Journal of Integer Sequences, Vol. 12 (2009),
Article 09.5.2

On the Bi-Unitary Analogues of Euler's Arithmetical Function and the Gcd-Sum Function

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Abstract

We give combinatorial-type formulae for the bi-unitary analogues of Euler's arithmetical function and the gcd-sum function and prove asymptotic formulae for the latter one and for another related function.

1 Introduction

Euler's arithmetical function and the gcd-sum function (or Pillai's function) are defined by

$$\phi(n) := \#\{k \in \mathbb{N} : 1 \le k \le n, \gcd(k,n) = 1\} \tag{1}$$

and

$$P(n) := \sum_{k=1}^{n} \gcd(k,n), \tag{2}$$

respectively. The function P was investigated in recent papers of this journal; it is sequence A018804 in Sloane's On-Line Encyclopedia of Integer Sequences. A direct connection between these two functions is given by

$$P(n) = \sum_{d \mid n} d\phi(n/d). \tag{3}$$

Let n be a positive integer. Recall that a positive integer d is said to be a *unitary divisor* of n if $d \mid n$ and $\gcd(d, n/d) = 1$, and we write $d \mid\mid n$. Let $(k, n)_* := \max\{d \in \mathbb{N} : d \mid k, d \mid\mid n\}$, and consider the function ϕ^* (A047994) defined by

$$\phi^*(n) := \#\{k \in \mathbb{N} : 1 \leq k \leq n, (k, n)_* = 1\}, \tag{4}$$

which is multiplicative and $\phi^*(p^\nu) = p^\nu - 1$ for every prime power p^ν ($\nu \geq 1$). Also,

$$\phi^*(n) = \sum_{d \mid\mid n} d\mu^*(n/d), \quad \sum_{d \mid\mid n} \phi^*(d) = n, \tag{5}$$

where $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. Various properties of the function ϕ^* were investigated by several authors, cf. [1, 6, 9, 11, 12, 13, 14].

Furthermore, consider the gcd-sum type function given by

$$P^*(n) := \sum_{k=1}^{n} (k, n)_*, \tag{6}$$

which was introduced in our paper [17]; also see [18, 19]. The function P^* (A145388) is also multiplicative; we have $P^*(p^\nu) = 2p^\nu - 1$ for every prime power p^ν ($\nu \geq 1$) and

$$P^*(n) = \sum_{d \mid\mid n} d\phi^*(n/d). \tag{7}$$

The functions ϕ^* and P^* defined by (4) and (6) are called the unitary Euler function and unitary gcd-sum function (or unitary Pillai function), respectively. Many of their properties are analogous to the functions ϕ and P. Note that μ^* is the unitary analogue of the Möbius function μ.

A natural question is the following: Why do we not consider the greatest common unitary divisor of k and n when defining these functions? The answer is, because if we do so, the resulting functions are not multiplicative, and their properties are not so close to those of Euler's function (1) and of the gcd-function (2), as it turns out from the results of the present paper.

Let $(k, n)_{**} = \max\{d \in \mathbb{N} : d \mid\mid k, d \mid\mid n\}$ stand for the greatest common unitary divisor of k and n. The function

$$\phi^{**}(n) := \#\{k \in \mathbb{N} : 1 \leq k \leq n, (k, n)_{**} = 1\}, \tag{8}$$

called the bi-unitary Euler function, was introduced by Subbarao and Suryanarayana in 1971, cf. [16] and [2]. We have

$$\phi^{**}(n) = \sum_{d \mid\mid n} \mu^*(d)\phi(n/d, d), \tag{9}$$

where $\phi(x, n) := \#\{k \in \mathbb{N} : 1 \leq k \leq x, \gcd(k, n) = 1\}$ is the Legendre function. Note that $\phi(n, n) = \phi(n)$.

Here $(k,n)_{**} \le (k,n)_* \le (k,n)$ for every $k,n \ge 1$, and hence

$$\phi(n) \le \phi^*(n) \le \phi^{**}(n). \tag{10}$$

The average order of ϕ^{**} is given by

$$\sum_{n\le x} \phi^{**}(n) = \frac{Ax^2}{2} + \mathcal{O}(x \log^2 x), \tag{11}$$

where

$$A := \prod_p \left(1 - \frac{p-1}{p^2(p+1)}\right) = \zeta(2) \prod_p \left(1 - \frac{2}{p^2} + \frac{2}{p^3} - \frac{1}{p^4}\right), \tag{12}$$

the products being over the primes and ζ denoting the Riemann zeta function; see [16, Corollary 3.6.2]. Here $A \approx 0.8073$, this value is the asymptotic density of ordered pairs $(m,n) \in \mathbb{N}^2$ such that $(m,n)_{**} = 1$, i.e., there is no prime power p^ν appearing in the prime factorizations of both m and n; see [16, Corollary 3.6.3], [20, Theorems 2.2, 2.3].

