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Integer Partitions and Convexity

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Abstract

Let n be an integer ≥ 1, and let $p(n,k)$ and $P(n,k)$ count the number of partitions of n into k parts, and the number of partitions of n into parts less than or equal to k, respectively. In this paper, we show that these functions are convex. The result includes the actual value of the constant of Bateman and Erdős.

1 Introduction

The k^{th} difference $\Delta^k f$ of any function f of the nonnegative integers is defined recursively by $\Delta^k f = \Delta(\Delta^{k-1} f)$, with $\Delta f(n) = f(n) - f(n-1)$ for $n \geq 1$ and $\Delta f(0) = f(0)$. Good [5] studied the behavior of $\Delta^k p(n)$, where $p(n)$ denotes the total number of partitions of n. He initially conjectured [5] that if $k > 3$, then the sequence $\Delta^k p(n)$, $n \geq 0$ alternates in sign. However, computations by Razen, and, independently, by Good [5], found counterexamples to this conjecture, and led to a new conjecture, namely that $\Delta^k p(n) > 0$ for each fixed k. Good [5] even made a stronger conjecture that for each k, there is an $n_0(k)$ such that $\Delta^k p(n)$ alternates in sign for $n < n_0(k)$, and $\Delta^k p(n) \geq 0$ for $n \geq n_0(k)$. He also suggested that $6(k-1)(k-2) + k^3/2$ might be a good approximation to $n_0(k)$. Some further computations by Gaskin led Good to revise his conjecture about the size of $n_0(k)$, and suggest that $\pi k^{5/2}$ might be a good approximation to it [6].

At about the same time as the first publication of Good's problem, the same question about the sign of $\Delta^k p(n)$ was also raised independently by Andrews, and was answered by Gupta [7]. Gupta noted that $\Delta p(n) > 0$ for all n, and gave a simple proof of the result that $\Delta^2 p(n) \geq 0$ for $n \geq 2$, while $\Delta^2 p(0) = 1$, $\Delta^2 p(1) = -1$; in other words, he showed that the function $p(n)$ is convex for $n \geq 2$.

Another easy proof that $\Delta^k p(n)$ is positive for large n can be obtained by applying the result of the theorem of Beteman and Erdős [2]. They showed that if $p(\mathcal{A}, n)$ is the number of partitions of n into parts taken from $\mathcal{A} \subset \{1, 2, 3, \ldots\}$, then $\Delta^k p(\mathcal{A}, n) \geq 0$ for all n large enough iff the greatest common divisor of each subset $\mathcal{B} \subseteq \mathcal{A}$ with $\mid \mathcal{A} \setminus \mathcal{B} \mid= k$ is equal to 1. In particular, the theorem of Beteman and Erdős asserts that there is $n_0 = n_0(\mathcal{A})$ such that the function $p(\mathcal{A}, n)$ is convex for $n \geq n_0$ iff for all pairs $\{a, b\}$ of $\mathcal{A}$, $\gcd(\mathcal{A} \setminus \{a, b\}) = 1$.

For more historical details see [8]. The aim of this paper is to give the actual form of this result when $\mathcal{A} = \{1, 2, \ldots, k\}$.

2 Definitions and notation

A *partition* of an integer n into k parts $(1 \leq k \leq n)$ is an integer solution of the system:

$$\begin{cases} n = a_1 + 2a_2 + \cdots + na_n, \\ k = a_1 + a_2 + \cdots + a_n, \\ a_i \geq 0, \ i = 1, \ldots, n, \end{cases} \tag{1}$$

where a_i counts the number of parts i.

