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On Fibonacci-Like Sequences

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Abstract

In this note, we study Fibonacci-like sequences that are defined by the recurrence $S_k = a$, $S_{k+1} = b$, $S_{n+2} \equiv S_{n+1} + S_n \pmod{n+2}$ for all $n \geq k$, where $k, a, b \in \mathbb{N}$, $0 \leq a < k$, $0 \leq b < k+1$, and $(a, b) \neq (0, 0)$. We will show that the number $\alpha = 0.S_k S_{k+1} S_{k+2} \cdots$ is irrational. We also propose a conjecture on the pattern of the sequence $\{S_n\}_{n \geq k}$.

1 Introduction

Given a sequence of natural numbers $a_1, a_2, \ldots$, the question of determining the irrationality of the number $\alpha = 0.a_1 a_2 \cdots$ is a classical and interesting question. For example, if $a_1, a_2, \ldots$ is the sequence of all prime numbers, then α is irrational ([5]). Another well-known example is the set of generalized Mahler sequences. Let $m \geq 1$, $h \geq 2$ be integers, and

$$(m)_h = m_1 h^{r-1} + m_2 h^{r-2} + \cdots + m_r$$

for some integer $r > 0$ and $0 \leq m_i < h$ for all $1 \leq i \leq r$. Mahler [6] showed that for $t \geq 2$ then the number

$$a(t) = 0.(t^0)_{10}(t^1)_{10}(t^2)_{10} \cdots$$

is irrational. Bundschuh [4] generalized this result to arbitrary bases. More precisely, he showed that for any $t, r \geq 2$ then the number

$$a_r(t) = 0.(t^0)_r(t^1)_r(t^2)_r \cdots$$

is irrational. Readers can find several proofs of this result in [7, 9]. In the most general form, one studies the number

$$a_r(t) = a_r^{(n_i)}(t) = 0.(t^{n_0})_r(t^{n_1})_r(t^{n_2})_r \cdots$$

for given $r, t \geq 2$ and sequence $(n_i)_{i\geq 0}$ of non-negative integers. In [10], Shan and Wang showed that $a_r(t)$ is irrational if (n_i) is an unbounded sequence. Several criteria for irrationality of $a_r(t)$ for bounded (n_i) were obtained by Sander [8], and Shorey and Tijdeman [11]. Motivated by these papers, we will study an analogous result for some Fibonacci-like sequences.

Recall that the Fibonacci sequence is defined by the following recurrence:

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n \ \text{for all}\ \ n \geq 0.$$

In this note, we will study some properties of Fibonacci-like sequences that are defined by the following recurrence:

$$S_k = a,\ S_{k+1} = b,\ S_{n+2} \equiv S_{n+1} + S_n \pmod{n+2} \qquad \text{for all} \qquad n \geq k, \tag{1}$$

for some $k, a, b \in \mathbb{N}$, $0 \leq a < k$ and $0 \leq b < k+1$. For any triple $(k, a, b) \in \mathbb{N}^3$ with $0 \leq a < k$ and $0 \leq b < k+1$, we denote $S^k_{a,b} = \{S^k_{a,b}(n)\}^\infty_{n=k}$ the sequence defined by recurrence (1).

The main result of this note is the following theorem.

Theorem 1. *Suppose that a, b, k are natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Then*

$$\alpha^k_{a,b} = 0.S^k_{a,b}(k)S^k_{a,b}(k+1)S^k_{a,b}(k+2)\ldots \tag{2}$$

is irrational. Here expression (2) means that the decimal expansion of $\alpha^k_{a,b}$ is obtained by the concatenation of the integers $S^k_{a,b}(n)$ written in decimal form.

It is worth noticing that most of papers inspired by Mahler deal with exponentially increasing sequences, while $S^k_{a,b}$ is always less than n. Furthermore, while the Fibonacci sequence is well-known and has been studied extensively in the literature, it seems that the sequence $S^k_{a,b}$ has not been studied before. The only reference we found about these sequences refers to $S^0_{0,1}$. This sequence is known as sequence A056542 in Sloane's Online Encyclopedia of Integer Sequences [12].

2 Irrationality

In order to give a proof for Theorem 1, we first need some lemmas.

