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Expansion of Fibonacci and Lucas Polynomials: An Answer to Prodinger's Question

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Abstract

We give an answer to a recent question of Prodinger, which consists of finding q-analogues of identities related to Fibonacci and Lucas polynomials.

1 Introduction

The bivariate polynomials of Fibonacci and Lucas, denoted respectively by (U_n) and (V_n) , are defined by

$$\begin{cases} U_0 = 0, \ U_1 = 1, \\ U_n = tU_{n-1} + zU_{n-2} \ (n \ge 2), \end{cases} \text{ and } \begin{cases} V_0 = 2, \ V_1 = t, \\ V_n = tV_{n-1} + zV_{n-2} \ (n \ge 2). \end{cases}$$

It is well-known (see, for example, [3]) that

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} t^{n-2k} z^k, \qquad V_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} t^{n-2k} z^k \ (n \ge 1).$$

In [1], Belbachir and Bencherif proved the following five formulae:

$$2U_{2n+1} = \sum_{k=0}^{n} a_{n,k} t^k V_{2n-k}, \text{ with } a_{n,k} = 2\sum_{j=0}^{n} (-1)^{j+k} \binom{j}{k} - (-1)^{n+k} \binom{n}{k}, \tag{1}$$

$$V_{2n} = \sum_{k=1}^{n} b_{n,k} t^k V_{2n-k}, \text{ with } b_{n,k} = (-1)^{k+1} \binom{n}{k},$$
(2)

$$V_{2n-1} = \sum_{k=1}^{n} c_{n,k} t^{k} U_{2n-k}, \text{ with } c_{n,k} = 2 \left(-1\right)^{k+1} \binom{n}{k} - \left[k=1\right],$$
(3)

$$2V_{2n-1} = \sum_{\substack{k=1\\n}}^{n} d_{n,k} t^k V_{2n-1-k}, \text{ with } d_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k},$$
(4)

$$2U_{2n} = \sum_{k=1}^{n} e_{n,k} t^k V_{2n-k}, \quad \text{with}$$
(5)

$$e_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k} + \sum_{j=0}^{n-1} (-1)^{j+k-1} \binom{j}{k-1} - \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1}$$

and they also gave the following simple sixth one:

$$V_{2n} = 2U_{2n+1} - xU_{2n}. (6)$$

The results presented in [1] are companion generalizations for Chebyshev polynomials [2].

Without loss of generality, we suppose that t = 1. We refer here to the modified polynomials given by Prodinger in the introduction of [5].

In order to give q-analogues of these instances, Prodinger considered Cigler's [4] suggestion to replace U_n and V_n , respectively, with

$$\mathbf{F}_{n+1}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} {\binom{n-k}{k}}_q z^k, \quad \mathbf{L}_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} {\binom{n-k}{k}}_q \frac{[n]_q}{[n-k]_q} z^k,$$

with the following q-notations

$$[n]_q = 1 + q + \dots + q^{n-1}, \quad [n]_q! = [1]_q[2]_q \dots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Prodinger [5] found the following two q-analogues for identities (4) and (6) respectively

$$\mathbf{F}_{2n}(z) = \sum_{k=1}^{n} \beta_{n,k} \mathbf{F}_{2n-k}(z), \text{ with } \beta_{n,k} = (-1)^{k+1} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q},$$
(7)

$$\mathbf{L}_{2n-1} = \sum_{k=1}^{n} \delta_{n,k} \mathbf{L}_{2n-1-k} \left(z \right), \quad \text{with} \quad \delta_{n,k} = (-1)^{k+1} \frac{q^{\binom{k}{2}}}{1+q^{n-1}} \left(\begin{bmatrix} n-1\\k \end{bmatrix}_q + q^{n-1} \begin{bmatrix} n\\k \end{bmatrix}_q \right),$$
(8)

and left the three others as a challenge.

We give an answer to the remaining three instances and we find a q-analogue for identity (6).

2 An answer to Prodinger's question

Here, we find a q-analogue for the identities (6), (3), (1) and (5), which can be considered as an answer to Prodinger's question [5].

Theorem 1. The following q-identities hold

$$\mathbf{L}_{2n}\left(z\right) = \mathbf{F}_{2n+1}\left(\frac{z}{q}\right) + \mathbf{F}_{2n+1}\left(z\right) - \mathbf{F}_{2n}\left(z\right),\tag{9}$$

$$\mathbf{L}_{2n-1}(z) = \sum_{k=1}^{n} \beta_{n,k} \left(\mathbf{F}_{2n-k}(z) + \mathbf{F}_{2n-k}\left(\frac{z}{q}\right) \right) - \mathbf{F}_{2n-1}(z), \qquad (10)$$

$$\mathbf{F}_{2n+1}(z) = \frac{1}{1 - q^{2n}} \left(\mathbf{L}_{2n}(qz) - q^{2n} \mathbf{L}_{2n}(z) \right),$$
(11)

$$\mathbf{F}_{2n}(z) = \frac{1}{1 - q^{2n-1}} \left(\mathbf{L}_{2n-1}(qz) - q^{2n-1} \mathbf{L}_{2n-1}(z) \right).$$
(12)

Proof. We have

$$\begin{split} \mathbf{F}_{n+1}\left(\frac{z}{q}\right) + \mathbf{F}_{n+1}\left(z\right) - \mathbf{F}_{n}\left(z\right) &= \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{q} \left(1 + q^{k} - q^{k} \frac{[n-2k]_{q}}{[n-k]_{q}}\right) z^{k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{q} \left(1 + q^{k} - q^{k} \frac{1-q^{n-2k}}{1-q^{n-k}}\right) z^{k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{q} \frac{1-q^{n}}{1-q^{n-k}} z^{k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{q} \frac{[n]_{q}}{[n-k]_{q}} z^{k} = \mathbf{L}_{n}\left(z\right). \end{split}$$

For the second identity $\sum_{k=1}^{n} \beta_{n,k} \left(\mathbf{F}_{2n-k}(z) + \mathbf{F}_{2n-k}\left(\frac{z}{q}\right) \right) - \mathbf{F}_{2n-1}(z) = \mathbf{F}_{2n}(z) + \mathbf{F}_{2n}\left(\frac{z}{q}\right) - \mathbf{F}_{2n-1}(z) = \mathbf{L}_{2n-1}(z).$

To establish (11) and (12), we have

$$\mathbf{L}_{n}(qz) - q^{n}\mathbf{L}_{n}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}} {\binom{n-k}{k}}_{q} \frac{[n]_{q}}{[n-k]_{q}} (q^{k}-q^{n}) z^{k}$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}} {\binom{n-k}{k}}_{q} (1-q^{n}) z^{k}$$
$$= (1-q^{n}) \mathbf{F}_{n+1}(z),$$

and then take n odd and n even respectively.

Remark 2. We have to notice that the expressions of relations (11) and (12) do not give, for q = 1, the initial formulae as given in relations (1) and (5). We obtain, when $q \to 1$, the well known identity $U_{n+1}(z) = V_n(z) - zU_{n-1}(z)$, for n odd and n even respectively. However, they express, as in Belbachir's and Bencherif's work [1], an odd Fibonacci polynomial in terms of Lucas polynomials starting with a smallest even index, and an even Fibonacci polynomial in terms of Lucas polynomials starting with a smallest odd index.

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