Thus, a partition of n into parts less than or equal to k is an integer solution of the following system:

$$\{ \ n = a_1 + 2a_2 + \cdots + ka_k, a_i \geq 0, \ i = 1, \ldots, k. \tag{2}$$

Let $p(n)$, $p(n, k)$ and $P(n, k)$ be respectively the total number of partitions of n, the number of partitions of n into exactly k parts and the number of partitions of n into parts less than or equal to k. According to Bouroubi [3] and Comtet [4], we have

$$p(n) = P(n, n), \tag{3}$$

$$p(n, k) = p(n-1, k-1) + p(n-k, k), \tag{4}$$

$$p(n, k) = P(n-k, k), \tag{5}$$

and

$$P(n, k) = P(n, k-1) + P(n-k, k). \tag{6}$$

3 Convexity of the functions $(P(n,k))_n$ and $(p(n,k))_n$

Theorem 1. *The function $P(n,k)$ is convex for $n \geq 2$ and $k \geq 7$.*

Proof. Setting,

$$\gamma(n,k) = P(n,k) + P(n-2,k) - 2P(n-1,k).$$

First we note that if $n \leq k$ then

$$\gamma(n,k) = P(n,n) + P(n-2,n-2) - 2P(n-1,n-1).$$

From (3), we get $\gamma(n,k) = p(n) + p(n-2) - 2p(n-1) > 0$.
Suppose now $n > k$, since $\gamma(7,6) = \gamma(13,6) = -1$, let us show by mathematical induction on k that $\gamma(n,k)$ is positive for every n, $n > k \geq 7$. For that we consider g_k the generating function of $P(n,k)$ [4], i.e.,

$$g_k(z) = \frac{1}{(1-z)\cdots(1-z^k)}, \ \mid z \mid < 1.$$

Thus, the generating function of $\gamma(n,k)$ equals

$$h_k(z) = \frac{(1-z)^2}{\prod_{i=1}^{k}(1-z^i)}.$$

Hence

$$h_k(z) = \frac{1}{1-z^k}\ h_{k-1}(z).$$

Consequently

$$\gamma(n,k) = \sum_{j=0}^{n} \alpha(j,k)\ \gamma(n-j,k-1),$$

where $\alpha(j,k) = 1$ if k divides j and $\alpha(j,k) = 0$ otherwise.

Now let us show that $\gamma(n,7) \geq 0$ for every $n \geq 8$.
By the decomposition of the rational function of $h_7(z)$ into partial fractions, we get

$$\begin{aligned} h_7(z) \quad = \quad & \tfrac{1}{5040}\tfrac{1}{(1-z)^5} + \tfrac{1}{480}\tfrac{1}{(1-z)^4} + \tfrac{47}{4320}\tfrac{1}{(1-z)^3} + \tfrac{161}{4320}\tfrac{1}{(1-z)^2} + \tfrac{16051}{172800}\tfrac{1}{1-z} + \\ & + \tfrac{1}{192}\tfrac{1}{(1+z)^3} + \tfrac{23}{384}\tfrac{1}{(1+z)^2} + \tfrac{713}{2304}\tfrac{1}{1+z} + \tfrac{1}{7}\tfrac{(1-z)^2}{1-z^7} + \tfrac{1}{108}\tfrac{(21-2z)(1-z)}{1-z^3} + \\ & + \tfrac{1}{54}\tfrac{(2+z)(1-z)^2}{(1-z^3)^2} + \tfrac{1}{36}\tfrac{(1-2z)(1+z)}{1+z^3} + \tfrac{1}{25}\tfrac{(2-z+z^2-2z^3)(1-z)}{1-z^5} - \tfrac{1}{16}\tfrac{z}{1+z^2}. \end{aligned}$$

By taking lower bounds of each of the coefficients of z^n for the power series expansions of the above functions we find:

$$\begin{aligned}\gamma(n,7) \geq\ & \tfrac{1}{5040}\left(\tfrac{1}{24}n^4+\tfrac{5}{12}n^3+\tfrac{35}{24}n^2+\tfrac{25}{12}n+1\right)+\tfrac{1}{480}\left(\tfrac{1}{6}n^3+n^2+\tfrac{11}{6}n+1\right)+\\ &+\tfrac{47}{4320}\left(\tfrac{1}{2}n^2+\tfrac{3}{2}n+1\right)+\tfrac{161}{4320}(n+1)+\tfrac{16051}{172800}-\tfrac{1}{192}(\tfrac{1}{2}n^2+\tfrac{3}{2}n+1)-\\ &-\tfrac{23}{384}(n+1)-\tfrac{713}{2304}-\tfrac{2}{7}-\tfrac{23}{108}-\tfrac{1}{54}(n+2)-\tfrac{1}{18}+\tfrac{2}{25}+\tfrac{1}{16}\cdot\end{aligned}$$

i.e.,

$$\begin{aligned}\gamma(n,7) &\geq \tfrac{1}{120960}n^4+\tfrac{13}{30240}n^3+\tfrac{1}{192}n^2-\tfrac{859}{30240}n-\tfrac{16451}{24192}\\ &= 0.8267195767\ 10^{-5}\times(n+30.63520805)\times(n-9.699836835)\\ &\times\ (n^2+31.064628784\ n+276.8069841)\cdot\end{aligned}$$