Lemma 2. *Suppose that $a, b, k \in \mathbb{N}$ such that $0 \leq a < k, 0 \leq b < k+1$, $(a, b) \neq (0, 0)$. Then the sequence $S^k_{a,b}$ is not bounded.*

Proof. Suppose that $S^k_{a,b}$ is bounded for some a, b, k. Let $M = \max_{n\geq k}\{S^k_{a,b}(n)\}$. Then, for every $n > 2M$, we have $S^k_{a,b}(n) = S^k_{a,b}(n-1) + S^k_{a,b}(n-2)$, since $S^k_{a,b}(n-1) \leq M$ and $S^k_{a,b}(n-2) \leq M$. Thus, the sequence $S^k_{a,b}$ eventually coincides with a usual linear recurrence sequence taking non-negative values. Since $S^k_{a,b}$ is bounded it immediately follows that

$$S^k_{a,b}(n-1) = S^k_{a,b}(n-2) = 0.$$

By backward induction, we have $S^k_{a,b}(n) = 0$ for all n, which is a contradiction. This concludes the proof of the lemma. □

Lemma 3. *For any sufficiently large m, there exists n such that $S^k_{a,b}(n)$ has exactly m digits. In other words, there exists n such that $10^{m-1} \leq S^k_{a,b}(n) < 10^m$.*

Proof. From Lemma 2, the sequence $\{S^k_{a,b}(n)\}_{n\geq k}$ is unbounded. Hence there exists n such that $S^k_{a,b}(n) \geq 10^{m-1}$. We choose n as small as possible. Then $S^k_{a,b}(n-1), S^k_{a,b}(n-2) < 10^{m-1}$. This implies that

$$S^k_{a,b}(n) \leq S^k_{a,b}(n-1) + S^k_{a,b}(n-2) < 2 \times 10^{m-1} < 10^m.$$

This concludes the proof of the lemma. □

Using Lemma 2 and Lemma 3, we get the following proof of Theorem 1.

Proof. (of Theorem 1) Suppose that $\alpha^k_{a,b}$ is a rational number for some a, b, k. Then it has an eventually periodic decimal expansion. Thus we can write

$$\alpha^k_{a,b} = 0.a_1 \ldots a_s b_1 \ldots b_t b_1 \ldots b_t \ldots$$

We choose n large enough such that $S^k_{a,b}(n)$ starts from a position after a_s. Then for any $r \geq n$, the number $\alpha_r = S^k_{a,b}(r)S^k_{a,b}(r+1)S^k_{a,b}(r+2)\ldots$ is periodic of period wt for any positive integer w. We choose $m = vt$ for some large positive integer v such that $10^{m-1} > S^k_{a,b}(i)$ for all $i \leq n$. From Lemma 3, there exists l such that $S^k_{a,b}(l)$ has exactly m digits. We choose l to be as small as possible; then $l > n$.

If $S^k_{a,b}(l-1) = 0$, then $S^k_{a,b}(l-2) = S^k_{a,b}(l)$ has exactly m digits, which is a contradiction. Hence $0 < S^k_{a,b}(l-1) < 10^{m-1}$. Similarly, we have $0 < S^k_{a,b}(l-2) < 10^{m-1}$. Hence

$$\begin{aligned} S^k_{a,b}(l) &\leq S^k_{a,b}(l-2) + S^k_{a,b}(l-1) < 2 \times 10^{m-1}, \\ S^k_{a,b}(l+1) &\leq S^k_{a,b}(l-1) + S^k_{a,b}(l) < 3 \times 10^{m-1}. \end{aligned}$$

Therefore, $S^k_{a,b}(l+1)$ has no more than m digits. We have two separate cases.

1. Suppose that $S^k_{a,b}(l+1) \equiv S^k_{a,b}(l-1) + S^k_{a,b}(l) \pmod{l+1}$ has m digits. But $\alpha_l = S^k_{a,b}(l)S^k_{a,b}(l+1)S^k_{a,b}(l+2)\ldots$ is periodic of period $m = vt$ so $S^k_{a,b}(l+1) = S^k_{a,b}(l)$. This implies that $S^k_{a,b}(l-1) = 0$ which is a contradiction.