Now let

$$P^{**}(n) := \sum_{k=1}^{n} (k,n)_{**} \tag{13}$$

be the bi-unitary gcd-sum function, introduced recently by Haukkanen [4]. It is known (see [4, Corollary 3.1]) that

$$P^{**}(n) = \sum_{d||n} \phi^*(d)\phi(n/d,d). \tag{14}$$

Note that for every $n \ge 1$,

$$P^{**}(n) \le P^*(n) \le P(n). \tag{15}$$

Haukkanen [4] considered also the function

$$P_f^{**}(n) := \sum_{k=1}^{n} f((k,n)_{**}), \tag{16}$$

and showed that for any arithmetical function f,

$$P_f^{**}(n) = \sum_{d||n} (f \times \mu^*)(d)\phi(n/d,d), \tag{17}$$

where $\times$ stands for the unitary convolution. The unitary convolution of the functions f and g is given by

$$(f \times g)(n) = \sum_{d||n} f(d)g(n/d). \tag{18}$$

For the properties of convolution (18) and for background material on arithmetical functions involving unitary divisors we refer to the book [9].

Furthermore, let S^{**} denote the function given by (16) in case $f(n)=\tau(n)$, the number of divisors of n, i.e.,

$$S^{**}(n)=\sum_{k=1}^{n}\tau((k,n)_{**}). \tag{19}$$

It is the aim of this paper to give combinatorial-type formulae for $\phi^*(n)$, $\phi^{**}(n)$, $P^{**}(n)$, $S^{**}(n)$ and to investigate the average orders of the functions P^{**} and S^{**}.

2 Combinatorial-type formulae

Let $\phi^*(x,n)=\#\{k\in\mathbb{N}:1\le k\le x,(k,n)_*=1\}$ be the unitary Legendre function, considered by Haukkanen [3], and let $\phi^{**}(x,n)=\#\{k\in\mathbb{N}:1\le k\le x,(k,n)_{**}=1\}$ stand for the bi-unitary Legendre function, investigated in [16]. Observe that $\phi^*(n,n)=\phi^*(n)$ and $\phi^{**}(n,n)=\phi^{**}(n)$.

Let $n=p_1^{\nu_1}\cdots p_r^{\nu_r}>1$ be a fixed integer and let $A=A_n=\{1,\ldots,r\}$, $r=\omega(n)$.

We give explicit formulae for the values $\phi(x,n)$, $\phi^*(x,n)$ and $\phi^{**}(x,n)$ which are of combinatorial type and do not involve the Möbius function μ, its unitary analogue μ^* or other similar arithmetical functions.

In the following formulae $\lfloor y\rfloor$ stands for the largest integer less than or equal to y (floor function) and we use the convention that the empty products are equal to 1.

Theorem 1.

$$\phi(x,n)=\sum_{B\subseteq A}(-1)^{\#B}\left\lfloor x\prod_{j\in B}p_j^{-1}\right\rfloor, \tag{20}$$

$$\phi^*(x,n)=\sum_{B\subseteq A}(-1)^{\#B}\left\lfloor x\prod_{j\in B}p_j^{-\nu_j}\right\rfloor, \tag{21}$$

$$\phi^{**}(x,n)=\sum_{B\subseteq A}(-1)^{\#B}\sum_{C\subseteq B}(-1)^{\#C}\left\lfloor x\prod_{j\in B}p_j^{-\nu_j}\prod_{j\in C}p_j^{-1}\right\rfloor. \tag{22}$$

Proof. It is well known that $\phi(x,n)=\sum_{d\mid n}\mu(d)\lfloor x/d\rfloor$. We give its proof for the sake of completeness and in order to compare it with the following proofs. Using that $d\mid\gcd(k,n)$ iff $d\mid k$, $d\mid n$,

$$\phi(x,n)=\sum_{k\le x}\sum_{d\mid\gcd(k,n)}\mu(d)=\sum_{k\le x}\sum_{\substack{d\mid k\\ d\mid n}}\mu(d)=\sum_{d\mid n}\mu(d)\sum_{\substack{k\le x\\ d\mid k}}1=\sum_{d\mid n}\mu(d)\lfloor x/d\rfloor.$$

Now let $d=p_1^{\beta_1}\cdots p_r^{\beta_r}$ stand for a square-free divisor of n, i.e., $\beta_j=1$ or 0 for every j, and let B be the set of those j for which $\beta_j=1$ and we obtain (20) by the definition of the Möbius function.

