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Unimodality of Certain Sequences Connected With Binomial Coefficients

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Abstract

This paper is devoted to the study of certain unimodal sequences related to binomial coefficients. Although the paramount purpose is to prove unimodality, in a few cases we even determine the maxima of the sequences. Our new results generalize some earlier theorems on unimodality. The proof techniques are quite varied.

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1 Introduction

Let $\{H_n\}_{n=0}^{\infty}$ denote a binary recurrence sequence defined by the initial values $H_0 \in \mathbb{R}$ and $H_1 \in \mathbb{R}$, not both zero, and the recurrence relation

$$H_n = AH_{n-1} + BH_{n-2}, \qquad (n \geq 2),$$

where the coefficients A and B are also real numbers. Moreover, let

$$h_{n,k} = A^{n-2k}B^k \binom{n-k}{k} \left(H_1 + \frac{k}{n-2k+1}AH_0\right), \qquad k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

The sequences $\{H_n\}_n$ and $\{h_{n,k}\}_k$ are strongly linked since

$$H_{n+1} = (n \bmod 2) \cdot B^{\frac{n+1}{2}} H_0 + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h_{n,k}, \quad (n \geq 0).$$

In what follows, for the natural numbers n and k we denote the floor $\left\lfloor \frac{n}{k} \right\rfloor$ by n_k.

For instance, if $A = B = H_1 = 1$ and $H_0 = 0$ then the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$, and the well-known $F_{n+1} = \sum_{k=0}^{n_2} \binom{n-k}{k}$ identity are obtained.

One of the purposes of this paper is to investigate the unimodality of the sequence $\{h_{n,k}\}_k$. A finite sequence of real numbers $\{a_k\}_{k=0}^{m}$ $(m \geq 1)$ is *unimodal* if there exists an integer $l \in \{0, \dots, m\}$ such that the subsequence $\{a_k\}_{k=0}^{l}$ increases, while $\{a_k\}_{k=l}^{m}$ decreases. If $a_0 \leq a_1 \leq \cdots \leq a_{l_0-1} < a_{l_0} = \cdots = a_{l_1} > a_{l_1+1} \geq \cdots \geq a_m$ then the integers $l_0, \dots, l_1$ are the *modes* of $\{a_k\}_{k=0}^{m}$. In case of $l_0 = l_1$ we talk about a *peak*, otherwise the set of modes is called a *plateau*. Naturally, these definitions can even be extended to infinite sequences. For positive sequences, unimodality is implied by strict log-concavity. A sequence $\{a_k\}_{k=0}^{m}$ is said to be *strictly log-concave* (SLC for short) if $a_l^2 > a_{l-1}a_{l+1}$, $1 \leq l \leq m-1$.

Theorems 2.1-2.3 discuss three classes of (1) and generalize most of the earlier results on unimodality of the sequences related to binomial coefficients. Generally, $\{h_{n,k}\}_k$ is not unimodal, not even if A and B are positive. For example, let $A = B = 1$, further $H_0 = -8$, $H_1 = 5$. Now

$$h_{8,k} = 5,\ 27,\ 27,\ -30,\ -27, \quad (k = 0, \dots, 4)$$

is not unimodal.

The first result dealing with unimodality of the elements of the Pascal triangle is due to Tanny & Zuker [3], who showed that $\binom{n-k}{k}$ $(k = 0, \dots, n_2)$ is unimodal. They [4, 5] also investigated the unimodality of $\binom{n-\alpha k}{k}$. In this work we also treat certain cases of the binomial sequence $\binom{n+\alpha k}{\beta k}$.

Benoumhani [2] proved the unimodality of the sequence $\frac{n}{n-k}\binom{n-k}{k}$ connected to Lucas numbers. Recently, Belbachir and Bencherif [1] proved that the sequences $2^{n-2k}\binom{n-k}{k}$ and $2^{n-2k}\frac{n}{n-k}\binom{n-k}{k}$ linked to Pell sequence and its companion sequence are unimodal. In all the aforesaid cases the authors descibe the peaks and the plateaus with two elements, the elements in which the monotonity changes.

2 Results

Theorem 2.1. *Suppose that A and B are given real numbers with $A > 0$, moreover $H_0 = 0$ and $H_1 = 1$. Assuming that $n \geq 2$, the sequence*

$$h_{n,k} = A^{n-2k} B^k \binom{n-k}{k}, \quad (k = 0, \dots, n_2)$$

is unimodal if and only if $B \geq 0$. In this case the peak $k = p_n$ of $\{h_{n,k}\}$ satisfies

$$p_n \in \left\{ \left\lfloor \frac{n}{2} \left(1 - \frac{1}{\sqrt{4c+1}} \right) \right\rfloor, \left\lceil \frac{n}{2} \left(1 - \frac{1}{\sqrt{4c+1}} \right) \right\rceil \right\},$$

and the plateau with two elements may occur at the places p_n and $p_n + 1$.

