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Sum Relations for Lucas Sequences[1]

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Abstract

In this paper, we establish four sum relations for Lucas sequences. As applications, we derive some combinatorial identities involving Lucas sequences that extend some known results.

1 Introduction

Given two integer parameters P and Q, the Lucas sequences of the first kind $U_n = U_n(P, Q)$ $(n \in \mathbb{N})$ and of the second kind $V_n = V_n(P, Q)$ $(n \in \mathbb{N})$ are defined by the recurrence relations

$$U_0 = 0, \quad U_1 = 1, \quad \text{and} \quad U_n = PU_{n-1} - QU_{n-2} \ (n \geq 2), \tag{1}$$

$$V_0 = 2, \quad V_1 = p, \quad \text{and} \quad V_n = PV_{n-1} - QV_{n-2} \ (n \geq 2). \tag{2}$$

The characteristic equation $x^2 - Px + Q = 0$ of the sequences U_n and V_n has two roots $\alpha = (P + \sqrt{D})/2$ and $\beta = (P - \sqrt{D})/2$ with the discriminant $D = P^2 - 4Q$. Note that $D^{1/2} = \alpha - \beta$. Furthermore, $D = 0$ means $x^2 - Px + Q = 0$ has the repeated root $\alpha = \beta = P/2$. It is well known that for any $n \in \mathbb{N}$ (see [4, pp. 41–44]),

$$PU_n + V_n = 2U_{n+1}, \quad (\alpha - \beta)U_n = \alpha^n - \beta^n, \quad V_n = \alpha^n + \beta^n. \tag{3}$$

The Lucas sequences U_n and V_n can be regarded as the generalization of many integer sequences, for example, F_n, L_n, P_n, Q_n, J_n, and j_n, known as the Fibonacci, Lucas, Pell,

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Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas numbers, according to whether $P=1, Q=-1$, $P=2, Q=-1$, or $P=1, Q=-2$; see [5] for a good introduction. These numbers play important roles in many different areas of mathematics, so their numerous elegant properties have been studied by many authors, see for example, [3, 4].

The idea of the present paper stems from the familiar combinatorial theorem about sets called the principle of cross-classification. We establish four sum relations for the Lucas sequences as follows.

Theorem 1. *Let n be a positive integer, and let $s_1, s_2, \ldots, s_n$ be any non-negative integers. Then*

$$-\sum_{1\le i\le n}\frac{U_{s_i}}{\alpha^{s_i}}+\sum_{1\le i<j\le n}\frac{U_{s_i+s_j}}{\alpha^{s_i+s_j}}-\sum_{1\le i<j<k\le n}\frac{U_{s_i+s_j+s_k}}{\alpha^{s_i+s_j+s_k}}+\cdots+(-1)^n\frac{U_{s_1+s_2+\cdots+s_n}}{\alpha^{s_1+s_2+\cdots+s_n}}$$
$$=-D^{\frac{n-1}{2}}\frac{U_{s_1}U_{s_2}\cdots U_{s_n}}{\alpha^{s_1+s_2+\cdots+s_n}}, \tag{4}$$

$$-\sum_{1\le i\le n}\frac{U_{s_i}}{\beta^{s_i}}+\sum_{1\le i<j\le n}\frac{U_{s_i+s_j}}{\beta^{s_i+s_j}}-\sum_{1\le i<j<k\le n}\frac{U_{s_i+s_j+s_k}}{\beta^{s_i+s_j+s_k}}+\cdots+(-1)^n\frac{U_{s_1+s_2+\cdots+s_n}}{\beta^{s_1+s_2+\cdots+s_n}}$$
$$=(-1)^nD^{\frac{n-1}{2}}\frac{U_{s_1}U_{s_2}\cdots U_{s_n}}{\beta^{s_1+s_2+\cdots+s_n}}, \tag{5}$$

$$\sum_{1\le i\le n}\frac{V_{s_i}}{\alpha^{s_i}}+\sum_{1\le i<j\le n}\frac{V_{s_i+s_j}}{\alpha^{s_i+s_j}}+\sum_{1\le i<j<k\le n}\frac{V_{s_i+s_j+s_k}}{\alpha^{s_i+s_j+s_k}}+\cdots+\frac{V_{s_1+s_2+\cdots+s_n}}{\alpha^{s_1+s_2+\cdots+s_n}}$$
$$=2^n-2+\frac{V_{s_1}V_{s_2}\cdots V_{s_n}}{\alpha^{s_1+s_2+\cdots+s_n}}, \tag{6}$$

$$\sum_{1\le i\le n}\frac{V_{s_i}}{\beta^{s_i}}+\sum_{1\le i<j\le n}\frac{V_{s_i+s_j}}{\beta^{s_i+s_j}}+\sum_{1\le i<j<k\le n}\frac{V_{s_i+s_j+s_k}}{\beta^{s_i+s_j+s_k}}+\cdots+\frac{V_{s_1+s_2+\cdots+s_n}}{\beta^{s_1+s_2+\cdots+s_n}}$$
$$=2^n-2+\frac{V_{s_1}V_{s_2}\cdots V_{s_n}}{\beta^{s_1+s_2+\cdots+s_n}}. \tag{7}$$

