Journal of Integer Sequences, Vol. 16 (2013),
Article 13.4.1

# The Central Component of a Triangulation

Alon Regev
Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115
USA
regev@math.niu.edu

**Abstract**

We define the central component of a triangulation of a regular convex polygon as the diameter or triangle containing its geometric center. This definition yields a new recursion relation for Catalan numbers, which can be used to derive congruence relations. We generalize this idea to $k$-angulations, giving congruences of $k$-Catalan numbers. We also enumerate the triangulations that include a fixed vertex in their central components.

## 1 Introduction

Considering a triangulation of a regular convex polygon as a subset of $\mathbb{R}^2$ centered at the origin, define its *central component* to be the diameter or triangle that contains the origin (see Figure 1). More generally, every dissection of a polygon can be associated with its set of components, including one central component. Bowman and the author used components and central components to enumerate symmetry classes of dissections in a paper [2], where these notions are more formally defined. In this note we use central components to obtain new recursion relations for Catalan and $k$-Catalan numbers, and use these recursions to prove congruence relations of these numbers. We also enumerate the triangulations that include a fixed vertex in their central components.

Let $C_n$ be the $n$-th Catalan number, so $C_{n-2}$ is the number of triangulations of an $n$-gon, and let $C_x = 0$ unless $x$ is a nonnegative integer. The Catalan numbers satisfy the recursion

$$\sum_{k=0}^{n} C_k C_{n-k} = C_{n+1}. \tag{1}$$

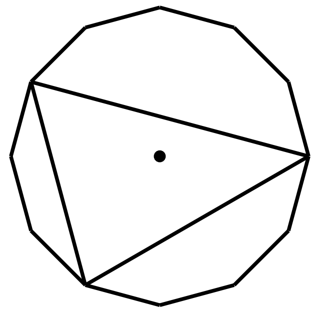

Figure 1: A central triangle with $n = 12$, $i = 3$, $j = 4$ and $k = 5$.

Consider a triangulation of an $n$-gon as a labeled graph with vertices $0, 1, \dots, n-1$ and edges denoted $xy$ for distinct vertices $x$ and $y$. The edges include $n$ sides $01, 12, \dots, (n-1)0$ and $n-3$ diagonals. The *cyclic length* of an edge $xy$, with $x < y$, is defined as

$$\min\{y - x, n + x - y\}.$$

By enumerating the triangulations of an $n$-gon according to their central components, we obtain the following new recursion relation, which is one of the main results of this paper.

**Lemma 1.** *For any* $n \geq 3$,

$$C_{n-2} = \frac{n}{2} C_{n/2-1}^2 + \sum_{\substack{i+j+k=n \\ i \leq j \leq k < n/2}} m_{ijk} C_{i-1} C_{j-1} C_{k-1}, \tag{2}$$

*where*

$$m_{ijk} = \begin{cases} \frac{n}{3}, & \text{if } i = j = k; \\ n, & \text{if } i < j = k \text{ or } i = j < k; \\ 2n, & \text{if } i < j < k. \end{cases}$$

*Proof.* The first term of (2) enumerates the triangulations whose central component is a diameter: there are $n/2$ possible positions for a diameter, and for each of these there are $C_{n/2-1}$ triangulations of each of the two resulting $(n/2+1)$-gons. In the summation, $m_{ijk}$ is the number of ways to position a triangle whose sides have cyclic lengths $i, j, k$ inside an $n$-gon (see Figure 1). The conditions under the summation ensure that indeed this is a central triangle. The three cases determining $m_{ijk}$ correspond to the central triangle being equilateral, isosceles or scalene. Each position of the triangle results in an $(i+1)$-gon, a $(j+1)$-gon and a $(k+1)$-gon, and these can be triangulated in $C_{i-1}$, $C_{j-1}$, and $C_{k-1}$ ways, respectively. □

## 2 Congruence relations

Congruence relations of Catalan numbers $C_n$ and related sequences have been the object of extensive study (see [1, 3] and the references therein). We next show that Lemma 1 can be used to derive some results of this nature. Note that the following result can also be proved using (1). In what follows we use the notation

$$t_{ijk} = m_{ijk}C_{i-1}C_{j-1}C_{k-1},$$

with $m_{ijk}$ as in Lemma 1.

