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A Gcd-Sum Function Over Regular Integers Modulo n

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Abstract

We introduce a gcd-sum function involving regular integers (mod n) and prove results giving its minimal order, maximal order and average order.

1 Introduction

Let $n > 1$ be an integer with prime factorization $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$. An integer k is called *regular* (mod n) if there exists an integer x such that $k^2 x \equiv k \pmod{n}$, i.e., the residue class of k is a regular element (in the sense of J. von Neumann) of the ring $\mathbb{Z}_n$ of residue classes (mod n). In general, an element k of a ring R is said to be (von Neumann) regular if there is an $x \in R$ such that $k = kxk$. If every $k \in R$ has this property, then R is called a *von Neumann regular ring*, cf. for example [9, p. 110].

It can be shown that $k \geq 1$ is regular (mod n) if and only if for every $i \in \{1, \ldots, r\}$ either $p_i \nmid k$ or $p_i^{\nu_i} \mid k$. Also, $k \geq 1$ is regular (mod n) if and only if $\gcd(k, n)$ is a unitary divisor of n. We recall that d is said to be a *unitary divisor* of n if $d \mid n$ and $\gcd(d, n/d) = 1$, and use the notation $d \mid\mid n$. These and other characterizations of regular integers are given in our paper [15].

Let $\mathrm{Reg}_n = \{k : 1 \leq k \leq n \text{ and } k \text{ is regular (mod } n)\}$, and let $\varrho(n) = \# \mathrm{Reg}_n$ denote the number of regular integers k (mod n) such that $1 \leq k \leq n$. The function $\varrho(n)$ was investigated in paper [15]. It is multiplicative and $\varrho(p^\nu) = \phi(p^\nu) + 1 = p^\nu - p^{\nu-1} + 1$ for every prime power p^ν ($\nu \geq 1$), where ϕ is the Euler function. Note that $(\varrho(n) : n \in \mathbb{N})$ is the sequence A055653 in Sloane's On-Line Encyclopedia of Integer Sequences.

In this paper we introduce the function

$$\widetilde{P}(n) := \sum_{k \in \mathrm{Reg}_n} \gcd(k, n). \tag{1}$$

This is analogous to the gcd-sum function, called also Pillai's arithmetical function,

$$P(n) := \sum_{k=1}^{n} \gcd(k, n), \tag{2}$$

investigated in the recent papers [1, 2, 3, 4, 13] of this journal. This is sequence A018804 in Sloane's Encyclopedia.

Note that the function $P(n)$ was introduced by S. S. Pillai [11], showing that

$$P(n) = \sum_{d|n} d\phi(n/d), \quad \sum_{d|n} P(d) = n\tau(n) = \sum_{d|n} \sigma(d)\phi(n/d),$$

where $\tau(n)$ and $\sigma(n)$ denote, as usual, the number and the sum of divisors of n, respectively.

Note also, that for any arithmetical function f,

$$P_f(n) := \sum_{k=1}^{n} f(\gcd(k, n)) = \sum_{d|n} f(d)\phi(n/d), \tag{3}$$

which is a result of E. Cesàro [5]. See also the book [7, p. 127].

O. Bordellès [1] showed that

$$P = \mu * (E \cdot \tau), \tag{4}$$

where $*$ denotes the Dirichlet convolution, where $E(n) = n$ and μ is the Möbius function. Then, using this representation he proved the following asymptotic formula: for every $\varepsilon > 0$,

$$\sum_{n \le x} P(n) = \frac{x^2}{2\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + \mathcal{O}(x^{1+\theta+\varepsilon}), \tag{5}$$

where γ is Euler's constant and θ is the number appearing in Dirichlet's divisor problem, that is

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + \mathcal{O}(x^{\theta+\varepsilon}). \tag{6}$$

It is known that $1/4 \le \theta \le 131/416 \approx 0.3149$.

Formulae (4) and (5) were obtained, even in a more general form, in the paper [6], where the authors proved also the following result concerning the maximal order of $P(n)$,

$$\limsup_{n \to \infty} \frac{\log(P(n)/n) \log\log n}{\log n} = \log 2, \tag{7}$$

which is well known for the function $\tau(n)$ instead of $P(n)/n$.

Asymptotic formulae for generalized gcd-sum functions of type (3), concerning sets of polynomials with integral coefficients and regular convolutions were given in our paper [14].

We investigate in what follows arithmetical and asymptotical properties of the function $\widetilde{P}(n)$. We show that it is multiplicative and for every $n \geq 1$,

$$\widetilde{P}(n) = n \prod_{p|n} \left(2 - \frac{1}{p}\right). \tag{8}$$

Further, we show that the result (7) holds also for the function $\widetilde{P}(n)$, its minimal order is $3n/2$ and prove an asymptotic formula for its summatory function both without assuming the Riemann hypothesis and assuming the Riemann hypothesis (RH), based on a convolution identity analogous to (4).

