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Powers of Two as Sums of Two Lucas Numbers

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Abstract

Let $(L_n)_{n\geq 0}$ be the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. In this paper, we are interested in finding all powers of two which are sums of two Lucas numbers, i.e., we study the Diophantine equation $L_n + L_m = 2^a$ in nonnegative integers n, m, and a. The proof of our main theorem uses lower bounds for linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in diophantine approximation. This paper continues our previous work where we obtained a similar result for the Fibonacci numbers.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. The Fibonacci numbers are famous for possessing wonderful and amazing properties. They are accompanied by the sequence of *Lucas numbers*, which is as important as the Fibonacci sequence. The Lucas sequence $(L_n)_{n\geq 0}$ follows the same recursive pattern as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$.

The study of properties of the terms of such sequences, or more generally, linear recurrence sequences, has a very long history and has generated a huge literature. For the beauty and rich applications of these numbers and their relatives, one can see Koshy's book [9].

For example, a remarkable property of the Fibonacci sequence is that 1, 2 and 8 are the only Fibonacci numbers which are powers of 2. One proof of this fact follows from Carmichael's primitive divisor theorem [6], which states that for n greater than 12, the nth Fibonacci number F_n has at least one prime factor that is not a factor of any previous Fibonacci number (see the paper of Bilu, Hanrot, and Voutier [2] for the most general version of the above statement). Similarly, it is well known that 1, 2 and 4 are the only powers of 2 that appear in the Lucas sequence.

The problem of finding all perfect powers in the Fibonacci sequence and the Lucas sequence was a famous open problem finally solved in 2006 in a paper in *Annals of Mathematics* by Bugeaud, Mignotte, and Siksek [3]. In their work, they applied a combination of Baker's method, the modular approach and some classical techniques to show that the only perfect powers in the Fibonacci sequence are 0, 1, 8 and 144, and the only perfect powers in the Lucas sequence are 1 and 4. A detailed account of this problem can be found in [3, Section 10].

In our recent paper [5], we found all powers of 2 which are the sums of at most two Fibonacci numbers. Specifically, we proved the following.

Theorem 1. *The only solutions of the Diophantine equation $F_n + F_m = 2^a$ in positive integers n, m and a with $n \geq m$ are given by*

$$2F_1 = 2, \quad 2F_2 = 2, \quad 2F_3 = 4, \quad 2F_6 = 16,$$

and

$$F_2 + F_1 = 2, \quad F_4 + F_1 = 4, \quad F_4 + F_2 = 4, \quad F_5 + F_4 = 8, \quad F_7 + F_4 = 16.$$

In this paper, we prove an analogue of Theorem 1 when the sequence of Fibonacci numbers is replaced by the sequence of the Lucas numbers, i.e., we extend our previous work [5] and determine all the solutions of the Diophantine equation

$$L_n + L_m = 2^a \tag{1}$$

in nonnegative integers $n \geq m$ and a.

Similar problems have recently been investigated. For example, repdigits which are sums of at most three Fibonacci numbers were found by Luca [12]; Fibonacci numbers which are sums of two repdigits were obtained by Díaz and Luca [7], while factorials which are sums of at most three Fibonacci numbers were found by Luca and Siksek [11].

We prove the following result.

Theorem 2. *All solutions of the Diophantine equation (1) in nonnegative integers $n \geq m$ and a, are*

$$2L_0 = 4, \quad 2L_1 = 2, \quad 2L_3 = 8, \quad L_2 + L_1 = 4, \quad L_4 + L_1 = 8 \quad \textit{and} \quad L_7 + L_2 = 32.$$

Let us give a brief overview of our strategy for proving Theorem 2. First, we rewrite equation (1) in suitable ways in order to obtain two different linear forms in logarithms which are both nonzero and small. Next, we use a lower bound on such nonzero linear forms in two logarithms due to Laurent, Mignotte, and Nesterenko as well as a general lower bound due to Matveev to find an absolute upper bound for n; hence, an absolute upper bound for m and a, which we then reduce using standard facts concerning continued fractions.