**Theorem 2.** *$C_n$ is odd if and only if $n = 2^a - 1$ for some integer $a \geq 0$.*

*Proof.* We use induction on $n$, with the base cases easily verified. The proof will follow from the next observation: the term $t_{ijk}$ (with $i \leq j \leq k$) is odd if and only if $i = 1$ and $j = k = 2^c$ for some $c \geq 1$. To see this, first note that by induction, any term of the form $t_{1,2^c,2^c}$ is odd. Conversely, if $t_{ijk}$ is odd then by induction, $i = 2^b$, $j = 2^c$ and $k = 2^d$ for some $0 \leq b \leq c \leq d$. Since $n = 2^b + 2^c + 2^d$ is odd it follows that $b = 0$, and since $m_{ijk}$ is odd then $c = d$ as claimed.

Now let $n = 2^a + 1$, with $a \geq 1$. On the right hand side of (2) we have $C_{n/2-1} = 0$, and the observation above implies that $t_{ijk}$ is odd for exactly one term of the summation, so $C_{n-2}$ is odd.

Conversely, suppose that $C_{n-2}$ is odd. If $n$ is even then so are the terms $t_{ijk}$, so that $\frac{n}{2}C^2_{n/2-1}$ must be odd. Therefore $n/2$ is odd, and at the same time by induction $n/2 = 2^b$ for some $b \geq 0$. Thus $b = 0$ and $n = 0 = 2^0 - 1$. If $n$ is odd then $C_{n/2-1} = 0$, so at least one of the $t_{ijk}$ must be odd, and by the observation above this implies $n = 1 + 2^c + 2^c = 1 + 2^{c+1}$, completing the proof. □

We next use Lemma 1 to prove a recent result of Eu, Liu and Yeh [4]. Another proof is given by Xin and Xu [8, Theorem 5].

**Theorem 3.** *[4, Theorem 2.3] For all $n \geq 0$,*

$$C_n \equiv_4 \begin{cases} 1, & \text{if } n = 2^a - 1 \text{ for some } a \geq 0; \\ 2, & \text{if } n = 2^a + 2^b - 1 \text{ for some } b > a \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

*Proof.* In what follows, we repeatedly make use without mention of Theorem 2 and of the uniqueness of binary representation. By verifying the base cases we may assume $n \geq 10$ and proceed by induction on $n$. Consider the three cases on the right hand side of (3).

1. The case $n = 2^a - 1$:

   Let $n = 2^a + 1$ with $a \geq 4$, and we show that $C_{n-2} \equiv_4 1$. Reducing (2) modulo 4 gives

$$C_{n-2} \equiv_4 \sum_{\substack{i+j+k=n \\ i \leq j \leq k < n/2}} m_{ijk}C_{i-1}C_{j-1}C_{k-1}. \tag{4}$$

Since $C_{2^{a-1}-1}$ is odd,

$$t_{1,2^{a-1},2^{a-1}} = nC_0C_{2^{a-1}}C_{2^{a-1}} \equiv_4 1, \tag{5}$$

and in fact this is the only term not divisible by 4. To see this, consider the terms for which $t_{ijk} \not\equiv_4 0$.

(a) Suppose $i = j = k$. Then $C_{i-1}$ must be odd, so $i = 2^c$ for some $c \geq 0$. Therefore

$$3(2^c - 1) = i + j + k = n = 2^a + 1,$$

or equivalently, $2^{c+1} + 2^c = 2^a + 2^2$. Therefore $a = 3$, $c = 2$ and $n = 9$, contrary to the assumption that $n \geq 10$.

(b) Suppose $i = j < k$. Then $C_{i-1}$ must be odd, so $i = j = 2^c$ for some $c \geq 0$. Since $C_{k-1} \not\equiv_4 0$, by induction either $k = 2^d$ with $d > c$, or $k = 2^d + 2^e$ with $e > d \geq c$. In the former case, this implies

$$2^a + 1 = 2i + k = 2^{c+1} + 2^d,$$

and in the latter case

$$2^a + 1 = 2^{c+1} + 2^d + 2^e,$$

both cases resulting in a contradiction.

