Journal of Integer Sequences, Vol. 17 (2014),
Article 14.9.3

Higher Order Derivatives of Trigonometric Functions, Stirling Numbers of the Second Kind, and Zeon Algebra

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Abstract

In this work we provide a new short proof of closed formulas for the n-th derivative of the cotangent and secant functions using simple operations in the context of the Zeon algebra. Our main ingredients in the proof comprise a representation of the ordinary derivative as an integration over the Zeon algebra, a representation of the Stirling numbers of the second kind as a Berezin integral, and a change of variables formula under Berezin integration. The approach described here is also suitable to give closed expressions for higher order derivatives of tangent, cosecant and all the aforementioned functions hyperbolic analogues.

1 Introduction

In this work, using basic operations on the Zeon algebra [17, 18, 29, 30], we will give a simple and short proof of the following closed formulas for the n-th derivative of the cotangent and secant functions [2, 8, 13, 14, 15].

[1]This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) under grant 307617/2012-2.

Theorem 1. *Let $n \geq 1$ be an integer. Then*

$$\frac{d^n \cot(x)}{dx^n} = (2i)^n\big(\cot(x) - i\big)\sum_{k=1}^{n}\frac{k!}{2^k}S(n,k)\big(i\cot(x) - 1\big)^k \tag{1}$$

and

$$\frac{d^n \sec(x)}{dx^n} = \sec(x)i^n\sum_{j=0}^{n}(-1)^j j!\sum_{k=j}^{n}\binom{n}{k}S(k,j)2^{k-j}\big(i\tan(x) + 1\big)^j, \tag{2}$$

where $i := \sqrt{-1}$, $\binom{n}{k} := n!/\big(k!(n-k)!\big)$ and $S(n,k)$ denotes the Stirling numbers of the second kind.

The determination of closed expressions for higher order derivatives of trigonometric functions is a subject of recurrent interest [2, 3, 7, 8, 10, 11, 12, 13, 14, 15, 19, 20, 21, 22, 23, 24, 25]. As remarked earlier [13, 14, 15, 19], the closed expression in (1) remained an open problem for several years [3, 7, 21, 22, 23]. The proof given here is worth reporting, because, besides being an independent short proof of a non-trivial problem [19], a natural interpretation of the proof from the point of view of the approach addressed here is at our disposal. Simple operations on the Zeon algebra (see Definitions 2, 3 and Lemma 4 in Section 2) and the representation of the ordinary derivative and the Stirling numbers of the second kind as a Berezin integral (see Lemma 6 and Section 3) comprise our main ingredients.

Although we will focus on the proof of Theorem 1, our approach is also suitable to prove (following the steps described in Section 3) closed expressions for the n-th derivative of tangent (tan), cosecant (csc), and the hyperbolic analogues of all the functions cited.

Before we continue, we establish the basic underlying algebraic setup needed to give the proof of Theorem 1.

2 Basics of the Zeon algebra and Berezin integration

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ be the complex numbers, real numbers, and integers, respectively.

Definition 2. The *Zeon algebra* $\mathcal{Z}_n \supset \mathbb{C}$ is defined as the associative algebra generated by the collection $\{\varepsilon_j\}_{j=1}^n$ $(n < \infty)$ and the scalar $1 \in \mathbb{C}$, such that $1\varepsilon_j = \varepsilon_j = \varepsilon_j 1$, $\varepsilon_j\varepsilon_k = \varepsilon_k\varepsilon_j$ $\forall\ j, k$ and $\varepsilon_j^2 = 0\ \forall\ j$.

Note that only linear elements in $\mathcal{Z}_n$ contribute to the calculations.

For $\{j, k, \ldots, l\} \subset \{1, 2, \ldots, n\}$ and $\varepsilon_{jk\cdots l} \equiv \varepsilon_j\varepsilon_k\cdots\varepsilon_l$ the most general element with n generators ε_j can be written as (with the convention of sum over repeated indices implicit)

$$\phi_n = a + a_j\varepsilon_j + a_{jk}\varepsilon_{jk} + \cdots + a_{12\cdots n}\varepsilon_{12\cdots n} = \sum_{\mathbf{j}\in 2^{[n]}} a_{\mathbf{j}}\varepsilon_{\mathbf{j}}, \tag{3}$$

