Journal of Integer Sequences, Vol. 10 (2007),
Article 07.4.6

Some Observations on the Lah and Laguerre Transforms of Integer Sequences

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

We study the Lah and related Laguerre transforms within the context of exponential Riordan arrays. Links to the Stirling numbers are explored. Results for finite matrices are generalized, leading to a number of useful matrix factorizations.

1 Integer sequences and transforms on them

In this note, we shall consider integer sequences

$$a : \mathbf{N}_0 \to \mathbf{Z}$$

with general term $a_n = a(n)$. Normally, sequences will be described by either by their ordinary generating function (o.g.f.), that is, the function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} a_n x^n,$$

or by their exponential generating function (e.g.f.) $f(x)$, where

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

We shall encounter transformations that operate on integer sequences during the course of this note. An example of such a transformation that is widely used in the study of integer sequences is the so-called Binomial transform [14], which associates to the sequence with general term a_n the sequence with general term b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \tag{1}$$

If we consider the sequence with general term a_n to be the column vector $\mathbf{a} = (a_0, a_1, \ldots)'$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector by the lower-triangle matrix $\mathbf{B}$ whose (n, k)-th element is equal to $\binom{n}{k}$:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{2}$$

We note that $\mathbf{B}$ corresponds to Pascal's triangle. Its row sums are 2^n, while its diagonal sums are the Fibonacci numbers $F(n + 1)$. If $\mathbf{B}^m$ denotes the $m-$th power of $\mathbf{B}$, then the $n-$th term of $\mathbf{B}^m\mathbf{a}$ where $\mathbf{a} = \{a_n\}$ is given by $\sum_{k=0}^{n} m^{n-k} \binom{n}{k} a_k$.

If $\mathcal{A}(x)$ is the ordinary generating function of the sequence a_n, then the ordinary generating function of the transformed sequence b_n is $\frac{1}{1-x}\mathcal{A}(\frac{x}{1-x})$. Similarly, if $\mathcal{G}(x)$ is the exponential generating function (e.g.f.) of the sequence a_n, then the exponential generating function of the binomial transform of a_n is $\exp(x)\mathcal{G}(x)$.

The binomial transform is an element of the exponential Riordan group, which can be defined as follows.

The *exponential Riordan group* [2], [3], [4], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \ldots$ and $f(x) = f_1x + f_2x^2 + \ldots$ where $f_1 \neq 0$. The associated matrix is the matrix whose k-th column has exponential generating function $g(x)f(x)^k/k!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. The group law is then given by

$$[g, f] * [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{a} = \{a_n\}$ is an integer sequence with exponential generating function $\mathcal{A}\ (x)$, then the sequence $\mathbf{Ma}$ has exponential generating function $g(x)\mathcal{A}(f(x))$.

We shall use the notation $(g(x), f(x))$ to denote an element of the (ordinary) Riordan group [5]. Riordan group techniques have been used to provide alternative proofs of many binomial identities that originally appeared in works such as [8], [9]. See for instance [12], [13].

Example 1. The Binomial matrix $\mathbf{B}$ is the element $[e^x, x]$ of the exponential Riordan group. More generally, $\mathbf{B}^k$ is the element $[e^{kx}, x]$ of the exponential Riordan group. It is easy to show that the inverse $\mathbf{B}^{-k}$ of $\mathbf{B}^k$ is given by $[e^{-kx}, x]$. Note that as an element of the (ordinary) Riordan group, $\mathbf{B}$ is given by $(\frac{1}{1-x}, \frac{x}{1-x})$. Similarly $\mathbf{B}^k$ is the element $(\frac{1}{1-kx}, \frac{x}{1-kx})$ of the Riordan group.

Example 2. The exponential generating function of the row sums of the matrix $[g, f]$ is obtained by applying $[g, f]$ to e^x, the e.g.f. of the sequence $1, 1, 1, \ldots$. Hence the row sums of $[g, f]$ have e.g.f. $g(x)e^{f(x)}$.

We shall frequently refer to sequences by their sequence number in the On-Line Encyclopedia of Integer Sequences [10], [11]. For instance, Pascal's triangle is A007318 while the Fibonacci numbers $F(n)$ are A000045.