For the proof of (21) we use that $d\,||\,(k,n)_*$ iff $d\mid k$, $d\,||\,n$ and have

$$\phi^*(x,n)=\sum_{k\le x}\sum_{d||(k,n)_*}\mu^*(d)=\sum_{k\le x}\sum_{\substack{d\mid k\\ d||n}}\mu^*(d)=\sum_{d||n}\mu^*(d)\sum_{\substack{k\le x\\ d\mid k}}1=\sum_{d||n}\mu^*(d)\lfloor x/d\rfloor.$$

Now let $d = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ be a unitary divisor of n, i.e., $\gamma_j = \nu_j$ or 0 for every j, and let B be the set of those j for which $\gamma_j = \nu_j$ and we obtain the formula (21).

For the proof of (22) we apply that $d \mid\mid (k,n)_{**}$ iff $d \mid\mid k$, $d \mid\mid n$ and obtain

$$\phi^{**}(x,n) = \sum_{k \le x} \sum_{d \mid\mid (k,n)_{**}} \mu^*(d) = \sum_{k \le x} \sum_{\substack{d \mid\mid k \\ d \mid\mid n}} \mu^*(d) = \sum_{d \mid\mid n} \mu^*(d) \sum_{\substack{k \le x \\ d \mid k \\ \gcd(d,k/d)=1}} 1$$

$$= \sum_{d \mid\mid n} \mu^*(d) \phi(x/d, d) = \sum_{d \mid\mid n} \mu^*(d) \sum_{e \mid d} \mu(e) \lfloor x/(de) \rfloor.$$

Let $d = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ be a unitary divisor of n i.e., $\gamma_j = \nu_j$ or 0 for every j, and let B be the set of those j for which $\gamma_j = \nu_j$. Furthermore, let $e = p_1^{\eta_1} \cdots p_r^{\eta_r}$ be a square-free divisor of d, where $\eta_k = 1$ or 0 for every k and let C be the set of those k for which $\eta_k = 1$, where $C \subseteq B$. $\square$

Corollary 1.

$$\phi^{**}(n) = \sum_{B \subseteq A} (-1)^{\#B} \sum_{C \subseteq B} (-1)^{\#C} \left\lfloor \prod_{j \in A \setminus B} p_j^{\nu_j} \prod_{j \in C} p_j^{-1} \right\rfloor. \tag{23}$$

This is obtained from (22) for $x = n$. If $r = 1$, then we deduce that $\phi^{**}(p_1^{\nu_1}) = p_1^{\nu_1} - 1 = \phi^*(p_1^{\nu_1})$, which can be seen directly from the definition (8). For $r = 2$ we obtain

$$\phi^{**}(p_1^{\nu_1} p_2^{\nu_2}) = (p_1^{\nu_1} - 1)(p_2^{\nu_2} - 1) + \left\lfloor p_1^{\nu_1} p_2^{-1} \right\rfloor + \left\lfloor p_2^{\nu_2} p_1^{-1} \right\rfloor = \phi^*(p_1^{\nu_1} p_2^{\nu_2}) + \left\lfloor p_1^{\nu_1} p_2^{-1} \right\rfloor + \left\lfloor p_2^{\nu_2} p_1^{-1} \right\rfloor, \tag{24}$$

therefore the function ϕ^{**} is not multiplicative.

For $P^{**}(n)$ and $S^{**}(n)$ we have the following formulae. First note that $\tau \times \mu^* = \beta$, where $\beta(n)$ is the number of square-full divisors of n (A005361), β is multiplicative and $\beta(p^\nu) = \nu$ for any prime power p^ν ($\nu \ge 1$). Therefore we have from (17),

$$S^{**}(n) = \sum_{d \mid\mid n} \beta(d) \phi(n/d, d). \tag{25}$$

Theorem 2.