In this paper, we follow the approach and the presentation described in [5].

2 Auxiliary results

Before proceeding further, we recall that the Binet formula

$$L_n = \alpha^n + \beta^n \quad \text{holds for all} \quad n \geq 0,$$

where

$$\alpha := \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta := \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$ of $(L_n)_{n\geq 0}$. This will be an important ingredient in what follows. In particular, the inequality

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \tag{2}$$

holds for all $n \geq 0$.

In order to prove Theorem 2, we need a result of Laurent, Mignotte, and Nesterenko [10] about linear forms in two logarithms. But first, some notation.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(X - \eta^{(i)}),$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|, 1\}\right)\right)$$

is called the *logarithmic height* of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$.

The following properties of the logarithmic height, which will be used in the next section without special reference, are also known:

- $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$.
- $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$.
- $h(\eta^s) = |s|h(\eta)$.

With the above notation, Laurent, Mignotte, and Nesterenko [10, Corollary 1] proved the following theorem.

Theorem 3. *Let γ_1, γ_2 be two non-zero algebraic numbers, and let $\log\gamma_1$ and $\log\gamma_2$ be any determinations of their logarithms. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$, and*

$$\Gamma := b_2\log\gamma_2 - b_1\log\gamma_1,$$

where b_1 and b_2 are positive integers. Further, let A_1, A_2 be real numbers > 1 such that

$$\log A_i \geq \max\left\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\}, \quad i = 1, 2.$$

Then, assuming that γ_1 and γ_2 are multiplicatively independent, we have

$$\log|\Gamma| > -30.9 \cdot D^4\left(\max\left\{\log b', \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log A_1 \cdot \log A_2,$$

where

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

We shall also need the following general lower bound for linear forms in logarithms due to Matveev [13] (see also the paper of Bugeaud, Mignotte, and Siksek [3, Theorem 9.4]).

Theorem 4 (Matveev's theorem). *Assume that $\gamma_1, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad \textit{for all} \quad i = 1, \ldots, t.$$

In 1998, Dujella and Pethő in [8, Lemma 5 (a)] gave a version of the reduction method based on the Baker-Davenport lemma [1]. To conclude this section of auxiliary results, we present the following lemma from [4], which is an immediate variation of the result due to Dujella and Pethő from [8], and will be one of the key tools used in this paper to reduce the upper bounds on the variables of the equation (1).

Lemma 5. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $|| \cdot ||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

3 The Proof of Theorem 2

Assume throughout that equation (1) holds. First of all, observe that if $n = m$, then the original equation (1) becomes $L_n = 2^{a-1}$. But the only solutions of this latter equation are $(n, a) \in \{(0, 2), (1, 1), (3, 3)\}$ and this fact has already been mentioned in the Introduction. So, from now on, we assume that $n > m$.

If $n \leq 200$, then a brute force search with *Mathematica* in the range $0 \leq m < n \leq 200$ gives the solutions $(n, m, a) \in \{(2, 1, 2), (4, 1, 3), (7, 2, 5)\}$. This took a few seconds. Thus, for the rest of the paper we assume that $n > 200$.

Let us now get a relation between n and a. Combining (1) with the right inequality of (2), one gets that

$$2^a \leq 2\alpha^n + 2\alpha^m < 2^{n+1} + 2^{m+1} = 2^{n+1}(1 + 2^{m-n}) \leq 2^{n+1}(1 + 2^{-1}) < 2^{n+2},$$

which leads to $a \leq n+1$. This estimate is essential for our purpose.