Our results and proofs involve properties of arithmetical functions defined by unitary divisors and of the unitary convolution. For background material in this topic we refer to the book [10].

As an open question we formulate the following: What is the minimal order of the Pillai function $P(n)$?

A common generalization of the functions $P(n)$ and $\widetilde{P}(n)$ is outlined at the end of Section 2.

2 Arithmetical properties

Let f be an arbitrary arithmetical function and consider the more general function

$$\widetilde{P}_f(n) := \sum_{k \in \mathrm{Reg}_n} f(\gcd(k,n)).$$

Proposition 1. *For every $n \geq 1$,*

$$\widetilde{P}_f(n) = \sum_{d||n} f(d)\, \phi(n/d). \tag{9}$$

Proof. The integer $k \geq 1$ is regular iff $\gcd(k,n) \,||\, n$, cf. the Introduction, and obtain

$$\widetilde{P}_f(n) = \sum_{k=1}^{n} \sum_{d||n} f(d) = \sum_{d||n} f(d) \sum_{\substack{1 \leq j \leq n/d \\ (j,n/d)=1}} 1 = \sum_{d||n} f(d)\phi(n/d).$$

□

Corollary 1. *If f is multiplicative, then $\widetilde{P}_f$ is also multiplicative and $\widetilde{P}_f(p^\nu) = f(p^\nu) + p^\nu - p^{\nu-1}$ for every prime power p^ν ($\nu \geq 1$).*

In particular, $\widetilde{P}$ is multiplicative and $\widetilde{P}(p^\nu) = 2p^\nu - p^{\nu-1}$ for every prime power p^ν ($\nu \geq 1$).

Proof. According to (9), $\widetilde{P}_f$ is the unitary convolution of the functions f and ϕ. It is known that the unitary convolution preserves the multiplicativity of functions. In particular, for $f(n) = n$ we obtain that $\widetilde{P}$ is multiplicative and the explicit formula (8). □

Let $\omega(n,k)$ denote the number of distinct prime factors of n which do not divide k. For $k=1$, $\omega(n):=\omega(n,1)$ is the number distinct factors of n. Also, let $\tau^*(n,k)$ denote the number of unitary divisors of n which are relatively prime to k. Here $\tau^*(n):=\tau^*(n,1)$ is the number of unitary divisors of n. We have $\tau^*(n,k)=2^{\omega(n,k)}$ and $\tau^*(n)=2^{\omega(n)}$.

Another representation of $\widetilde{P}$ is given by

Proposition 2. *For every $n \geq 1$,*

$$\widetilde{P}(n) = \sum_{de=n} \mu(d)e \cdot 2^{\omega(e,d)}.$$

Proof. By Proposition 1,

$$\widetilde{P}(n) = \sum_{\substack{de=n \\ (d,e)=1}} d\phi(e) = \sum_{\substack{de=n \\ (d,e)=1}} d \sum_{ab=e} \mu(a)b = \sum_{\substack{dab=n \\ (d,a)=1 \\ (d,b)=1}} d\mu(a)b$$

$$= \sum_{ac=n} \mu(a)c \sum_{\substack{bd=c \\ (b,d)=1 \\ (d,a)=1}} 1 = \sum_{ac=n} \mu(a)c\,\tau^*(c,a).$$

□

Proposition 3. *For every $n \geq 1$ we have $\widetilde{P}(n) \leq P(n)$, with equality iff n is square-free, and $2^{\omega(n)}\phi(n) \leq \widetilde{P}(n) \leq 2^{\omega(n)}n$, with equality iff $n=1$.*

Proof. This follows at once by (1), (2) and (8). □

Remark 1. Let $\widehat{P}(n) := \sum_{k \in \mathrm{Reg}_n} \mathrm{lcm}[k,n]$. Then it follows, similar to the "usual" lcm-sum function that for every $n \geq 1$,

$$\widehat{P}(n) = \frac{n}{2}\left(1 + \sum_{d||n} d\phi(d)\right).$$

Remark 2. For every $n \in \mathbb{N}$ let $A(n)$ be an arbitrary nonempty subset of the set of positive divisors of n. For the system of divisors $A=(A(n): n \in \mathbb{N})$ and for an arbitrary arithmetical function f consider the following restricted summation of the gcd's:

$$P_{A,f}(n) = \sum_{\substack{1 \leq k \leq n \\ \mathrm{gcd}(k,n) \in A(n)}} f(\mathrm{gcd}(k,n)).$$

It follows, similar to the proof of Proposition 1, that

$$P_{A,f}(n) = \sum_{d \in A(n)} f(d)\phi(n/d).$$

If $A(n)$ is the set of all (positive) divisors of n, then we have the function (3) and if $A(n)$ is the set of the unitary divisors of n, then we reobtain (9).