$$P^{**}(n) = \sum_{B \subseteq A} \left(\sum_{D \subseteq B} (-1)^{\#D} \prod_{j \in B \setminus D} p_j^{\nu_j} \right) \left(\sum_{C \subseteq B} (-1)^{\#C} \left\lfloor \prod_{j \in A \setminus B} p_j^{\nu_j} \prod_{j \in C} p_j^{-1} \right\rfloor \right), \tag{26}$$

$$S^{**}(n) = \sum_{B \subseteq A} \left(\prod_{j \in B} \nu_j \right) \sum_{C \subseteq B} (-1)^{\#C} \left\lfloor \prod_{j \in A \setminus B} p_j^{\nu_j} \prod_{j \in C} p_j^{-1} \right\rfloor. \tag{27}$$

Proof. From (14) and (5) we obtain

$$P^{**}(n) = \sum_{d \mid\mid n} \phi^*(d) \phi(n/d, d) = \sum_{d \mid\mid n} \left(\sum_{e \mid\mid d} \mu^*(e) \frac{d}{e} \right) \left(\sum_{\delta \mid d} \mu(\delta) \lfloor n/(d\delta) \rfloor \right),$$

and use similar arguments as in Theorem 1. To obtain (27) use (25).

For $r = 1$ we deduce from (26) that $P^{**}(p_1^{\nu_1}) = 2p_1^{\nu_1} - 1 = P^*(p_1^{\nu_1})$, which can be seen also directly from the definitions. For $r = 2$ we obtain

$$P^{**}(p_1^{\nu_1}p_2^{\nu_2}) = (2p_1^{\nu_1} - 1)(2p_2^{\nu_2} - 1) - (p_1^{\nu_1} - 1)\left\lfloor p_2^{\nu_2}p_1^{-1}\right\rfloor - (p_2^{\nu_2} - 1)\left\lfloor p_1^{\nu_1}p_2^{-1}\right\rfloor$$

$$= P^*(p_1^{\nu_1}p_2^{\nu_2}) - (p_1^{\nu_1} - 1)\left\lfloor p_2^{\nu_2}p_1^{-1}\right\rfloor - (p_2^{\nu_2} - 1)\left\lfloor p_1^{\nu_1}p_2^{-1}\right\rfloor.$$

Also, for $r = 1$ we have from (27), $S^{**}(p_1^{\nu_1}) = p_1^{\nu_1} + \nu_1$. For $r = 2$ we obtain

$$S^{**}(p_1^{\nu_1}p_2^{\nu_2}) = (p_1^{\nu_1} + \nu_1)(p_2^{\nu_2} + \nu_2) - \nu_1\left\lfloor p_2^{\nu_2}p_1^{-1}\right\rfloor - \nu_2\left\lfloor p_1^{\nu_1}p_2^{-1}\right\rfloor$$

$$= S^{**}(p_1^{\nu_1}p_2^{\nu_2}) - \nu_1\left\lfloor p_2^{\nu_2}p_1^{-1}\right\rfloor - \nu_2\left\lfloor p_1^{\nu_1}p_2^{-1}\right\rfloor.$$

□

Therefore the functions P^{**} and S^{**} are not multiplicative.

3 The average order of P^{**}

We show that

Theorem 3.

$$\sum_{n\le x} P^{**}(n) = \frac{1}{2}Bx^2\log x + \mathcal{O}(x^2), \tag{28}$$

where

$$B := \prod_p \left(1 - \frac{3p-1}{p^2(p+1)}\right) = \zeta(2)\prod_p\left(1 - \frac{(2p-1)^2}{p^4}\right) \approx 0.35823. \tag{29}$$

Proof. We will use the fact that for any real numbers $s \ge 0$ and $0 \le \varepsilon < 1$,

$$\sum_{\substack{n\le x\\ \gcd(n,k)=1}} n^s = \frac{x^{s+1}\phi(k)}{(s+1)k} + \mathcal{O}(x^s\min(x^\varepsilon\sigma_{-\varepsilon}(k), 2^{\omega(k)})), \tag{30}$$

this estimate being uniform for x and k, where $\sigma_{-\varepsilon}(k)$ is the sum of ε-th powers of the reciprocals of divisors of k. A similar treatment was applied also in [17]. We have by (14), and by (30) applied for $s = 0$ and $s = 1$, respectively,

$$\sum_{n\le x} P^{**}(n) = \sum_{\substack{de\le x\\ \gcd(d,e)=1}} \phi^*(d)\phi(e,d) = \sum_{\substack{de\le x\\ \gcd(d,e)=1}} \phi^*(d)\left(\frac{e\phi(d)}{d} + \mathcal{O}\left(e^\varepsilon\sigma_{-\varepsilon}(d)\right)\right)$$