(c) Suppose $i < j = k$. As above, it follows that $j = k = 2^e$ and either $i = 2^c$, with $e > c$, or $i = 2^c + 2^d$, with $e \geq d > c$. In the former case,

$$2^a + 1 = i + 2j = 2^c + 2^{e+1},$$

which implies $c = 0$ and $e = a - 1$, yielding the term given by (5). The latter case again results in a contradiction.

(d) Suppose $i < j < k$. Since $m_{ijk}$ is even, $C_{i-1}C_{j-1}C_{k-1}$ must be odd. Therefore $i = 2^c$, $j = 2^d$ and $k = 2^e$ for some $e > d > c \geq 0$, but then $2^e + 2^d + 2^c = 2^a + 1$, which is impossible.

2. The case $n = 2^a + 2^b - 1$:

Let $n = 2^a + 2^b + 1$, and we show that $C_{n-2} \equiv_4 2$. If $a = 0$, then $n/2 - 1 = 2^{b-1}$, so by induction on $b$ we have $C_{n/2-1} \equiv_4 2$. If $a \geq 1$, then by definition $C_{n/2-1} = 0$. In either case, (4) still holds. Note that $C_{2^0} = 1$ and if $a \geq 1$ then by induction $C_{2^a} \equiv_4 2$. Therefore

$$t_{2^a+1,2^{b-1},2^{b-1}} = nC_{2^a}C_{2^{b-1}-1}C_{2^{b-1}-1} \equiv_4 2. \tag{6}$$

Again, we show that all other $t_{ijk}$ are divisible by 4, by considering the terms for which $t_{ijk} \not\equiv_4 0$.

(a) Suppose $i = j = k$. Then we must have $i = 2^c$ for some $c \geq 0$, but this implies

$$2^{c+1} + 2^c = n = 2^a + 2^b + 1,$$

which is impossible for sufficiently large $n$.

(b) Suppose $i < j < k$. Then $C_{i-1}C_{j-1}C_{k-1}$ is odd, so $i = 2^c$, $j = 2^d$ and $k = 2^e$ for some $e > d > c \geq 0$. Therefore

$$2^c + 2^d + 2^e = n = 2^a + 2^b + 1.$$

Since we may assume $a \geq 1$ (otherwise $2n \equiv_4 0$), it follows that $c = 0$, $d = a$ and $e = b$. But this implies $k \geq n/2$, contrary to the conditions under the summation.

(c) Suppose that $i < j = k$. Since $C_{k-1}$ must be odd, $k = 2^e$ for some $e \geq 0$. Now if $i = 2^d$ for some $d < e$ then

$$2^a + 2^b + 1 = 2^d + 2^{e+1},$$

which is impossible. Therefore by induction we have $i = 2^c + 2^d$, with $e > d > c \geq 0$. This gives

$$2^a + 2^b + 1 = i + 2k = 2^c + 2^d + 2^{e+1},$$

so $c = 0$ and $2^a + 2^b = 2^d + 2^{e+1}$. It follows that $e = b - 1$ and $d = a$, which is the term given by (6). A similar analysis of the case $i = j < k$ results in a contradiction.

3. Otherwise:

We show that $C_{n-2} \equiv_4 0$ unless $n - 2$ has one of the forms above. If $C^2_{n/2-1} \not\equiv_4 0$ then $n/2 = 2^a$ for some $a \geq 0$. For $n$ sufficiently large this implies $8|n$ so that $\frac{n}{2}C^2_{n/2-1} \equiv_4 0$. Next, consider the terms for which $t_{ijk} \not\equiv_4 0$.

(a) Suppose $i = j = k$. Since $C^3_{n/3-1} \not\equiv_4 0$, we must have $n/3 = 2^a$ for some $a \geq 0$. Thus $n = 3 \cdot 2^a$. However, for sufficiently large $n$ this would imply that $m_{ijk} = n/3 \equiv_4 0$.