with a, a_j, a_{jk}, …, $a_{12\cdots n} \in \mathbb{C}$, $2^{[n]}$ being the power set of $[n] := \{1, 2, \ldots, n\}$, and $1 \leq j < k < \cdots \leq n$. We refer to a as the body of ϕ_n and write $b(\phi_n) = a$ and to $\phi_n - a$ as the soul such that $s(\phi_n) = \phi_n - a$. The top-term is given by $\varepsilon_{12\cdots n} = \varepsilon_1\varepsilon_2\cdots\varepsilon_n$, since $\varepsilon_{12\cdots n}$ contains all the elements of $\{\varepsilon_j\}_{j=1}^n$. Note that $\varepsilon_{12\cdots n}\varepsilon_j = 0$ for all $j = 1,\ldots,n$. In Lemma 4 we will also use the notation $\phi_n \equiv \phi_n\big(\boldsymbol{\varepsilon} \equiv (\varepsilon_1,\ldots,\varepsilon_n)\big)$ to indicate directly the dependence of ϕ_n on the generators of the Zeon algebra $\mathcal{Z}_n$.

Any sufficiently smooth function $f(z)$ in the complex domain admits an extension to the context of the Zeon algebra from previous results due to DeWitt [16, Chapter 1] and Rogers [27, Chapter 4]. More precisely, we have [16, Equation (1.1.6)]

$$f(\phi_n) := \sum_{j=0}^{n} \frac{f^{(j)}\big(b(\phi_n)\big)}{j!} s^j(\phi_n) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!} s^j(\phi_n), \tag{4}$$

where $f^{(j)}(a) = d^j f(z)/dz^j|_{z=a}$ is the j-th ordinary derivative of $f(z)$ at a. Note that $f\big(a+s(\phi_n)\big)|_{s(\phi_n)=0} = f(a)$ and $f\big(a+s(\phi_n)\big) \in \mathcal{Z}_n$, because $s^{n+1}(\phi_n) = 0$.

Concrete examples of (3), which will be important in the proof of Theorem 1, are the generalization of the ordinary exponential and the generalized inverse ($a \equiv b(\phi_n) \neq 0$ for the generalized inverse) given by

$$e^{\phi_n} = e^a \sum_{j=0}^{n} \frac{s^j(\phi_n)}{j!} \text{ and } \phi_n^{-1} = \frac{1}{a}\sum_{j=0}^{n}(-1)^j \frac{s^j(\phi_n)}{a^j}, \tag{5}$$

respectively. Particular cases of (5) are given by $e^{1+\sqrt{2}\varepsilon_1-\varepsilon_3+i\varepsilon_{23}} = e\big(1-\varepsilon_3+\sqrt{2}\varepsilon_1+i\varepsilon_{23} -\sqrt{2}\varepsilon_{13}+i\sqrt{2}\varepsilon_{123}\big)$ with $n \geq 3$ and $(1-\varepsilon_2+\varepsilon_{146})^{-1} = 1+\varepsilon_2-\varepsilon_{146}-2\varepsilon_{1246}$ with $n \geq 6$.

Using (5) we can also define more complex functions, which will be needed in this work, such as $\cot(\phi_n)$. The generalization of the cotangent function is defined by

$$\cot(\phi_n) := i\frac{e^{i\phi_n}+e^{-i\phi_n}}{e^{i\phi_n}-e^{-i\phi_n}} \tag{6}$$

with $b(\phi_n) \equiv a \in \mathbb{R}\backslash\{k\pi : k \in \mathbb{Z}\}$. Note that $b\big(e^{i\phi_n}-e^{-i\phi_n}\big) = e^{ia}-e^{-ia} \neq 0$. Therefore, $\cot(\phi_n)$ is well-defined using (5) and, as expected, $\cot(\phi_n)|_{s(\phi_n)=0} = \cot(a)$.