$$= \sum_{d\le x}\frac{\phi(d)\phi^*(d)}{d}\sum_{\substack{e\le x/d\\ \gcd(e,d)=1}} e + \mathcal{O}\left(\sum_{d\le x}\phi^*(d)\sigma_{-\varepsilon}(d)\sum_{e\le x/d} e^\varepsilon\right)$$

$$= \sum_{d\le x} \frac{\phi(d)\phi^*(d)}{d} \left(\frac{x^2\phi(d)}{2d^3} + \mathcal{O}\left((x/d)^{\varepsilon+1}\sigma_{-\varepsilon}(d)\right)\right) + \mathcal{O}\left(\sum_{d\le x} d\sigma_{-\varepsilon}(d)(x/d)^{\varepsilon+1}\right)$$

$$= \frac{x^2}{2}\sum_{d\le x}\frac{\phi^2(d)\phi^*(d)}{d^4} + \mathcal{O}\left(x^{\varepsilon+1}\sum_{d\le x}\frac{\sigma_{-\varepsilon}(d)}{d^{\varepsilon}}\right),$$

where the error term is $\mathcal{O}(x^{\varepsilon+1}\cdot x^{1-\varepsilon}) = \mathcal{O}(x^2)$, cf. [17, Lemma 2.2].

Now let $E_k(n) = n^k$ and $\phi^2\phi^* = f * E_3$ in terms of the Dirichlet convolution. Thus $f = \phi^2\phi^* * \mu E_3$. Here f is multiplicative and direct computations show that $f(p) = -3p^2+3p-1$ and $f(p^\nu) = p^{2\nu+1}(1-1/p)^3$ for every prime p and every $\nu \ge 2$. We obtain that the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

is absolutely convergent for $\operatorname{Re} s > 3$ and by $\phi^2\phi^*/E_4 = f/E_4 * E_{-1}$,

$$\sum_{n\le x}\frac{\phi^2(n)\phi^*(n)}{n^4} = \sum_{d\le x}\frac{f(d)}{d^4}\sum_{e\le x/d}\frac{1}{e} = \sum_{d\le x}\frac{f(d)}{d^4}\left(\log(x/d) + \mathcal{O}(1)\right)$$

$$= \log x\sum_{d=1}^{\infty}\frac{f(d)}{d^4} + \mathcal{O}\left(\log x\sum_{d>x}\frac{|f(d)|}{d^4}\right) + \mathcal{O}\left(\sum_{d\le x}\frac{|f(d)|\log d}{d^4}\right) = B\log x + \mathcal{O}(1),$$

therefore

$$\sum_{n\le x}\frac{\phi^2(n)\phi^*(n)}{n^4} = B\log x + \mathcal{O}(1), \tag{31}$$

by the Euler product formula. This completes the proof. □

Remark 1. The asymptotic formula (31) can be obtained by using known theorems on mean values of certain multiplicative functions. The following useful theorem, which is a particular case of a more general result, has been proved in [7] using Iwaniec's ideas.

Theorem [7, Proposition A.3]. *Let f be any nonnegative multiplicative function such that $f(n) \ll n^\alpha$ for some $\alpha < 1/2$ and satisfying*

$$\sum_{p\le x}\frac{f(p)\log p}{p} = \kappa\log x + \mathcal{O}_f(1) \quad (x \ge 2),$$

where $\kappa = \kappa_f > 0$. Then we have uniformly for all $x \ge 2$,

$$\sum_{n\le x}\frac{f(n)}{n} = \mathcal{M}_{f,\kappa}(\log x)^\kappa + \mathcal{O}_f((\log x)^{\kappa-1}), \tag{32}$$

where

$$\mathcal{M}_{f,\kappa} := \frac{1}{\Gamma(\kappa+1)}\prod_p\left(1-\frac{1}{p}\right)^\kappa\left(1+\sum_{\nu=1}^{\infty}\frac{f(p^\nu)}{p^\nu}\right).$$

Now (31) follows at once by applying (32) to $f(n) = \frac{\phi^2(n)\phi^*(n)}{n^3}$ with $\kappa = 1$ and noticing that $f(p) = (1-1/p)^3 = 1 + \mathcal{O}(1/p)$.