Hence

$$\gamma(n,7)\geq 0, \forall n\geq 10.$$

For $n\in\{8,9\}$, we have

$$\gamma(8,7)=2\ ;\ \gamma(9,7)=1.$$

Suppose now that $\gamma(n,j)\geq 0$, for $7\leq j\leq k-1$ and show that $\gamma(n,k)\geq 0$.

On the one hand, we have

$$\begin{aligned}\gamma(n,k) &= \alpha(n,k)-\alpha(n-1,k)+\alpha(n-k-1,k)\ \gamma(k+1,k-1)+\\ &+\sum_{j=0:j\neq n-k-1}^{n-2}\alpha(j,k)\ \gamma(n-j,k-1).\end{aligned}$$

Hence by the induction assumption, we get

$$\gamma(n,k)\geq\alpha(n,k)-\alpha(n-1,k)+\alpha(n-k-1,k)\ \gamma(k+1,k-1).$$

On the other hand from (6), we have

$$\gamma(n,k)=\gamma(n,k-1)+\gamma(n-k,k).$$

Therefore

$$\gamma(k+1,k-1)=\gamma(k+1,k-2)+\gamma(2,k-1)=1+\gamma(k+1,k-2).$$

- if $k-2\geq 7$ then $\gamma(k+1,k-2)\geq 0$, by the induction assumption.

- if $k-2=6$ then $\gamma(k+1,k-2)=\gamma(9,6)=0$.

Consequently

$$\gamma(n,k) \geq \alpha(n,k) - \alpha(n-1,k) + \alpha(n-k-1,k) \geq 0.$$

Indeed
- if k divides n then $\alpha(n,k) - \alpha(n-1,k) + \alpha(n-k-1,k) = 1$,
- if k divides $n-1$ then $\alpha(n,k) - \alpha(n-1,k) + \alpha(n-k-1,k) = 0$,
- if k divides neither n nor $n-1$ then $\alpha(n,k) - \alpha(n-1,k) + \alpha(n-k-1,k) = 0$. □

Corollary 2. *The function $p(n,k)$ is convex for $n \geq k+2$ and $k \geq 7$.*

Proof. Using (5), we have

$$p(n,k) + p(n-2,k) - 2p(n-1,k) = P(n-k,k) + P(n-k-2,k) - 2P(n-k-1,k),$$

and the result follows immediately, using Theorem 1. □

Remark 3. *Using the same method we can show that the function $P(n,5)$ and $P(n,6)$ are convex for $n \geq 2$ and $n \geq 14$ respectively. We give below the value of $\gamma(n,5)$ and $\gamma(n,6)$, for $0 \leq n \leq 20$.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\gamma(n,5)$	1	-1	1	0	1	0	1	0	2	0	2	0	3	2	3	1	3	1	4	1	5
$\gamma(n,6)$	1	-1	1	0	1	0	2	-1	3	0	3	0	5	-1	6	1	6	1	9	0	11

Table 1: The value of $\gamma(n,5)$ and $\gamma(n,6)$, for $0 \leq n \leq 20$.

4 Conclusion

Let $\mathcal{A} = \{1, 2, \ldots, k\}$, $k \geq 2$. In this paper we showed that the partition function $P(\mathcal{A}, n)$ is convex for $k \geq 5$ and the constant of Bateman and Erdős, $n_0(\mathcal{A})$ equals 2 if $k=5$ or $k \geq 7$, however for $\mathcal{A} = \{1,2,3,4,5,6\}$, $n_0(\mathcal{A}) = 14$.

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