(b) Suppose $i = j < k$ or $i < j = k$. If $C_{i-1}C_{j-1}C_{k-1}$ is odd then

$$n = 2^{b+1} + 2^c \tag{7}$$

for some $b, c \geq 0$ . Now

$$2^c < n/2 = 2^b + 2^{c-1},$$

so $c - 1 < b$ and in fact $c < b$ (since $c - 1 = b$ would imply $n = 2^c \equiv_4 0$). Therefore (7) implies that $c = 0$ or $c = 1$, so $n - 2 = 2^{b+1} - 1$ or $n - 2 = 2^{b+1} + 2^0 - 1$.

The only case left to check is when $n$ is odd and $C_{i-1}C_{j-1}C_{k-1} \equiv_4 2$. In this case, by induction $n = 2 \cdot 2^b + (2^c + 2^d)$ with $b, c, d \geq 0$ and $d > c$. Since $n$ is odd then $c = 0$, so that $n - 2 = 2^{b+1} + 2^d - 1$.

(c) Suppose $2nC_iC_jC_k \not\equiv_4 0$. Since $C_i$, $C_j$ and $C_k$ are odd, $n = 2^b + 2^c + 2^d$ for some $d > c > b \geq 0$. Since $n$ is odd, $b = 0$ and $n - 2 = 2^c + 2^d - 1$.

□

Another congruence relation follows immediately from reducing (2) modulo a prime $p \geq 5$.

**Theorem 4.** *If $p \geq 5$ is prime and $n \equiv_p -2$ then $C_n \equiv_p 0$.*

## 3 Generalization to $k$-angulations

Lemma 1 can be generalized to give a recursion for the number of *$k$-angulations*, which are partitions of a polygon into $k$-gons. Let $f_{n,k}$ be the number of $k$-angulations of an $n$-gon. It is well known (see, for example, the paper of Przytycki and Sikora [5]) that

$$f_{(k-1)n+2,k+1} = C_{n,k} \tag{8}$$

where

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

are the $k$-Catalan numbers [7, A137211]. Define $f_{n,k} = 0$ unless $n = (k-2)m+2$ for some integer $m \geq 0$. The proof of the following Lemma is completely analogous to that of Lemma 1.

**Lemma 5.** *For any $n \geq 2$ and $k \geq 3$,*

$$f_{n,k} = \frac{n}{2} f^2_{n/2+1,k} + \sum_{\substack{i_1+\ldots+i_k=n \\ i_1 \leq \ldots \leq i_k < n/2}} m_{i_1 \ldots i_k} f_{i_1+1,k} \cdots f_{i_k+1,k}, \tag{9}$$

*where $m_{i_1 \ldots i_k}$ is the number of ways to position a $k$-gon with sides of cyclic lengths $i_1, \ldots, i_k$ inside an $N$-gon for $N = i_1 + \ldots + i_k$.*

For example, let

$$Q_n = \frac{1}{2n+1}\binom{3n}{n}$$

be the number of quadrangulations of a $(2n+2)$-gon, and let $Q_x = 0$ unless $x$ is a nonnegative integer. Then

$$Q_n = (n+1)Q^2_{n/2} + \sum_{\substack{i+j+k+l=2n+2 \\ i \leq j \leq k \leq l < n+1}} m_{ijkl} Q_{(i-1)/2} Q_{(j-1)/2} Q_{(k-1)/2} Q_{(l-1)/2}, \tag{10}$$

where

$$m_{ijkl} = \begin{cases} \frac{N}{4}, & \text{if } i = l; \\ N, & \text{if } i = k < l \text{ or } i < j = l; \\ \frac{3N}{2}, & \text{if } i = j < k = l; \\ 3N, & \text{if } i = j < k < l \text{ or } i < j = k < l \text{ or } i < j < k = l; \\ 6N, & \text{if } i < j < k < l \end{cases} \tag{11}$$

for $N = i + j + k + l$.

Theorem 4 can be generalized by reducing equation (9) modulo a prime $p \geq 3$.

**Theorem 6.** *If $p \geq 3$ is prime with $p \nmid k$ and $p | n$ then $f_{n,k} \equiv_p 0$.*

*Proof.* For a given $k$-gon, the number of cyclic permutations of the $k$ sides that leave the $k$-gon unchanged is divisible by $k$. Therefore the number of inequivalent rotations of the $k$-gon inside the $n$-gon is divisible by $n/k$. It follows that $m_{i_1 \ldots i_k}$ is divisible by $n/k$, and so the given assumptions imply that $p$ divides $f_{n,k}$. □

# 4 Triangulations with a fixed vertex in their central component

L. Shapiro [6] proposed the following question: how many triangulations include the vertex 0 in their central component? The following theorem answers this question.