On the other hand, we rewrite equation (1) as

$$\alpha^n - 2^a = -\beta^n - L_m.$$

We now take absolute values in the above relation obtaining

$$|\alpha^n - 2^a| \leq |\beta|^n + L_m < \frac{1}{2} + 2\alpha^m.$$

Dividing both sides of the above expression by α^n and taking into account that $n > m$, we get

$$\left|1 - 2^a \cdot \alpha^{-n}\right| < \frac{3}{\alpha^{n-m}}. \tag{3}$$

In order to apply Theorem 3, we take $\gamma_1 := \alpha$, $\gamma_2 := 2$, $b_1 := n$ and $b_2 := a$. So,

$$\Gamma := b_2 \log \gamma_2 - b_1 \log \gamma_1,$$

and therefore estimation (3) can be rewritten as

$$\left|1 - e^{\Gamma}\right| < \frac{3}{\alpha^{n-m}}. \tag{4}$$

The algebraic number field containing γ_1, γ_2 is $\mathbb{Q}(\sqrt{5})$, so we can take $D := 2$. By using (1) and the Binet formula for the Lucas sequence, we have

$$\alpha^n = L_n - \beta^n < L_n + 1 \leq L_n + L_m = 2^a.$$

Consequently, $1 < 2^a\, \alpha^{-n}$ and so $\Gamma > 0$. This, together with (4), gives

$$0 < \Gamma < \frac{3}{\alpha^{n-m}}, \tag{5}$$

where we have also used the fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$. Hence,

$$\log \Gamma < \log 3 - (n-m) \log \alpha. \tag{6}$$

Note further that $h(\gamma_1) = (\log \alpha)/2 = 0.2406\cdots$ and $h(\gamma_2) = \log 2 = 0.6931\cdots$; thus, we can choose $\log A_1 := 0.5$ and $\log A_2 := 0.7$. Finally, by recalling that $a \leq n+1$, we get

$$b' = \frac{n}{1.4} + a < 1.71429\, n + 1 < 2n.$$

Since α and 2 are multiplicatively independent, we have, by Theorem 3, that

$$\begin{aligned} \log \Gamma \geq & -30.9 \cdot 2^4 \cdot \left(\max\{\log(2n), 21/2, 1/2\}\right)^2 \cdot 0.5 \cdot 0.7 \\ > & -174 \cdot \left(\max\{\log(2n), 21/2, 1/2\}\right)^2. \end{aligned} \tag{7}$$

We now combine (6) and (7) to obtain

$$(n-m)\log\alpha < 180\cdot\left(\max\left\{\log(2n), 21/2\right\}\right)^2. \tag{8}$$

Let us now get a second linear form in logarithms. To this end, we now rewrite (1) as follows:

$$\alpha^n(1+\alpha^{m-n}) - 2^a = -\beta^n - \beta^m.$$

Taking absolute values in the above relation and using the fact that $\beta = (1-\sqrt{5})/2$, we get

$$|\alpha^n(1+\alpha^{m-n}) - 2^a| = |\beta|^n + |\beta|^m < 2$$

for all $n > 200$ and $m \geq 0$. Dividing both sides of the above inequality by the first term of the left-hand side, we obtain

$$\left|1 - 2^a\cdot\alpha^{-n}\cdot(1+\alpha^{m-n})^{-1}\right| < \frac{2}{\alpha^n}. \tag{9}$$

We are now ready to apply Matveev's result Theorem 4. To do this, we take the parameters $t := 3$ and

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := 1+\alpha^{m-n}.$$

We take $b_1 := a$, $b_2 := -n$ and $b_3 := -1$. As before, $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. To see why the left-hand side of (9) is not zero, note that otherwise, we would get the relation

$$2^a = \alpha^n + \alpha^m. \tag{10}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$2^a = \beta^n + \beta^m. \tag{11}$$

Combining (10) and (11), we obtain

$$\alpha^n < \alpha^n + \alpha^m = |\beta^n + \beta^m| \leq |\beta|^n + |\beta|^m < 2,$$

which is impossible for $n > 200$. Hence, indeed the left-hand side of inequality (9) is nonzero.