Theorem 1 has some applications and can be deduced as the generalization of some known results. In section 2, we shall make use of Theorem 1 to illustrate its effectiveness. In section 3, we shall give the proof of Theorem 1.

2 Some applications of Theorem 1

Theorem 2. *Let n be a positive integer, and let $C_n=U_n/Q^n$, $D_n=V_n/Q^n$, $E_n=U_n^2/Q^n$, $F_n=U_nV_n/Q^n$, $G_n=V_n^2/Q^n$. Suppose that the discriminant D is not equal to 0. Then, for non-negative integers $s_1, s_2, \ldots, s_n$,*

$$-\sum_{1\le i\le n}E_{s_i}+\sum_{1\le i<j\le n}E_{s_i+s_j}-\sum_{1\le i<j<k\le n}E_{s_i+s_j+s_k}+\cdots+(-1)^nE_{s_1+s_2+\cdots+s_n}$$
$$=\begin{cases}D^{\frac{n-2}{2}}D_{s_1+s_2+\cdots+s_n}U_{s_1}U_{s_2}\cdots U_{s_n}, & 2\mid n,\\ -D^{\frac{n-1}{2}}C_{s_1+s_2+\cdots+s_n}U_{s_1}U_{s_2}\cdots U_{s_n}, & 2\nmid n,\end{cases} \tag{8}$$

$$
\begin{aligned}
&- \sum_{1\le i\le n} F_{s_i} + \sum_{1\le i<j\le n} F_{s_i+s_j} - \sum_{1\le i<j<k\le n} F_{s_i+s_j+s_k} + \cdots + (-1)^n F_{s_1+s_2+\cdots+s_n} \\
&= \begin{cases} D^{\frac{n}{2}} C_{s_1+s_2+\cdots+s_n} U_{s_1} U_{s_2} \cdots U_{s_n}, & 2 \mid n, \\ -D^{\frac{n-1}{2}} D_{s_1+s_2+\cdots+s_n} U_{s_1} U_{s_2} \cdots U_{s_n}, & 2 \nmid n, \end{cases}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&\sum_{1\le i\le n} F_{s_i} + \sum_{1\le i<j\le n} F_{s_i+s_j} + \sum_{1\le i<j<k\le n} F_{s_i+s_j+s_k} + \cdots + F_{s_1+s_2+\cdots+s_n} \\
&= C_{s_1+s_2+\cdots+s_n} V_{s_1} V_{s_2} \cdots V_{s_n},
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\sum_{1\le i\le n} G_{s_i} + \sum_{1\le i<j\le n} G_{s_i+s_j} + \sum_{1\le i<j<k\le n} G_{s_i+s_j+s_k} + \cdots + G_{s_1+s_2+\cdots+s_n} \\
&= 2^{n+1} - 4 + D_{s_1+s_2+\cdots+s_n} V_{s_1} V_{s_2} \cdots V_{s_n}.
\end{aligned} \tag{11}
$$

Proof. Adding and subtracting (4) and (5), and (6) and (7), respectively, we are done by applying the last two identities of (3). □

Corollary 3. *Let n be a positive integer, and let k be a non-negative integer. Suppose that the discriminant D is not equal to 0. Then*

$$\sum_{i=0}^{n} \binom{n}{i} Q^{(n-i)k} U_{ik} V_{ik} = U_{kn} V_k^n, \tag{12}$$

$$\sum_{i=0}^{n} \binom{n}{i} Q^{(n-i)k} V_{ik}^2 = 2^{n+1} Q^{nk} + V_{kn} V_k^n, \tag{13}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i Q^{(n-i)k} U_{ik}^2 = \begin{cases} D^{\frac{n-2}{2}} V_{kn} U_k^n, & 2 \mid n, \\ -D^{\frac{n-1}{2}} U_{kn} U_k^n, & 2 \nmid n, \end{cases} \tag{14}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i Q^{(n-i)k} U_{ik} V_{ik} = \begin{cases} D^{\frac{n}{2}} U_{kn} U_k^n, & 2 \mid n, \\ -D^{\frac{n-1}{2}} V_{kn} U_k^n, & 2 \nmid n. \end{cases} \tag{15}$$