Obviously, $h_{n,k}$ cannot be unimodal with negative B, and trivially unimodal when $B = 0$. Considering A and B as two natural numbers, a combinatorial interpretation of $A^{n-2k}B^k\binom{n-k}{k}$ is the number of words formed with the letters $R, S_1, \dots, S_A, T_1, \dots, T_B$ of length n, beginning vith k consecutive R's and containing exactly k letters choosing among $\{T_1, \dots, T_B\}$.

A similar theorem to Theorem 2.1 is true for the companion sequence of H_n.

Theorem 2.2. *Let $A > 0$ and B denote real numbers, $H_0 = 2$, $H_1 = A$ and let $n \geq 2$ be an integer. The sequence*

$$h_{n-1,k} = A^{n-2k} B^k \frac{n}{n-k} \binom{n-k}{k}, \quad (k = 0, \dots, n_2)$$

is unimodal if and only if $B \geq 0$. The description of peaks and plateaus coincide as we have in Theorem 2.1.

The choice $H_0 = 0$ and $H_1 = 1$ makes the formula (1) as simple as possible. The following theorem does not fix the initial values, but only the coefficients.

Theorem 2.3. *Assume that the initial values H_0 and H_1 are positive, $A = B = 1$ and $n \geq 2$. Under these conditions the sequence*

$$h_{n,k} = \binom{n-k}{k} \left(H_1 + \frac{k}{n-2k+1} H_0 \right), \quad (k = 0, \dots, n_2)$$

is also unimodal.

Another direction is to investigate the unimodality of the sequence $\binom{n+\alpha k}{\beta k}$, where α and $\beta \geq 0$ are integers. If either α or β is zero then unimodality is trivial, as well as with $(\alpha, \beta) = (1, 1)$. The case $(\alpha, \beta) = (-1, 1)$ has been treated in [3], while $\alpha < -1$, $\beta = 1$ in [4] and [5].

The first pair worth considering is $(\alpha, \beta) = (-1, 2)$, when $h_{n,k} = \binom{n-k}{2k}$. Note that if $a_n = \sum_{k=0}^{n_3} \binom{n-k}{2k}$ then the terms of $\{a_n\}$ satisfy the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-4}$ with the initial values $a_0 = a_1 = a_2 = 1$ and $a_3 = 2$ (Sloane: A005251).

Theorem 2.4. *The sequence $h_{n,k} = \binom{n-k}{2k}$ is unimodal ($k = 0, \ldots, n_3$).*

A more interesting pair is $(\alpha, \beta) = (1, 2)$. Now $F_{2n+1} = \sum_{k=0}^{n} \binom{n+k}{2k}$ (Sloane: A001519). Note that the sequence $a_n = F_{2n+1}$ satisfies $a_n = 3a_{n-1} - a_{n-2}$ ($n \geq 2$).

Theorem 2.5. *The sequence $h_{n,k} = \binom{n+k}{2k}$ is unimodal ($k = 0, \ldots, n$). A plateau exists if and only if $n = (F_{12u+4} - 1)/2$ or $n = (F_{12u+8} - 1)/2$ with $k = (L_{12u+4} - 7)/10$ or $k = (L_{12u+8} - 7)/10$ ($u \in \mathbb{N}$), respectively.*

For numerical examples, see the following two tables.

u	F_{12u+4}	n	L_{12u+4}	k	$\binom{n+k}{2k} = \binom{n+k+1}{2k+2}$
0	3	1	7	0	$\binom{1}{0} = \binom{2}{2}$
1	987	493	2207	220	$\binom{713}{440} = \binom{714}{442}$
2	317811	158905	710647	71064	$\binom{229969}{142128} = \binom{229970}{142130}$

u	F_{12u+8}	n	L_{12u+8}	k	$\binom{n+k}{2k} = \binom{n+k+1}{2k+2}$
0	21	10	47	4	$\binom{14}{8} = \binom{15}{10}$
1	6765	3382	15127	1512	$\binom{4894}{3024} = \binom{4895}{3026}$
2	2178309	1089154	4870847	487084	$\binom{1576238}{974168} = \binom{1576239}{974170}$

Finally, let us consider the case $(1, \beta)$. For instance, if $\beta = 3$ then the numbers $\sum_{k=0}^{n_2} \binom{n+k}{3k}$ give every third term of the Padovan sequence $\{p_n\}$ determined by $p_n = p_{n-2} + p_{n-3}$ and $p_0 = 1$, $p_1 = p_2 = 0$ (Sloane: A003522). Theorem 2.6 partially generalizes Theorem 2.5.

