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Semihappy Numbers

H. G. Grundman
Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010
USA
hgrundma@brynmawr.edu

Abstract

We generalize the concept of happy number as follows. Let $\mathbf{e} = (e_0, e_1, ...)$ be a sequence with $e_0 = 2$ and $e_i = \{1, 2\}$ for $i > 0$. Define $S_{\mathbf{e}} : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$S_{\mathbf{e}}\left(\sum_{i=0}^{n} a_i 10^i\right) = \sum_{i=0}^{n} a_i^{e_i}.$$

If $S_{\mathbf{e}}^k(a) = 1$ for some $k \in \mathbb{Z}^+$, then we say that a is a *semihappy number* or, more precisely, an $\mathbf{e}$-*semihappy number*. In this paper, we determine fixed points and cycles of the functions $S_{\mathbf{e}}$ and discuss heights of semihappy numbers. We also prove that for each choice of $\mathbf{e}$, there exist arbitrarily long finite sequences of consecutive $\mathbf{e}$-semihappy numbers.

1 Introduction

Let $S_2 : \mathbb{Z}^+ \to \mathbb{Z}^+$ denote the function that takes a positive integer to the sum of the squares of its decimal digits. A *happy number* is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$. (See sequence A007770 in Sloane [5].) In [2], happy numbers were generalized as follows: For $e \geq 2$, $b \geq 2$, and $0 \leq a_i < b$, define $S_{e,b} : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

If $S_{e,b}^m(a) = 1$ for some $m \geq 0$, then a is an *e-power b-happy number.*

Here we generalize the concept of happy numbers in a different direction. Rather than summing the squares of the digits, we take a sum, squaring only some of the digits, according to a preset pattern. More precisely, let $\mathbf{e} = (e_0, e_1, ...)$ be a sequence with $e_0 = 2$ and $e_i \in \{1, 2\}$, for $i > 0$. Define $S_{\mathbf{e}} : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$S_{\mathbf{e}}\left(\sum_{i=0}^{n} a_i 10^i\right) = \sum_{i=0}^{n} a_i^{e_i}.$$

If $S_{\mathbf{e}}^k(a) = 1$ for some $k \in \mathbb{Z}^+$, then we say that a is a *semihappy number* or, more precisely, an $\mathbf{e}$-*semihappy number*.

Although the functions $S_{\mathbf{e}}$ depend heavily on the value $\mathbf{e}$, we can still prove a number of results about semihappy numbers in general.

In Section 2, we find all fixed points and cycles for the functions $S_{\mathbf{e}}$ and the smallest semihappy numbers of various heights. Then in Section 3, we prove that for each choice of $\mathbf{e}$, there exist arbitrarily long finite sequences of $\mathbf{e}$-semihappy numbers.

2 Fixed Points, Cycles, and Heights

Theorem 1 gives the fixed points and cycles of the function $S_{\mathbf{e}}$, for each $\mathbf{e}$. The proof follows standard techniques. We first show that for each $a \geq 100$, $S_{\mathbf{e}}(a) < a$ and so, for each $a \in \mathbb{Z}^+$, there exists some $m \in \mathbb{Z}^+$ such that $S_{\mathbf{e}}^m(a) < 100$. The cycles can then be determined by applying powers of $S_{\mathbf{e}}$ to each $a < 100$. Note that the second part of the theorem generalizes the well known result for the function used in defining happy numbers.

Theorem 1. *Given* $\mathbf{e}$ *with* $e_1 = 1$, *the function* $S_{\mathbf{e}}$ *has two fixed points,* 1 *and* 89, *and one nontrivial cycle,* $9 \mapsto 81 \mapsto 9$.

Given $\mathbf{e}$ *with* $e_1 = 2$, *the function* $S_{\mathbf{e}}$ *has one fixed point,* 1, *and one nontrivial cycle,* $4 \mapsto 16 \mapsto 37 \mapsto 58 \mapsto 89 \mapsto 145 \mapsto 42 \mapsto 20 \mapsto 4$.