4 The average order of S^{**}

For the function S^{**} we have

Theorem 4.

$$\sum_{n\le x} S^{**}(n) = \frac{1}{2}Cx^2 + \mathcal{O}(x\log^2 x), \tag{33}$$

where

$$C := \prod_p \left(1+\frac{1}{(p+1)^2}\right) = \zeta^2(2)\prod_p\left(1-\frac{1}{p^2}-\frac{2}{p^3}+\frac{2}{p^4}\right) \approx 1.266558. \tag{34}$$

Proof. Similar to the proof of Theorem 3, here it is enough to apply (30) for $\varepsilon = 0$. From (25) we obtain

$$\begin{aligned}
\sum_{n\le x} S^{**}(n) &= \sum_{\substack{de\le x\\ (d,e)=1}} \beta(d)\phi(e,d) = \sum_{\substack{de\le x\\ (d,e)=1}} \beta(d)\left(\frac{e\phi(d)}{d} + \mathcal{O}\left(2^{\omega(d)}\right)\right)\\
&= \sum_{d\le x}\frac{\phi(d)\beta(d)}{d}\sum_{\substack{e\le x/d\\ (e,d)=1}} e + \mathcal{O}\left(\sum_{de\le x}\beta(d)2^{\omega(d)}\right)\\
&= \sum_{d\le x}\frac{\phi(d)\beta(d)}{d}\left(\frac{x^2\phi(d)}{2d^3} + \mathcal{O}\left((x/d)2^{\omega(d)}\right)\right) + \mathcal{O}\left(x\sum_{d\le x}\frac{\beta(d)2^{\omega(d)}}{d}\right)\\
&= \frac{x^2}{2}\sum_{d=1}^{\infty}\frac{\beta(d)\phi^2(d)}{d^4} + \mathcal{O}\left(x^2\sum_{d>x}\frac{\beta(d)}{d^2}\right) + \mathcal{O}\left(x\sum_{d\le x}\frac{\beta(d)2^{\omega(d)}}{d}\right).
\end{aligned} \tag{35}$$

We need here the estimate $\sum_{n\le x}\beta(n) \ll x$. The function β is multiplicative and increasing on prime powers, i.e., $\beta(p^\nu) = \nu > \nu - 1 = \beta(p^{\nu-1})$ for every prime p and every $\nu \ge 1$. It follows by well known results on sums of multiplicative functions, that

$$\sum_{n\le x}\beta(n) \le x\prod_{p\le x}\left(1-\frac{1}{p}\right)\sum_{\nu=0}^{\infty}\frac{\beta(p^\nu)}{p^\nu},$$

cf. for ex. [5, p. 23], and obtain that

$$\sum_{n\le x}\beta(n) \le x\prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{\nu=1}^{\infty}\frac{\nu}{p^\nu}\right) = x\prod_{p\le x}\left(1+\frac{1}{p(p-1)}\right) \ll x.$$

By partial summation we obtain

$$\sum_{d>x}\frac{\beta(d)}{d^2} = -\frac{1}{x^2}\sum_{d\le x}\beta(d) + 2\int_x^\infty\left(\sum_{d\le t}\beta(d)\right)\frac{dt}{t^3} \ll \frac{1}{x},$$

therefore the first $\mathcal{O}$-term in (35) is $\mathcal{O}(x)$. The second $\mathcal{O}$-term in (35) is $\mathcal{O}(x\log^2 x)$. This follows from the estimate

$$\sum_{d\le x}\frac{\beta(d)2^{\omega(d)}}{d}\le\prod_{p\le x}\left(1+\sum_{\nu=1}^{\infty}\frac{\beta(p^\nu)2^{\omega(p^\nu)}}{p^\nu}\right)=\prod_{p\le x}\left(1+\frac{2}{p-1}+\frac{2}{(p-1)^2}\right)\ll\log^2 x,$$

by using Mertens' theorem.

The given representations of C follow by the Euler product formula. For the numerical value of C consult [8, item 56]. $\square$

Remark 2. It is known that $\sum_{n\le x}\beta(n)\sim cx$, where $c=\zeta(2)\zeta(3)/\zeta(6)$, which can be obtained using the representation $\beta(n)=\sum_{de=n}\sum_{a^2b^3=d}\mu^2(b)$. For an asymptotic formula with remainder term for $\beta(n)$, see [15].

5 Acknowledgement

The author thanks the referee for improving the error term of Theorem 4 and for Remark 1.

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2000 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37, 05A15.
Keywords: Euler's arithmetical function, Möbius function, divisor function, gcd-sum function, unitary divisor, average order.

(Concerned with sequences A000010, A001221, A005361, A008683, A018804, A047994, and A145388.)

Received April 24 2009; revised version received June 22 2009. Published in *Journal of Integer Sequences*, June 23 2009.

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