Theorem 2.6. *The sequence $h_{n,k} = \binom{n+k}{\beta k}$ is unimodal ($2 \leq \beta \in \mathbb{N}$, $k = 0, \ldots, n_{\beta-1}$).*

The strict log-concavity is only utilized in the proof of Theorem 2.6, because it really simplifies the treatment. In the rest of the cases we start from the definition of unimodality, which is useful mainly where the peaks and plateaus are also aimed to determined. Finally, we draft two conjectures on Pascal triangles.

Conjecture 1. *Let $\binom{n}{k}$ be a fixed element of the Pascal triangle crossed by a ray. The sequence of binomial coefficents located along this ray is unimodal.*

Conjecture 2. *Take a generalized Pascal triangle corresponded to the Tribonacci sequence given by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ ($n \geq 3$) and $T_0 = T_1 = 0$, $T_2 = 1$. This triangle contains the elements*

$$\binom{n}{k}_2 = \sum_{i=m}^{\lfloor \frac{k}{2} \rfloor} \frac{n!}{i! \cdot (k-2i)! \cdot (n-k+i)!} = \sum_{i=m}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{n-i} \binom{n-i}{k-2i}$$

with $m = \max\{0\ ,\ k-n\}$. We conjecture that the sequence $\binom{n-k}{k}_2$ is unimodal.

3 Proofs

Proof of Theorem 2.1. There remains only the case $B > 0$ to consider. The inequality

$$A^{n-2k}B^k\binom{n-k}{k} \le A^{n-2(k+1)}B^{k+1}\binom{n-(k+1)}{k+1}, \qquad (k = 0, \dots, n_2 - 1)$$

is equivalent to

$$0 \le (4c+1)k^2 - (4c+1)kn + cn^2 + (2c+1)k - (c+1)n \tag{2}$$

with $c = B/A^2 > 0$. Multiplying (2) by $4(4c+1)$ it can be rewritten as

$$(4c+1)(n+1)^2 + 4c^2 \le \left((4c+1)(2k-n) + (2c+1)\right)^2. \tag{3}$$

It is easy to check that $(4c+1)(2k-n) + (2c+1) < 0$. Consequently, by (3) we have

$$k \le t_n = \frac{(4c+1)n - (2c+1) - \sqrt{(4c+1)(n+1)^2 + 4c^2}}{2(4c+1)}. \tag{4}$$

Put $t_n^\star = (n/2)(1 - 1/\sqrt{4c+1})$. From $-1 < t_n - t_n^\star < 0$ it follows that the peak p_n of $\{h_{n,k}\}$ is one of $\lfloor t_n^\star \rfloor$ and $\lceil t_n^\star \rceil$. When t_n is a natural number then $\{h_{n,k}\}$ possesses a plateau with two elements at $k = p_n$ and $k = p_n + 1$. Obviously, plateaus can be identified by the positive integer solutions x, y of the Pell equation $(4c+1)(x+1)^2 + 4c^2 = y^2$.

Remark 3.1. Formula (4) returns with

$$k \le \frac{5n - 3 - \sqrt{5n^2 + 10n + 9}}{10} \quad \text{and} \quad k \le \frac{4n - 3 - \sqrt{8n^2 + 16n + 9}}{8}$$

in the particular case of Fibonacci and Pell sequences (see [3] and [1]), respectively.

Proof of Theorem 2.2. Suppose that $B > 0$ and follow the steps of the proof of Theorem 2.1. Thus we can transform $h_{n,k} \le h_{n,k+1}$ into

$$0 \le (4c+1)k^2 - (4c+1)kn + cn^2 + (2c+2)k - (c+1)n + 1, \tag{5}$$

$(c = B/A^2)$ and by a suitable multiplication of (5) we have

$$(4c+1)n^2 + 4c(c-2) \le \left((4c+1)(2k-n) + (2c+2)\right)^2.$$

Hence

$$k \le \frac{(4c+1)n - (2c+2) - \sqrt{(4c+1)n^2 + 4c(c-2)}}{2(4c+1)}. \tag{6}$$

Now we find again that p_n is $\lfloor t_n^\star \rfloor$ or $\lceil t_n^\star \rceil$ with the same $t_n^\star$ we had in Theorem 2.1. For plateaus the diophantine equation $(4c+1)x^2 + 4c(c-2) = y^2$ should be investigated.