If A is a regular system of divisors of Narkiewicz-type, including the previous two special cases, then $P_{A,f}$ is the A-convolution of the functions f and ϕ. It turns out, that $P_{A,f}$ is multiplicative provided f is multiplicative, cf. [10, Ch. 4].

Other special cases for A can also be considered, for ex. $A(n)$ the set of prime divisors of n or $A(n)$ the set of exponential divisors of n.

3 Asymptotic properties

Theorem 1. *The minimal order of* $\widetilde{P}(n)$ *is* $3n/2$ *and the maximal order of* $\log(\widetilde{P}(n)/n)$ *is* $\log 2 \log n/\log\log n$.

Proof. From (8) we have $\widetilde{P}(n) \geq n(3/2)^{\omega(n)} \geq 3n/2$ for every $n \geq 1$, with equality for $n = 2^\nu$ ($\nu \geq 1$), giving the minimal order of $\widetilde{P}(n)$.

For the maximal order take into account (7), where the limsup is attained for a sequence of square-free integers (more exactly for $n_k = \prod_{k/\log^2 k < p \leq k} p$, $k \to \infty$), see [6, Theorem 4.1], and use $\widetilde{P}(n) \leq P(n)$ for every $n \geq 1$, with equality iff n is square-free, by Proposition 3. □

In what follows we prove the following asymptotic formula for the function $\widetilde{P}$. Let $\psi(n) = n\prod_{p|n}(1 + 1/p)$ denote the Dedekind function,

$$\alpha(n) := \sum_{p|n} \frac{\log p}{p-1}, \quad \beta(n) = \sum_{p|n} \frac{\log p}{p^2-1},$$

$$\delta(x) := \exp\left(-A(\log x)^{3/5}(\log\log x)^{-1/5}\right),$$
$$\eta(x) := \exp\left(B\log x(\log\log x)^{-1}\right),$$

where A and B are positive constants and let θ be the exponent in (6).

Theorem 2. *We have*

$$\sum_{n\leq x} \widetilde{P}(n) = \frac{x^2}{2\zeta(2)}(K_1 \log x + K_2) + \mathcal{O}(x^{3/2}\delta(x)), \tag{10}$$

where the constants K_1 *and* K_2 *are given by*

$$K_1 := \sum_{n=1}^{\infty} \frac{\mu(n)}{n\psi(n)} = \prod_p \left(1 - \frac{1}{p(p+1)}\right),$$

$$K_2 := K_1\left(2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) - \sum_{n=1}^{\infty} \frac{\mu(n)(\log n - \alpha(n) + 2\beta(n))}{n\psi(n)}.$$

If RH is true, then the error term of (10) *is* $\mathcal{O}(x^{(7-5\theta)/(5-4\theta)}\eta(x))$.

Remark 3. Here $K_1 \approx 0.7042$. Note that $K_1 = \lim_{x\to\infty} \frac{2}{x^2}\sum_{n\le x}\gamma(n)$, where $\gamma(n) = \prod_{p\mid n} p$ is the greatest square-free divisor of n. Also, $K_1/\zeta(2) \approx 0.4282$ is the so called "carefree constant", cf. [8, Section 2.5.1]. For $\theta \approx 0.3149$ one has $(7-5\theta)/(5-4\theta) \approx 1.4505$.

Proof. We need the following auxiliary results. Let $\sigma'_s(n)$ be the sum of s-th powers of the square-free divisors of n.

Lemma 1. ([12, Theorems 4.3, 5.2]) *If $k \ge 1$ is an integer, then for every $\varepsilon > 0$,*

$$\sum_{n\le x} 2^{\omega(n,k)} = \frac{kx}{\zeta(2)\psi(k)}\left(\log x + \alpha(k) - 2\beta(k) + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + \mathcal{O}\left(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k)x^{1/2}\delta(x)\right), \tag{11}$$

the $\mathcal{O}$ estimate being uniform in x and k.

If RH is true, then $x^{1/2}\delta(x)$ in the error term of (11) can be replaced by $x^{(2-\theta)/(5-4\theta)}\eta(x)$.

Note that

$$\alpha(n) = \mathcal{O}(\log n), \quad \beta(n) = \mathcal{O}(1), \tag{12}$$

since $\alpha(n) \le \sum_{p\mid n}\log p = \log\gamma(n) \le \log n$ and $\beta(n) \ll \sum_{p\mid n}\frac{\log p}{p^2} \le \sum_p \frac{\log p}{p^2} < \infty$.