Proof. Setting $s_1 = s_2 = \cdots = s_n = k$ in Theorem 2, the desired results follow. □

Remark 4. By the last two identities of (3), one can easily check that if the discriminant $D \neq 0$ then $U_n V_n = U_{2n}$, $V_n^2 = V_{2n} + 2Q^n$, $U_n^2 = (V_{2n} - 2Q^n)/D$, which together with Corollary 3 deduce some interesting results. The case k being an even number in (12) gives a sophisticated identity for Fibonacci and Lucas numbers which was asked by Hoggatt as an advanced problem in [1]. The case $k = 1$ in (14) and (15) give the familiar combinatorial identities for Fibonacci and Lucas numbers, see for example, [2, 6].

Theorem 5. *Let n be a positive integer, let $s_1, s_2, \ldots, s_n$ be any non-negative integers such that $s_1 + s_2 + \cdots + s_n = s$, and let $C_{m,n} = U_m U_{n-m}$, $D_{m,n} = U_m V_{n-m}$, $E_{m,n} = V_m U_{n-m}$, $F_{m,n} = V_m V_{n-m}$. Suppose that the discriminant D is not equal to 0. Then*

$$\begin{aligned} &-\sum_{1\le i\le n} C_{s_i,s} + \sum_{1\le i<j\le n} C_{s_i+s_j,s} - \sum_{1\le i<j<k\le n} C_{s_i+s_j+s_k,s} + \cdots + (-1)^n C_{s,s} \\ &= \begin{cases} -2D^{\frac{n-2}{2}} U_{s_1} U_{s_2} \cdots U_{s_n}, & 2 \mid n, \\ 0, & 2 \nmid n, \end{cases} \end{aligned} \tag{16}$$

$$\begin{aligned} &-\sum_{1\le i\le n} D_{s_i,s} + \sum_{1\le i<j\le n} D_{s_i+s_j,s} - \sum_{1\le i<j<k\le n} D_{s_i+s_j+s_k,s} + \cdots + (-1)^n D_{s,s} \\ &= \begin{cases} 0, & 2 \mid n, \\ -2D^{\frac{n-1}{2}} U_{s_1} U_{s_2} \cdots U_{s_n}, & 2 \nmid n, \end{cases} \end{aligned} \tag{17}$$

$$\begin{aligned} &\sum_{1\le i\le n} E_{s_i,s} + \sum_{1\le i<j\le n} E_{s_i+s_j,s} + \sum_{1\le i<j<k\le n} E_{s_i+s_j+s_k,s} + \cdots + E_{s,s} \\ &= (2^n - 2) U_s, \end{aligned} \tag{18}$$

$$\begin{aligned} &\sum_{1\le i\le n} F_{s_i,s} + \sum_{1\le i<j\le n} F_{s_i+s_j,s} + \sum_{1\le i<j<k\le n} F_{s_i+s_j+s_k,s} + \cdots + F_{s,s} \\ &= (2^n - 2) V_s + 2 V_{s_1} V_{s_2} \cdots V_{s_n}. \end{aligned} \tag{19}$$

Proof. Multiplying α^s in the both sides of (4) and (6), β^s in the both sides of (5) and (7), and then adding and subtracting (4) and (5), and (6) and (7), respectively, the desired results immediately follow by applying the last two identities of (3). □

Corollary 6. *Let n be a positive integer, and let k be a non-negative integer. Suppose that the discriminant D is not equal to 0. Then*

$$\sum_{i=0}^{n} \binom{n}{i} Q^{ik} U_{(n-2i)k} = 0, \tag{20}$$

$$\sum_{i=0}^{n} \binom{n}{i} Q^{ik} V_{(n-2i)k} = 2V_k^n, \tag{21}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i Q^{ik} V_{(n-2i)k} = \begin{cases} 2D^{\frac{n}{2}} U_k^n, & 2 \mid n, \\ 0, & 2 \nmid n, \end{cases} \tag{22}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i Q^{ik} U_{(n-2i)k} = \begin{cases} 0, & 2 \mid n, \\ 2D^{\frac{n-1}{2}} U_k^n, & 2 \nmid n. \end{cases} \tag{23}$$

Proof. By the last two identities of (3), one can easily check that if the discriminant $D \neq 0$ then $V_{ik}U_{(n-i)k} = U_{kn} + Q^{ik}U_{(n-2i)k}$, $V_{ik}V_{(n-i)k} = V_{nk} + Q^{ik}V_{(n-2i)k}$, $U_{ik}U_{(n-i)k} = (V_{nk} - Q^{ik}V_{(n-2i)k})/D$, $U_{ik}V_{(n-i)k} = U_{nk} - Q^{ik}U_{(n-2i)k}$. Thus, by setting $s_1 = s_2 = \cdots = s_n = k$ in Theorem 5, Corollary 6 follows immediately. □

Remark 7. The equations (21)–(23) extend the powers of Fibonacci and Lucas numbers as sums, see for example, [2, 6].