Remark 3.2. The inequality (6) provides formulae

$$k \leq \frac{5n-4-\sqrt{5n^2-4}}{10} \quad \text{and} \quad k \leq \frac{4n-5-\sqrt{8n^2-7}}{8}$$

in case of Lucas numbers and the companion sequence of Pell numbers (see [2] and [1]), respectively.

Proof of Theorem 2.3. Now $A = B = 1$ and the inequality $h_{n,k} \leq h_{n,k+1}$ provides

$$\begin{aligned}(n-k)(k+1)\left(H_1(n-2k+1)+H_0k\right) &\leq \\ (n-2k+1)(n-2k)\left(H_1(n-2k-1)+H_0(k+1)\right),&\end{aligned}$$

which is equivalent to $0 \leq f_n(k) = \sum_{i=0}^{3} e_i(n)k^i$, where

$$\begin{aligned} e_0(n) &= n^3+(h-1)n^2+(h-2)n, \\ e_1(n) &= (h-7)n^2+(-4h+2)n+(-2h+3), \\ e_2(n) &= (-5h+15)n+(3h-1), \\ e_3(n) &= 5h-10, \end{aligned}$$

and $h = H_0/H_1 > 0$. The special case $h = 2$ has essentially been treated by Benoumhani [2]. Therefore, by the sign of $e_3(n)$, we must distinguish two cases.

Firstly, we assume that $h > 2$, which entails positive leading coefficient in the polynomial $f_n(k)$ of degree three. Since $f_n(0) = e_0(n) > 0$ and $f_n(n/2) = -(hn^3+(2h+2)n^2+4n)/8 < 0$, it follows that $f_n(k)$ possesses exactly one zero in the interval $[0; n/2]$. Hence $h_{n,k}$ is unimodal.

Now, the assertion $0 < h < 2$ implies $e_3(n) < 0$. The reader can easily verify that $f_n(0) = n(n^2+(h-1)n+(h-2))$ is positive if $n \geq 2$ and $f_n(n/2) < 0$ as it has already been occurred previously. But the negative leading coefficient $e_3(n)$ causes ambiguous structure for the zeros of $f_n(k)$. Therefore a deeper analysis is necessary to clarify the situation.

The polynomial

$$\begin{aligned} f_n'(k) = (15h-30)k^2 + \left((-10h+30)n+(6h-2)\right)k+ \\ + \big((h-7)n^2+(-4h+2)n+(-2h+3)\big) \end{aligned}$$

has two distinct real zeros because its discriminant

$$D = 20(2h^2-3h+3)n^2+40(3h^2-5h+3)n+4(39h^2-111h+91)$$

is positive when $0 < h < 2$. Indeed, $2h^2-3h+3 > 0$ and the discriminant

$$D_1 = -960(h-2)^2(11h^2-19h+19)$$

of D takes negative values on the interval $]0, 2[$.

To finish the proof it is sufficient to show that the larger real zero f_2 of $f'_n(k)$ satisfies $n/2 < f_2$, more exactly

$$f_2 = \frac{(10h-30)n + (2-6h) - \sqrt{D}}{30h-60} > \frac{n}{2}.$$

To do this it is enough to see that $2 - 6h - 5hn < \sqrt{D}$, which is trivially true when $h \geq 1/3$. Contrary, if $h < 1/3$, it suffices to confirm

$$0 < (-h+2)n^2 + (-4h+4)n + (-8h+12). \tag{7}$$

And (7) is fulfilled since its right hand side has no real zero under the given conditions.

Proof of Theorem 2.4. Starting with $\binom{n-k}{2k} \leq \binom{n-k-1}{2k+2}$, we obtain

$$0 \leq f(k) = -23k^3 + (23n-21)k^2 + (-9n^2+12n-4)k + (n^3-3n^2).$$

Since $f(0) = n^2(n-3) \geq 0$ if $n \geq 3$, further $f(n/3) = -4n(n+3)(2n+3)/27 < 0$, it is sufficient to verify that $f(k)$ is strictly monotone decreasing. And, really, if $n \geq 3$ then $f'(k) \neq 0$ implies the required monotonity.