Proof. Let $a \geq 100$. Then we have $a = \sum_{i=0}^{n} a_i 10^i$, with $n \geq 2$, $a_n \neq 0$, and $0 \leq a_i \leq 9$ for $0 \leq i < n$. So

$$\begin{aligned} a - S_{\mathbf{e}}(a) &= \sum_{i=0}^{n} a_i 10^i - \sum_{i=0}^{n} a_i^{e_i} \\ &= \sum a_i(10^i - a_i^{e_i - 1}) \\ &\geq \sum a_i(10^i - a_i) \\ &\geq a_n(10^n - a_n) + a_0(1 - a_0) \\ &\geq 99 - 72 > 0 \end{aligned}$$

So if $a \geq 100$, $a > S_{\mathbf{e}}(a)$. Since the equality is strict, this implies that for each $a \in \mathbb{Z}^+$, there exists some $m \in \mathbb{Z}^+$ such that $S_{\mathbf{e}}^m(a) < 100$. A direct calculation for each $a < 100$ completes the proof. $\square$

Generalizing the definition of the height of a happy number, the $\mathbf{e}$-*height* of an $\mathbf{e}$-semihappy number, a, is the number of iterations of $S_{\mathbf{e}}$ needed to reach 1:

$$h_{\mathbf{e}}(a) = \min\{k \in \mathbb{Z}^+ \mid S_{\mathbf{e}}^k(a) = 1\}.$$

For example, for $\mathbf{e}$ with $e_1 = 1$, $S_{\mathbf{e}}(92) = 13$, $S_{\mathbf{e}}^2(92) = 10$, and $S_{\mathbf{e}}^3(92) = 1$. So the $\mathbf{e}$-height of 92 is 3.

Using a simple computer search, we determined the least $\mathbf{e}$-semihappy numbers of given heights for each possible $\mathbf{e}$. The results are given in Table 1.

Table 1: Least $\mathbf{e}$-semihappy numbers of given $\mathbf{e}$-heights

Height	$e_1 = 1$	$e_1 = 2$
0	1	1
1	10	10
2	13	13
3	43	23
4	76	19
5	288 or 398	7

The *global height* of a semihappy number, a, is

$$h(a) = \min_{\mathbf{e}} \{k \in \mathbb{Z}^+ \mid S_{\mathbf{e}}^k(a) = 1\}.$$

In Table 2, we list the smallest numbers of each global height less than nine, along with the value of $\mathbf{e}$ at which the minimum is achieved. (In the table, trailing dots in the definition of $\mathbf{e}$ indicates that every value of $\mathbf{e}$ matching what is given up to that point works.) The results through height seven are easily verified with a computer search. We prove the result for height eight below.

Table 2: Least numbers of given global heights

Height	least number	$\mathbf{e}$
0	1	$(2, ...)$
1	10	$(2, ...)$
2	13	$(2, ...)$
3	23	$(2, 2, ...)$
4	19	$(2, 2, ...)$
5	7	$(2, 2, ...)$
6	212	$(2, 2, 1...)$
7	7199	$(2, 2, 1, 2...)$
8	$8\underbrace{99\ldots99}_{86}799$	$\mathbf{e} = (2, 2, 1, \underbrace{2, 2\ldots, 2, 2}_{87}, \ldots)$

Theorem 2. *The least semihappy number of height 8 is*

$$A = 8\underbrace{99\ldots99}_{86}799.$$

Proof. Suppose that $a \le A$ and has height 8. Since $a \le A$, either a has exactly 89 9's and at most one digit less than 8, so that $S_{\mathbf{e}}(a) \le 7^2 + 89 \cdot 9^2 = 7258$, or a has at most 90 digits and fewer than 89 9's, so that $S_{\mathbf{e}}(a) \le 2 \cdot 8^2 + 88 \cdot 9^2 = 7256$. Since a has height 8, $S_{\mathbf{e}}(a)$ has height 7 and thus $S_{\mathbf{e}}(a) \ge 7199$. But if $7200 \le S_{\mathbf{e}}(a) \le 7258$, then $S^2_{\mathbf{e}}(a) \le 7^2+2^2+5^2+9^2 = 159 < 212$, and so $S^2_{\mathbf{e}}(a)$ is not of height 6, which is a contradiction.

Thus $S_{\mathbf{e}}(a) = 7199$ and $\mathbf{e} = (2, 2, 1, 2...)$. Since at least one digit is not being squared, a direct calculation shows that a must have exactly 88 9's, one 8, and one 7, the 7 being the only digit that is not being squared. The smallest possible such number is A with $\mathbf{e} = (2, 2, 1, \underbrace{2, 2 \dots, 2, 2}_{87}, \dots)$. □

3 Consecutive Semihappy Numbers

In this section we prove that for each $\mathbf{e}$, there exist arbitrarily long finite sequences of consecutive $\mathbf{e}$-semihappy numbers. This result was proved in the special case of happy numbers by El-Sedy and Siksek [1]. (See [3], [4] and [6] for alternative proofs and proofs of similar results for generalized happy numbers.) Before stating and proving the main theorem, we need some definitions and lemmas, which parallel closely ones found in [3].

