(-z)^k e^{-z}\\
&= e^{-z}\sum_{N\ge 0}\frac{B_N}{N!}\sum_{k=0}^{N}\binom{N}{k}(-z)^k x^{N-k}\\
&= e^{-z}\sum_{N\ge 0}\frac{B_N}{N!}(x-z)^N\\
&= e^{-z}\frac{x-z}{e^{x-z}-1} = \frac{x-z}{e^x-e^z} = F(x,z).
\end{aligned}$$

This symmetry proves the identity. □

References

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