3 The proof of Theorem 1

The proof of Theorem 1. Clearly, the case $n = 1$ in Theorem 1 is complete.

Now, we consider the case $n \geq 2$. By (3), it is easy to see that if the discriminant $D \neq 0$ then $U_n = n\alpha^{n-1} = n\beta^{n-1}$, $V_n = 2\alpha^n = 2\beta^n$ for all $n \in \mathbb{N}$. Thus, in view of the fact $\sum_{i=1}^{n} \binom{n}{i} = 2^n - 1$ and for any integer $n \geq 2$,

$$-\sum_{1\leq i\leq n} s_i + \sum_{1\leq i<j\leq n} (s_i + s_j) - \sum_{1\leq i<j<k\leq n} (s_i + s_j + s_k) + \cdots + (-1)^n(s_1 + s_2 + \cdots + s_n) = 0,$$

Theorem 1 is complete when the discriminant $D = 0$. Next, we use induction on n to consider the discriminant $D \neq 0$. Applying the last two identities of (3), we derive

$$U_m U_n = -\frac{U_{m+n} - \alpha^m U_n - \alpha^n U_m}{\sqrt{D}}, \tag{24}$$

$$U_m U_n = \frac{U_{m+n} - \beta^m U_n - \beta^n U_m}{\sqrt{D}}, \tag{25}$$

$$V_m V_n = V_{m+n} + \alpha^m V_n + \alpha^n V_m - 2\alpha^{m+n}, \tag{26}$$

$$V_m V_n = V_{m+n} + \beta^m V_n + \beta^n V_m - 2\beta^{m+n}, \tag{27}$$

which imply the case $n = 2$ in Theorem 1 is complete. Assume that (4) holds for all $2 \leq n = m$. Then, by multiplying $U_{s_{m+1}}/\alpha^{s_{m+1}}$ in the both sides of (4), and applying (24), we obtain

$$\begin{aligned}
&-D^{\frac{m}{2}}\frac{U_{s_1}U_{s_2}\cdots U_{s_m}U_{s_{m+1}}}{\alpha^{s_1+s_2+\cdots+s_m+s_{m+1}}} \\
&= \sum_{1\leq i\leq m}\left(\frac{U_{s_i+s_{m+1}}}{\alpha^{s_i+s_{m+1}}} - \frac{U_{s_i}}{\alpha^{s_i}} - \frac{U_{s_{m+1}}}{\alpha^{s_{m+1}}}\right) \\
&\quad - \sum_{1\leq i<j\leq m}\left(\frac{U_{s_i+s_j+s_{m+1}}}{\alpha^{s_i+s_j+s_{m+1}}} - \frac{U_{s_i+s_j}}{\alpha^{s_i+s_j}} - \frac{U_{s_{m+1}}}{\alpha^{s_{m+1}}}\right) \\
&\quad + \sum_{1\leq i<j<k\leq m}\left(\frac{U_{s_i+s_j+s_k+s_{m+1}}}{\alpha^{s_i+s_j+s_k+s_{m+1}}} - \frac{U_{s_i+s_j+s_k}}{\alpha^{s_i+s_j+s_k}} - \frac{U_{s_{m+1}}}{\alpha^{s_{m+1}}}\right) \\
&\quad - \cdots - (-1)^m\left(\frac{U_{s_1+s_2+\cdots+s_m+s_{m+1}}}{\alpha^{s_1+s_2+\cdots+s_m+s_{m+1}}} - \frac{U_{s_1+s_2+\cdots+s_m}}{\alpha^{s_1+s_2+\cdots+s_m}} - \frac{U_{s_{m+1}}}{\alpha^{s_{m+1}}}\right),
\end{aligned}$$

which together with $\sum_{i=1}^{m} \binom{m}{i}(-1)^i = -1$ means (4) holds for all $n = m + 1$. In a similar consideration, (5), (6) and (7) hold for all $n = m + 1$. This concludes the induction step. We are done. $\square$

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