Let $I : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $I(n) = n + 1$. For a fixed $\mathbf{e}$, we say that a set D of integers is $\mathbf{e}$-*good,* if there exist $n, k \in \mathbb{Z}^+$ such that for each $d \in D$, $S^k_{\mathbf{e}}(d + n) = 1$.

Lemma 3. *Fix* $\mathbf{e}$ *and a finite set* D. *Let* $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ *be the composition of a finite sequence of* $S_{\mathbf{e}}$ *and* I. *If the set* $F(D)$ *is* $\mathbf{e}$-*good, then* D *is* $\mathbf{e}$-*good.*

Proof. It follows immediately from the definition of $\mathbf{e}$-good that if $I(D)$ is $\mathbf{e}$-good, then D is $\mathbf{e}$-good. Using induction on the length of the sequence of functions, it suffices to show that if $S_{\mathbf{e}}(D)$ is $\mathbf{e}$-good, then D is $\mathbf{e}$-good.

Suppose that $S_{\mathbf{e}}(D)$ is $\mathbf{e}$-good. Then there exist n' and k' such that for all $s \in S_{\mathbf{e}}(D)$, $S^{k'}_{\mathbf{e}}(s + n') = 1$. Let

$$n = \underbrace{11 \dots 11}_{n'} 00 \dots 00,$$

where the number of zeros is the number of digits of the largest element of D. Then $S_{\mathbf{e}}(n) = n'$ and for each $d \in D$, $S_{\mathbf{e}}(d + n) = S_{\mathbf{e}}(d) + n'$. Let $k = k' + 1$. Then for all $d \in D$,

$$S^k_{\mathbf{e}}(d + n) = S^{k'}_{\mathbf{e}}(S_{\mathbf{e}}(d + n)) = S^{k'}_{\mathbf{e}}(S_{\mathbf{e}}(d) + n') = 1.$$

So D is $\mathbf{e}$-good. □

Theorem 4. *For each* $\mathbf{e}$, *there exist arbitrarily long sequences of consecutive* $\mathbf{e}$-*semihappy numbers.*

Proof. Given $m \in \mathbb{Z}^+$, set $D = \{1, 2, \dots, m\}$. We prove that D is $\mathbf{e}$-good for any $\mathbf{e}$.

First, suppose $e_1 = 1$. By Theorem 1, there exists some $k \in \mathbb{Z}^+$ such that for all $d \in D$, $S^k_{\mathbf{e}}(d) \in \{1, 9, 81, 89\}$. Let $F = S^{18}_{\mathbf{e}} I^{91}$. Then $F(1) = F(9) = 1$, $F(81) = 9$, and $F(89) = 81$. So

$$F^3 S^k_{\mathbf{e}}(D) = F^3(\{1, 9, 81, 89\}) = F^2(\{1, 9, 81\}) = F(\{1, 9\}) = \{1\}.$$

Now, suppose $e_1 = 2$. Again, by Theorem 1, there exists some $k \in \mathbb{Z}^+$ such that for all $d \in D$, $S_{\mathbf{e}}^k(d) \in \{1, 4, 16, 20, 37, 42, 58, 89, 145\}$. Let $F = S_{\mathbf{e}}^7 I^{22}$. Then

$$\begin{aligned}
F^5 S_{\mathbf{e}}^k(D) &= F^5(\{1, 4, 16, 20, 37, 42, 58, 89, 145\}) \\
&= F^4(\{1, 4, 20, 42, 58, 145\}) \\
&= F^3(\{1, 20, 42, 145\}) \\
&= F^2(\{1, 20, 145\}) \\
&= F(\{1, 145\}) \\
&= \{1\}
\end{aligned}$$

In either case, by Lemma 3, D is $\mathbf{e}$-good. Thus there exist n and $k \in \mathbb{Z}^+$ such that for each $d \in D$, $S_{\mathbf{e}}^k(d + n) = 1$ and, therefore, $n + 1, n + 2, \ldots, n + m$ are all $\mathbf{e}$-semihappy. So for any $m \in \mathbb{Z}^+$, there exists a sequence of m consecutive $\mathbf{e}$-semihappy numbers. $\square$

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