In this application of Matveev's theorem we take $A_1 := 1.4$ and $A_2 := 0.5$. Since $a \leq n+1$ it follows that we can take $B := n+1$. Let us now estimate $h(\gamma_3)$. We begin by observing that

$$\gamma_3 = 1+\alpha^{m-n} < 2 \quad \text{and} \quad \gamma_3^{-1} = \frac{1}{1+\alpha^{m-n}} < 1,$$

so that $|\log\gamma_3| < 1$. Next, notice that

$$h(\gamma_3) \leq |m-n|\left(\frac{\log\alpha}{2}\right) + \log 2 = \log 2 + (n-m)\left(\frac{\log\alpha}{2}\right).$$

Hence, we can take

$$A_3 := 2 + (n-m)\log\alpha > \max\{2h(\gamma_3), |\log\gamma_3|, 0.16\}.$$

Now Matveev's theorem implies that a lower bound on the left-hand side of (9) is

$$\exp\left(-C\cdot(1+\log(n+1))\cdot 1.4\cdot 0.5\cdot(2+(n-m)\log\alpha)\right)$$

where $C := 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2) < 9.7\times 10^{11}$. So, inequality (9) yields

$$n\log\alpha - \log 2 < 1.36\times 10^{12}\log n\cdot(2+(n-m)\log\alpha), \tag{12}$$

where we used the inequality $1+\log(n+1) < 2\log n$, which holds because $n > 200$.

Using now (8) in the right-most term of the above inequality (12) and performing the respective calculations, we arrive at

$$n < 6\times 10^{14}\log n\cdot\left(\max\{\log(2n), 21/2\}\right)^2. \tag{13}$$

If $\max\{\log(2n), 21/2\} = 21/2$, it then follows from (13) that $n < 8\times 10^{16}\log n$ giving $n < 3.5\times 10^{18}$. If on the other hand we have that $\max\{\log(2n), 21/2\} = \log(2n)$, then, from (13), we get $n < 6\times 10^{14}\log n\log^2(2n)$ and so $n < 5.9\times 10^{19}$. In any case, we have that $n < 5.9\times 10^{19}$ always holds. We summarize what we have proved so far in the following lemma.

Lemma 6. *If (n, m, a) is a solution in positive integers of equation (1) with $n > m$ and $n > 200$, then inequalities*

$$a \le n+1 < 6\times 10^{19}$$

hold.

4 Reducing the bound on n

After finding an upper bound on n the next step is to reduce it. To do this, we first use some properties of continued fractions to obtain a suitable upper bound on $n-m$, and secondly we use Lemma 5 to conclude that n must be relatively small. Let us see.

Turning back to inequality (5), we obtain

$$0 < a\log 2 - n\log\alpha < \frac{3}{\alpha^{n-m}}.$$

Dividing across by $\log\alpha$, we get

$$0 < a\gamma - n < \frac{7}{\alpha^{n-m}}, \quad \text{where} \quad \gamma := \frac{\log 2}{\log\alpha}. \tag{14}$$

Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [1, 2, 3, 1, 2, \ldots]$ be the continued fraction expansion of γ, and let denote p_k/q_k its kth convergent. Recall also that $a < 6 \times 10^{19}$ by Lemma 6.

A quick inspection using *Mathematica* reveals that

$$54475119544877440894 = q_{44} < 6 \times 10^{19} < q_{45} = 67219577652603468483.$$

Furthermore, $a_M := \max\{a_i \mid i = 0, 1, \ldots, 45\} = a_{17} = 134$. So, from the known properties of continued fractions, we obtain that

$$|a\gamma - n| > \frac{1}{(a_M + 2)a}. \tag{15}$$

Comparing estimates (14) and (15), we get right away that

$$\alpha^{n-m} < 7 \cdot 136 \cdot a < 6 \times 10^{22},$$

leading to $n - m \leq 110$. Let us now work a little bit on (9) in order to find an improved upper bound on n. Put

$$z := a \log 2 - n \log \alpha - \log \varphi(n - m), \tag{16}$$

where φ is the function given by the formula $\varphi(t) := 1 + \alpha^{-t}$. Therefore, (9) implies that

$$|1 - e^z| < \frac{2}{\alpha^n}. \tag{17}$$

Note that $z \neq 0$; thus, we distinguish the following cases. If $z > 0$, then, from (17), we obtain

$$0 < z \leq e^z - 1 < \frac{2}{\alpha^n}.$$

Replacing z in the above inequality by its formula (16) and dividing both sides of the resulting inequality by $\log \alpha$, we get

$$0 < a \left(\frac{\log 2}{\log \alpha} \right) - n - \frac{\log \varphi(n - m)}{\log \alpha} < 5 \cdot \alpha^{-n}. \tag{18}$$