Definition 3. The *Berezin integral* on $\mathcal{Z}_n$, denoted by $\int$, is the linear functional $\int : \mathcal{Z}_n \to \mathbb{C}$ such that (we use throughout this work the compact notation $d\mu_n := d\varepsilon_n\cdots d\varepsilon_1$)

$$d\varepsilon_j d\varepsilon_k = d\varepsilon_k d\varepsilon_j,\ \int \phi_n(\hat{\varepsilon}_j)d\varepsilon_j = 0 \text{ and} \int \phi_n(\hat{\varepsilon}_j)\varepsilon_j d\varepsilon_j = \phi_n(\hat{\varepsilon}_j),$$

where $\phi_n(\hat{\varepsilon}_j)$ means any element of $\mathcal{Z}_n$ with no dependence on ε_j. Multiple integrals are iterated integrals, i.e.,

$$\int f(\phi_n)d\mu_n = \int\cdots\left(\int\left(\int f(\phi_n)d\varepsilon_n\right)d\varepsilon_{n-1}\right)\cdots d\varepsilon_1. \tag{7}$$

For example, it follows directly from Definition 3 that $\int d\varepsilon_j = 0$, $\int \varepsilon_j d\varepsilon_j = 1$, $\int \varepsilon_j\varepsilon_k d\varepsilon_k = \varepsilon_j \ \forall \ j \neq k$ and $\int \phi_n d\mu_n = a_{12\cdots n}$. In other words, the Berezin integral of ϕ_n in (3) gives the coefficient of the top-term $\varepsilon_{12\cdots n}$. For more details on Berezin integration we refer the reader to the books of Berezin [5, Chapter 1] and [6, Chapter 2].

It is a direct consequence of Definition 3 that a change of variables formula holds for the Berezin integral. The proof of the change of variables formula is routine and a detailed proof is presented in, e.g., Rogers [27, Theorem 11.2.3] or DeWitt [16, Chapter 1] in the context of the Grassmann algebra and can be straightforwardly adapted to $\mathcal{Z}_n$. For our purposes it is sufficient to consider how a simple linear transformation acting on the generators of $\mathcal{Z}_n$ affects the Berezin integral. From now on $\mathrm{per}(A)$ means the permanent of the matrix A.

Lemma 4. *Let $\varepsilon'_j = a_{jk}\varepsilon_k$ with $a_{jk} \in \mathbb{C}$, and let $A = \big[a_{jk}\big]$ be a square matrix of order n such that $\det(A) \neq 0$ and $\mathrm{per}(A^{-1}) \neq 0$. With the previous assumptions and notation the following formula holds*

$$\int \phi_n(\boldsymbol{\varepsilon}) d\mu_n = \big(\mathrm{per}\big(A^{-1}\big)\big)^{-1} \int \phi_n(A^{-1}\boldsymbol{\varepsilon'}) d\mu'_n. \tag{8}$$

Remark 5. Note that $\big(\mathrm{per}\big(A^{-1}\big)\big)^{-1}$ instead of $\big(\det(A)\big)^{-1}$ (as it occurs in ordinary calculus) appears in (8).

The representation of the ordinary derivative as a Berezin integration can be traced back to previous papers [4, 9]. The proof of the aforementioned representation follows by calculating the Berezin integral of both sides of (4) with $s(\phi_n) \equiv \varphi_n := \sum_{j=1}^n \varepsilon_j$ and observing that $\int \varphi_n^k d\mu_n = k!\delta_{k,n}$ with $d\mu_n = d\varepsilon_n \cdots d\varepsilon_1$. Finally, we obtain the desired result using the multinomial theorem and Definition 3. For instance, see [4, Lemma 4.1 and Corollary 4.2].

Lemma 6. *Let f be a sufficiently smooth function, and let $\varphi_n = \sum_{j=1}^n \varepsilon_j$, where $\{\varepsilon_j\}_{j=1}^n$ is the set of generators of the Zeon algebra $\mathcal{Z}_n$. For $x \in \mathbb{C}$ and $d\mu_n = d\varepsilon_n \cdots d\varepsilon_1$, the following Berezin integral representation of the n-th ordinary derivative of f holds*

$$\int f(x + \varphi_n) d\mu_n = f^{(n)}(x).$$

3 Proof of Theorem 1

Before we prove Theorem 1 we will need some auxiliary results.