**Theorem 7.** *Let $n \geq 3$. The number $f(n)$ of triangulations of an $n$-gon with the vertex 0 outside their central component is*

$$f(n) = \frac{1}{2}C_{n-1} - C_{n-2} + \frac{1}{2}C^2_{n/2-1}.$$

*Proof.* Enumerate these triangulations according to the cyclic length $l$ of the shortest diagonal that separates 0 form the center (see Figure 2).

Given such $l$, suppose this diagonal is given by $k(n+k-l)$. Note that $1 \leq k \leq l-1$. Since this is the shortest such diagonal, the triangulation must also include the diagonals $0k$ and $0(n+k-l)$, forming a triangle. The regions outside of this triangle can be triangulated arbitrarily. Therefore

$$\begin{aligned} f(n) &= \sum_{l=2}^{\lfloor n/2 \rfloor} \sum_{k=1}^{l-1} C_{n-l-1}C_{l-k-1}C_{k-1} \\ &= \sum_{l=2}^{\lfloor n/2 \rfloor} C_{n-l-1}C_{l-1} \\ &= \sum_{m=1}^{\lfloor n/2 \rfloor -1} C_m C_{n-2-m} \\ &= \frac{1}{2}\sum_{m=1}^{n-3} C_m C_{n-2-m} + \frac{1}{2}C^2_{n/2-1}, \end{aligned}$$

where the second equality follows from (1). The result now follows by again applying (1). □

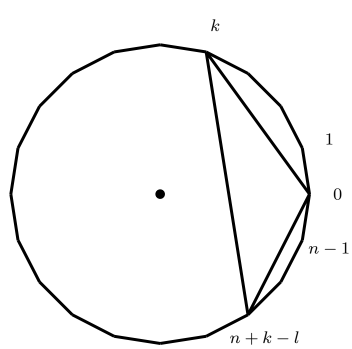


Figure 2: Triangulations with the vertex 0 outside the central component.

It seems that the sequence $a(n) = f(n-3)$ is given in an entry of Sloane's encyclopedia [7, A027302]:

$$a(n) = \sum_{0 \le k < n/2} T(n,k)T(n,k+1),$$

where

$$T(n,k) = \frac{n-2k+1}{n-k+1}\binom{n}{k}.$$

In this entry it also asserted that $a(n)$ is the number of Dyck $(n+2)$-paths with $UU$ spanning their midpoint. It would be interesting to determine whether any known bijection between triangulations and Dyck paths gives this correspondence.

# References


[1] R. Alter and K. Kubota, Prime and prime power divisibility of Catalan numbers, *J. Combin. Theory Ser. A* **15** (1973), 243–256.

[2] D. Bowman and A. Regev, Counting symmetry classes of dissections of a convex regular polygon, preprint, `http://arxiv.org/abs/1209.6270`.

[3] E. Deutsch and B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Number Theory* **117** (2006), 191–215.

[4] S. Eu, S. Liu, and Y. Yeh, Catalan and Motzkin numbers modulo 4 and 8, *European J. Combin.* **29** (2008), 1449–1466.

[5] J. H. Przytycki and A. Sikora, Polygon dissections and Euler, Fuss, Kirkman and Cayley numbers, *J. Comb. Theory A* **92** (1) (2000), 68–76.

[6] L. Shapiro, Private communication, 2012.

[7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[8] G. Xin and J. Xu, A short approach to Catalan numbers modulo $2^r$, *Electron. J. Combin.* **18** (2011), P177.


---


*2010 Mathematics Subject Classification:* Primary 05A15; Secondary 05A10.
*Keywords:* Catalan numbers, polygon triangulation, polygon dissection, congruence.


---

(Concerned with sequences A000108, A137211, and A027302.)

---


Received October 21 2012; revised version received March 1 2013. Published in *Journal of Integer Sequences*, March 9 2013.


---


Return to Journal of Integer Sequences home page.