Lemma 2. *For every $\varepsilon > 0$,*

$$\sum_{n\le x} 2^{\omega(n,k)} n = \frac{kx^2}{2\zeta(2)\psi(k)}\left(\log x + \alpha(k) - 2\beta(k) + 2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) + \mathcal{O}\left(\sigma'_{-1+\varepsilon}(k)\sigma'_{-\theta}(k)x^{3/2}\delta(x)\right), \tag{13}$$

If RH is true, then $x^{3/2}\delta(x)$ in the error term of (13) can be replaced by $x^{(7-5\theta)/(5-4\theta)}\eta(x)$.

Proof. By partial summation from Lemma 1. ☐

We can now complete the proof of Theorem 2. By Proposition 2 and Lemma 2,

$$\sum_{n\le x}\widetilde{P}(n) = \sum_{d\le x}\mu(d)\sum_{e\le x/d} 2^{\omega(e,d)}e$$

$$= \frac{x^2}{2\zeta(2)}\left(\sum_{d\le x}\frac{\mu(d)}{d\psi(d)}\left(\log x + 2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)}\right) - \sum_{d\le x}\frac{\mu(d)(\log d - \alpha(d) + 2\beta(d))}{d\psi(d)}\right) + \mathcal{O}\left(\sum_{d\le x}\sigma'_{-1+\varepsilon}(d)\sigma'_{-\theta}(d)(x/d)^{3/2}\delta(x/d)\right).$$

For every $\varepsilon > 0$ and x sufficiently large, $x^\varepsilon\delta(x)$ is increasing, therefore

$$(x/d)^{3/2}\delta(x/d) = (x/d)^{3/2-\varepsilon}(x/d)^\varepsilon\delta(x/d) \le (x/d)^{3/2-\varepsilon}x^\varepsilon\delta(x) = x^{3/2}\delta(x)/d^{3/2-\varepsilon}.$$

Furthermore, it is enough to use the inequalities $\sigma'_{-1+\varepsilon}(d) \le \tau(d)$ (for $\varepsilon < 1$) and $\sigma'_{-\theta}(d) \le \tau(d)$ and then obtain the given formula using (12) and the well known estimates

$$\sum_{d>x} \frac{1}{d^2} \ll \frac{1}{x}, \quad \sum_{d>x} \frac{\log d}{d^2} \ll \frac{\log x}{x}.$$

If we assume RH and in the error term use the property that $\eta(x)$ is increasing, so $\eta(x/d) \le \eta(x)$ for $d \ge 1$. $\square$

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References

[1] O. Bordellès, A note on the average order of the gcd-sum function, *J. Integer Sequences* **10** (2007), Article 07.3.3.

[2] O. Bordellès, Mean values of generalized gcd-sum and lcm-sum functions, *J. Integer Sequences* **10** (2007), Article 07.9.2.

[3] K. Broughan, The gcd-sum function, *J. Integer Sequences* **4** (2001), Article 01.2.2.

[4] K. Broughan, The average order of the Dirichlet series of the gcd-sum function, *J. Integer Sequences* **10** (2007), Article 07.4.2.

[5] E. Cesàro, Étude moyenne du plus grand commun diviseur de deux nombres, *Ann. Mat. Pura. Appl.* (1) **13** (1885), 235–250.

[6] J. Chidambaraswamy, R. Sitaramachandrarao, Asymptotic results for a class of arithmetical functions, *Monatsh. Math.* **99** (1985), 19–27.

[7] L. E. Dickson, *History of the Theory of Numbers*, vol. I, Chelsea, New York, 1971.

[8] S. R. Finch, *Mathematical Constants*, Cambridge University Press, 2003.

[9] I. Kaplansky, *Fields and Rings*, University of Chicago Press, 1972.

[10] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer, 1986.

[11] S. S. Pillai, On an arithmetic function, *J. Annamalai Univ.* **2** (1933), 243–248.

[12] D. Suryanarayana, V. Siva Rama Prasad, The number of k-free and k-ary divisors of m which are prime to n. *J. Reine Angew. Math.* 264 (1973), 56–75.

[13] Y. Tanigawa, W. Zhai, On the gcd-sum function, *J. Integer Sequences* **11** (2008), Article 08.2.3.

[14] L. Tóth, A generalization of Pillai's arithmetical function involving regular convolutions, *Acta Math. Inform. Univ. Ostraviensis* **6** (1998), 203–217, available at http://www.ttk.pte.hu/matek/ltoth/TothGeneralizationPillai(1998).pdf

[15] L. Tóth, Regular integers (mod n), *Annales Univ. Sci. Budapest., Sect. Comp.* **29** (2008), 263–275, available at http://front.math.ucdavis.edu/0710.1936

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