We now put

$$\gamma := \frac{\log 2}{\log \alpha}, \quad \mu := -\frac{\log \varphi(n - m)}{\log \alpha}, \quad A := 5 \quad \text{and} \quad B := \alpha.$$

Clearly γ is an irrational number. We also put $M := 6 \times 10^{19}$, which is an upper bound on a by Lemma 6. We therefore apply Lemma 5 to inequality (18) for all choices $n - m \in \{1, \ldots, 110\}$ except when $n - m = 1, 3$ and get that

$$n < \frac{\log(Aq/\epsilon)}{\log B},$$

where $q > 6M$ is a denominator of a convergent of the continued fraction of γ such that $\epsilon = ||\mu q|| - M||\gamma q|| > 0$. Indeed, with the help of *Mathematica* we find that if (n, m, a) is a possible solution of the equation (1) with $z > 0$ and $n - m \neq 1, 3$, then $n \leq 130$. This is false because our assumption that $n > 200$.

Suppose now that $z < 0$. First, note that $2/\alpha^n < 1/2$ since $n > 200$. Then, from (17), we have that $|1 - e^z| < 1/2$ and therefore $e^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{4}{\alpha^n}.$$

Then, by the same arguments used for proving (18), we obtain

$$0 < n\left(\frac{\log \alpha}{\log 2}\right) - a + \frac{\log \varphi(n-m)}{\log 2} < 6 \cdot \alpha^{-n}. \tag{19}$$

Here, we also take $M := 6 \times 10^{19}$, which is an upper bound on n by Lemma 6, and we apply Lemma 5 to inequality (19) for each $n - m \in \{1, \ldots, 110\}$ except for $n - m = 1, 3$. In this case, with the help of *Mathematica*, we find that if (n, m, a) is a possible solution of the equation (1) with $z < 0$ and $n - m \neq 1, 3$, then $n \leq 120$, which is false.

Finally, we deal with the cases when $n - m = 1$ and 3. We cannot study these cases as before because when applying Lemma 5 to the expressions (18) or (19) (according to whether z is positive or negative, respectively), the corresponding parameter μ appearing in Lemma 5 is either

$$-\frac{\log \varphi(t)}{\log \alpha} = \begin{cases} -1, & \text{if } t = 1; \\ 1 - \frac{\log 2}{\log \alpha}, & \text{if } t = 3. \end{cases} \quad \text{or} \quad \frac{\log \varphi(t)}{\log 2} = \begin{cases} \frac{\log \alpha}{\log 2}, & \text{if } t = 1; \\ 1 - \frac{\log \alpha}{\log 2}, & \text{if } t = 3. \end{cases}$$

But, in any case, one can see that the corresponding value of ϵ from Lemma 5 is always negative and therefore the reduction method is not useful for reducing the bound on n in these instances. For this reason we need to treat these cases differently.

All we want to do here is solve the equations

$$L_{m+1} + L_m = 2^a \quad \text{and} \quad L_{m+3} + L_m = 2^a \tag{20}$$

in positive integers m and a with $m + 1 > 200$ and $m + 3 > 200$, respectively. But, by definition $L_{m+1} + L_m = L_{m+2}$. Moreover, $L_{m+3} + L_m = 2L_{m+2}$, which is easily checked. We see from the above discussion that equations (20) are transformed into the simpler equations

$$L_{m+2} = 2^a \quad \text{and} \quad L_{m+2} = 2^{a-1} \tag{21}$$

to be resolved in positive integers m and a with $m > 199$ and $m > 197$, respectively. But, we quickly see that the above equations (21) have no solutions for $m > 1$ as mentioned earlier. This completes the analysis of the cases when $n - m = 1, 3$ and therefore the proof of Theorem 2.

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