Proof of Theorem 2.5. Assuming $h_{n,k} \leq h_{n,k+1}$, it provides

$$0 \leq f(k) = -5k^2 - 7k + (n^2+n-2).$$

Since $k = -0.7$ is the only solution of $f'(k) = 0$, it follows that $f(k)$ is strictly decreasing in the interval $[0; n-1]$. Further, the zeros of the polynomial $f(k)$ are

$$f_{1,2} = \frac{7 \pm \sqrt{20n^2+20n+9}}{-10}, \tag{8}$$

the smaller f_1 is negative, the larger f_2 is approximately $0.45n$ since

$$f\left(\frac{n}{\sqrt{5}} - 1\right) \cdot f\left(\frac{n}{\sqrt{5}}\right) = -\frac{4}{5}(4+\sqrt{5})n^2 - \frac{2}{5}(5+3\sqrt{5})n < 0$$

implies $n/\sqrt{5} - 1 < f_2 < n/\sqrt{5}$. Thus $h_{n,k}$ is unimodal, and its smallest mode $k_n = \lceil f_2 \rceil$ satisfies $\lfloor n/\sqrt{5} \rfloor \leq k_n \leq \lceil n/\sqrt{5} \rceil$.

Now we inspect the existence of plateaus. By (8), we need to solve the Pell equation $20x^2 + 20x + 9 = y^2$. It is well known that

$$y^2 - 5(2x+1)^2 = 4$$

implies $y = L_{2v}$ and $2x + 1 = F_{2v}$. A Fibonacci number with even suffix is odd if and only if 3 does not divide v. Put $v = 3u \pm 1$. A simple verification shows that $L_{6u\pm2} \equiv \pm 7 \pmod{10}$ depending on the parity of u. Choosing the appropriate cases we obtain f_2 as positive integer.

Remark 3.3. Another proof can be obtained by introducing the function $G_n(x) = \sum_{0\le k\le n} \binom{n+k}{2k} x^k$, for which

$$G_n(x) = \frac{1}{x^n\sqrt{x^2+4x}}\left(\left(\frac{x+\sqrt{x^2+4x}}{2}\right)^{2n+1} - \left(\frac{x-\sqrt{x^2+4x}}{2}\right)^{2n+1}\right).$$

It suffices to see that

$$G_n(x) = (x+2)\,G_{n-1}(x) - G_{n-2}(x)\,,\ G_0(x) = 1,\ G_1(x) = x+1.$$

We deduce that $G_n(x)$ admits n distinct real zeros given by

$$x_k = -2\left(1+\cos\frac{2k\pi}{2n+1}\right),\ k = 1,2,\dots,n,$$

and then the sequence $\binom{n+k}{2k}$ is SLC, which gives unimodality with a peak or a plateau with two elements.

For $m \ge 3$ let denote $\{\alpha_m\}_{m\ge -1}$ and $\{\beta_m\}_{m\ge -1}$ sequences defined by

$$\begin{cases} (\alpha_{-1},\alpha_0,\alpha_1,\alpha_2) = (-11,-2,1,10) \text{ and } \alpha_m = 322\alpha_{m-2} - \alpha_{m-4} + 160, \\ (\beta_{-1},\beta_0,\beta_1,\beta_2) = (4,0,0,4) \quad \text{ and } \beta_m = 322\beta_{m-2} - \beta_{m-4} + 224. \end{cases}$$

The integers n for which the sequences $\binom{n+k}{2k}$, $(k = 0,1,\dots,n)$ admit a plateau with two elements $\{k_n, k_n+1\}$ are exactly the integers α_s, $s \ge 1$ such that $k_{\alpha_s} = \beta_s$ for $s \ge 1$.

s	$n = \alpha_s$	$k_n = \beta_s$
1	1	0
2	10	4
3	493	220
4	3382	1512
5	158905	71064
6	1089154	487084

Proof of Theorem 2.6 We show that $h_{n,k}$ is strictly log-concave, or more precisely

$$\binom{n+k}{\beta k}^2 > \binom{n+k-1}{\beta k-\beta}\binom{n+k+1}{\beta k+\beta}. \tag{9}$$

Put $b = \beta - 1 \ge 1$. Thus (9) is equivalent to

$$\frac{(n+k)(n-bk+1)}{(n+k+1)(n-bk)} \cdot \prod_{i=0}^{\beta-1}\left(1+\frac{\beta}{\beta k-i}\right) \cdot \prod_{j=1}^{b-1}\left(1+\frac{b+1}{n-bk-j}\right) > 1.$$

Since $(n+k)(n-bk+1) > (n+k+1)(n-bk) > 0$, further the other multipliers are trivially greater than 1, the statement is proved.

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