It is a classical result in combinatorics that the following generating function holds for the Stirling numbers of the second kind [31, Section 1.9]:

$$G_S(x) = \frac{1}{k!}\big(e^x - 1\big)^k = \sum_{m=k}^{\infty} S(m,k)\frac{x^m}{m!}.$$

Using (4) and Lemma 6 with $f = G_S$ we find

$$S(n,k) = \frac{1}{k!}\int \left(e^{\varphi_n} - 1\right)^k d\mu_n, \tag{9}$$

where e^{φ_n} is defined by (5). Setting $A_n \equiv e^{\varphi_n} - 1 = \prod_{j=1}^n e^{\varepsilon_j} - 1 = \prod_{j=1}^n (1+\varepsilon_j) - 1$ (using (5) and $\varepsilon_j^2 = 0$) in (9) we get $S(n,k) = \int A_n^k d\mu_n / k!$, recovering the representation of $S(n,k)$ introduced by Schott and Staples [29] and proved there directly from the definition of $S(n,k)$ in terms of partitions of a finite set. In this way, [29, Definition 1.3] is compatible with Definition 3. Here we use a slightly different notation from that adopted by Schott and Staples [29], with a subscript n added to A_n.

Using the representation of the ordinary derivative in Lemma 6 we are ready to prove Theorem 1.

Proof of Theorem 1. Setting $f(x) = \cot(x)$ in Lemma 6, $x \in \mathbb{R}\backslash\{k\pi : k \in \mathbb{Z}\}$, we have the representation

$$\begin{aligned}
\cot^{(n)}(x) &= \int \cot(x+\varphi_n) d\mu_n = i\int \frac{e^{i\left(x+\varphi_n\right)} + e^{-i\left(x+\varphi_n\right)}}{e^{i\left(x+\varphi_n\right)} - e^{-i\left(x+\varphi_n\right)}} d\mu_n \\
&= i\int \frac{e^{i\left(x+2\varphi_n\right)} + e^{-ix}}{e^{i\left(x+2\varphi_n\right)} - e^{-ix}} d\mu_n = i(2i)^n \int \frac{e^{ix+\varphi_n} + e^{-ix}}{e^{ix+\varphi_n} - e^{-ix}} d\mu_n.
\end{aligned} \tag{10}$$

Equation (10) follows from (6) and the change of variables formula of Lemma 4 with $a_{jk} = a_j\delta_{jk}$ and $a_j = 2i\ \forall\ j$. Note that $\left(\mathrm{per}\left(A^{-1}\right)\right)^{-1} = \left(\mathrm{per}[\delta_{jk}/a_j]\right)^{-1} = (2i)^n$. Now we write $e^{\varphi_n} = \left(e^{\varphi_n} - 1\right) + 1$ and use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ to obtain

$$\frac{e^{ix+\varphi_n} + e^{-ix}}{e^{ix+\varphi_n} - e^{-ix}} = \frac{2\cos(x) + e^{ix}\left(e^{\varphi_n} - 1\right)}{2i\sin(x) + e^{ix}\left(e^{\varphi_n} - 1\right)}. \tag{11}$$

For $k = 0$ and $k > n$, we have

$$\int \left(e^{\varphi_n} - 1\right)^k d\mu_n = 0, \tag{12}$$

since $\int d\mu_n = 0$ $(n \geq 1)$ and

$$\left(e^{\varphi_n} - 1\right)^{n+1} = \varphi_n^{n+1}\left(\sum_{j=0}^{n-1} \frac{\varphi_n^j}{(j+1)!}\right)^{n+1} = 0.$$

Next, we use (5) with $\phi_n \equiv 2i\sin(x) + e^{ix}\left(e^{\varphi_n} - 1\right)$, (11), (12), and the linearity of the Berezin integral to get

$$\begin{aligned}
\int \tfrac{e^{ix+\varphi_n} + e^{-ix}}{e^{ix+\varphi_n} - e^{-ix}} d\mu_n &= \textstyle\sum_{k=0}^n (-1)^k e^{ikx} \int \frac{\left(2\cos(x) + e^{ix}\left(e^{\varphi_n} - 1\right)\right)\left(e^{\varphi_n} - 1\right)^k}{\left(2i\sin(x)\right)^{k+1}} d\mu_n \\
&= \left(\tfrac{\cos(x)}{i\sin(x)} - 1\right) \textstyle\sum_{k=1}^n (-1)^k e^{ikx} \int \frac{\left(e^{\varphi_n} - 1\right)^k}{\left(2i\sin(x)\right)^k} d\mu_n \\
&= -\left(i\cot(x) + 1\right) \textstyle\sum_{k=1}^n \frac{k!}{2^k}\left(i\cot(x) - 1\right)^k \frac{1}{k!}\int \left(e^{\varphi_n} - 1\right)^k d\mu_n.
\end{aligned}$$

Going back to (10) and using the Berezin integral representation of $S(n,k)$ of (9) we obtain the desired result, i.e., Equation (1).