2. Suppose that $S^k_{a,b}(l+1) \equiv S^k_{a,b}(l-1) + S^k_{a,b}(l) \pmod{l+1}$ has less than m digits. Let $p = S^k_{a,b}(l+1)$. Since $\alpha_l = S^k_{a,b}(l)S^k_{a,b}(l+1)S^k_{a,b}(l+2)\ldots$ is periodic of period $m = vt$ so $S^k_{a,b}(l) = p * q$ for some q where $p * q$ denotes the concatenation of p and q. We have

   $$S^k_{a,b}(l+2) \leq S^k_{a,b}(l+1) + S^k_{a,b}(l) < 10^{m-1} + 2 \times 10^{m-1} < 3 \times 10^{m-1}.$$

   So $S^k_{a,b}(l+2)$ has no more than m digits. We have two subcases.

   (a) Suppose that $S^k_{a,b}(l+2)$ has exactly m digits. Then by the periodicity of α_l we have $S^k_{a,b}(l)S^k_{a,b}(l+1)S^k_{a,b}(l+2) = p * q * p * q * p$. If $S^k_{a,b}(l) + S^k_{a,b}(l+1) \geq l+2$ then

   $$S^k_{a,b}(l) + S^k_{a,b}(l+1) < l + 10^{m-1} < l + 2 + 10^{m-1},$$

which implies that $S^k_{a,b}(l+2) < 10^{m-1}$ which is a contradiction. Hence

$$S^k_{a,b}(l) + S^k_{a,b}(l+1) < l+2.$$

This implies that $q * p = S^k_{a,b}(l+2) = S^k_{a,b}(l) + S^k_{a,b}(l+1) = p * q + p$. Suppose that $p * q = 10^h p + q$ and $q * p = 10^z q + p$. Then $q(10^z - 1) = 10^h p$. Thus, $10^h \mid q$. But $p * q = 10^h p + q$ so $q < 10^h$. Hence $q = 0$ and $p = 0$ which is a contradiction.

(b) Suppose that $S^k_{a,b}(l+2)$ has less than m digits. Then we can replace k by $l+1$. And we choose l' to be the smallest $l' > l$ such that $S^l_{a,b}(l')$ has exactly m digits. Apply the above argument for the new sequence $S^l_{a,b}$ until either we come up with a contradiction or we can choose l' large enough such that $l'+1 > 3 \times 10^{m-1}$. But in this case

$$S^k_{a,b}(l'+1) \leq S^k_{a,b}(l'-1) + S^k_{a,b}(l') < 3 \times 10^{m-1} < l'+1.$$

So $S^k_{a+b}(l'+1)$ has exactly m digits. And we go to the case 1 which implies a contradiction.

This concludes the proof of the theorem. □

We close this section by an open question.

Open Problem 1. *For a, b, k are natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Is $\alpha^k_{a,b}$ an algebraic or transcendental number?*

3 Occurrence of zeros

By examining several sequences for small values of a, b and k, we notice a curious property of the sequence $S^k_{a,b}$: this sequence always contains many zeros. We are unable to prove this statement. Precisely, we propose the following conjecture.

Conjecture 4. *Let a, b, k be natural numbers with $0 \leq a < k, 0 \leq b < k+1$. Then the sequence $S^k_{a,b}$ contains infinitely many zero elements.*

Suppose that the sequence $S^k_{a,b}$ contains only finitely many zero elements for some a, b, k. Let v be the largest index such that $S^k_{a,b}(v) = 0$. Let $c = S^k_{a,b}(v+1)$ and $d = S^k_{a,b}(v+2)$. Then the sequence $S^{v+1}_{c,d}$ contains no zero element. Therefore the conjecture is equivalent to the statement "there exists n such that $S^k_{a,b}(n) = 0$ for any a, b, k".

If Conjecture 4 holds, let $v_k(a,b)$ be the index of the first zero element in sequence $S^k_{a,b}$. We define

$$v_k = \max_{0 \leq a < k, 0 \leq b < k+1} v_k(a,b).$$

For any $0 \leq a < k$ and $0 \leq b < k+1$ then $S^k_{a,b} = \{a\} \cup S^{k+1}_{b,c}$ for some $0 \leq c < k+2$. Thus, $v_k \leq v_{k+1}$ for any k. Furthermore, $v_{v_k+1} \geq v_k + 1 > v_k$ for any k. Hence

$$\lim_{k \to \infty} v_k = \infty.$$

Using computer, we computed some values of the sequence $\{v_k\}_{k \in \mathbb{N}}$

$$\{v_k\}_{k \geq 1} = \{28, 28, 108, 108, 130, 130, 184, 184, 184, 1523, 1523, \ldots\}.$$

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