Using Lemma 6, $x \in \mathbb{R}\backslash\{(2k+1)\pi/2 : k \in \mathbb{Z}\}$, we get

$$\sec^{(n)}(x) = \int \sec(x+\varphi_n)d\mu_n = \int \frac{2}{e^{i(x+\varphi_n)} + e^{-i(x+\varphi_n)}}d\mu_n = \int \frac{2e^{i\varphi_n}}{e^{i(x+2\varphi_n)} + e^{-ix}}d\mu_n.$$

Now we write $e^{2i\varphi_n} = (e^{2i\varphi_n} - 1) + 1$ and use (5) to get

$$\sec^{(n)}(x) = 2\sum_{l=0}^{n}(-1)^l e^{ilx}\int e^{i\varphi_n}\frac{\left(e^{i2\varphi_n}-1\right)^l}{\left(2\cos(x)\right)^{l+1}}d\mu_n. \tag{13}$$

Next, we make the expansion $e^{i\varphi_n} = \prod_{j=1}^{n}(1+i\varepsilon_j)$ and, as a result, we need to analyze a general term such as

$$\begin{aligned}
\sum_{1\le j_1<\cdots<j_k\le n} i^k \int \varepsilon_{j_1}\cdots\varepsilon_{j_k}\left(e^{i2\varphi_n}-1\right)^l d\mu_n &= \binom{n}{k} i^k \int \varepsilon_n\cdots\varepsilon_{n-k+1}\left(e^{i2\varphi_n}-1\right)^l d\mu_n \\
&= \binom{n}{k} i^k \int \left(e^{i2\varphi_{n-k}}-1\right)^l d\mu_{n-k} \\
&= \binom{n}{k}(2i)^{n-k} i^k \int \left(e^{\varphi_{n-k}}-1\right)^l d\mu_{n-k}. \qquad (14)
\end{aligned}$$

The invariance of $e^{i2\varphi_n}$ under permutations of ε_j with $j = 1,\ldots,n$ was used to obtain the first equality. The second equality follows from

$$\varepsilon_n\cdots\varepsilon_{n-k+1}\left(e^{i2\varphi_n}-1\right)^l = \varepsilon_n\cdots\varepsilon_{n-k+1}\left(e^{i2\varphi_{n-k}}-1\right)^l$$

and (7). Finally, the change of variables formula of Lemma 4 was used to obtain the last equality. Note that the constraint $n-k \ge l$ follows from the properties of the Berezin integral. Using (14) and the representation of (9) in (13) we obtain the desired result, i.e., Equation (2). □

4 Concluding remarks

We have shown that closed formulas for the n-th derivative of the cotangent and secant functions in Theorem 1 follow from simple computations in the context of the Zeon algebra. Our approach is also suitable to give closed formulas for higher order derivatives of other trigonometric functions, i.e., csc, tan and hyperbolic functions such as coth, sech, csch and tanh. Our starting point was an extension of a function in the complex domain to the more general scenario of the Zeon algebra (see (4)). Along the way, the Berezin integral representation of the Stirling numbers of the second kind played a key role in our analysis. The aforementioned extension led us to prove Theorem 1 quite naturally using known results about the Zeon algebra (Lemma 4 and Lemma 6) and, at the same time, taking advantage of the computational power of the Zeon algebra, i.e., the fact that only linear terms on the generators appear in the calculations. The final message is that techniques based on superanalysis [5, 6], as it occurs in other contexts [1, 9, 26, 28], may provide a useful computational toolbox in representing combinatorial numbers, such as the Stirling numbers of the second kind, and in proving combinatorial identities of the type considered here.

5 Acknowledgments

The author thanks the anonymous referee for suggestions that improved the paper.

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2010 *Mathematics Subject Classification*: Primary 11B73; Secondary 33B10; 05A15; 05A18; 05A19.
Keywords: Zeon algebra, Berezin integration, cotangent, secant, Stirling number of the second kind, generating function.

(Concerned with sequence A008277).

Received April 13 2014; revised versions received July 8 2014; July 16 2014; August 6 2014. Published in *Journal of Integer Sequences*, August 12 2014.

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