Example 3. The Permutation matrix P and its inverse. We consider the matrix

$$\mathbf{P} = [\frac{1}{1-x}, x].$$

The general term $P(n, k)$ of this matrix is easily found:

$$\begin{aligned} P(n,k) &= \frac{n!}{k!}[x^n]\frac{x^k}{1-x} \\ &= \frac{n!}{k!}[x^{n-k}]\frac{1}{1-x} \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty} x^j \\ &= \frac{n!}{k!} \quad \text{if} \quad n-k \geq 0, \quad = 0, \quad \text{otherwise}, \\ &= [k \leq n]\frac{n!}{k!}. \end{aligned}$$

Here, we have used the Iverson bracket notation [6], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false. For instance, $\delta_{ij} = [i = j]$, while $\delta_n = [n = 0]$.

Clearly, the inverse of this matrix is $\mathbf{P}^{-1} = [(1 - x), x]$. The general term of this matrix is given by

$$\begin{aligned} P^{-1}(n,k) &= \frac{n!}{k!}[x^n](1-x)x^k \\ &= \frac{n!}{k!}[x^{n-k}](1-x) \\ &= \frac{n!}{k!}(\delta_{n-k} - \delta_{n-k-1}) \\ &= \delta_{n-k} - (k+1)\delta_{n-k-1} \end{aligned}$$

Thus

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2 The Lah transform

Introduced by Jovovic (see, for instance, A103194), the Lah transform is the transformation on integer sequences whose matrix is given by

$$\mathbf{Lah} = [1, \frac{x}{1-x}].$$

Properties of the matrix obtained from the $n \times n$ principal sub-matrix of **Lah**, and related matrices have been studied in [7]. From the above definition, we see that the matrix **Lah** has general term $\mathrm{Lah}(n,k)$ given by

$$\begin{aligned} \mathrm{Lah}(n,k) &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\ &= \frac{n!}{k!}[x^{n-k}]\sum j\binom{k+j-1}{j}x^j \\ &= \frac{n!}{k!}\binom{n-1}{n-k} \end{aligned}$$

Thus if b_n is the Lah transform of the sequence a_n, we have

$$b_n = \sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}a_k.$$

It is clear that the inverse of this matrix $\mathbf{Lah}^{-1}$ is given by $[1, \frac{x}{1+x}]$ with general term $\text{Lah}(n,k)(-1)^{n-k}$. Thus

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \binom{n-1}{n-k} b_k.$$

Numerically, we have

$$\mathbf{Lah} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 6 & 6 & 1 & 0 & 0 & \ldots \\ 0 & 24 & 36 & 12 & 1 & 0 & \ldots \\ 0 & 120 & 240 & 120 & 20 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Operating on the sequence with e.g.f. $f(x)$, it produces the sequence with e.g.f. $f(\frac{x}{1-x})$.

Example 4. The row sums of the matrix **Lah**, obtained by operating on the sequence $1, 1, 1 \ldots$ with e.g.f. e^x, is the sequence $1, 1, 3, 13, 73, 501, \ldots$ (A000262) with e.g.f. $e^{\frac{x}{1-x}}$. We observe that this is $n!L(n, -1, -1) = n!L_n^{(-1)}(-1)$ (see Appendix for notation). This sequence counts the number of partitions of $\{1, .., n\}$ into any number of lists, where a list means an ordered subset.

3 The generalized Lah transform

Extending the definition in [7], we can define, for the parameter t, the generalized Lah matrix

$$\mathbf{Lah}[t] = [1, \frac{x}{1-tx}].$$

It is immediate that $\mathbf{Lah}[0] = [1, x] = \mathbf{I}$, and $\mathbf{Lah}[1] = \mathbf{Lah}$. The general term of the matrix $\mathbf{Lah}[t]$ is easily computed:

$$\begin{aligned} \text{Lah}[t](n,k) &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-tx)^k} \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j t^j x^j \\ &= \frac{n!}{k!}[x^{n-k}]\sum j \binom{k+j-1}{j} t^j x^j \\ &= \frac{n!}{k!}\binom{n-1}{n-k} t^{n-k}. \end{aligned}$$

We can easily establish that $\mathbf{Lah}[t]^{-1} = [1, \frac{x}{1+tx}] = \mathbf{Lah}[-t]$ with general term $\frac{n!}{k!}\binom{n-1}{n-k}t^{n-k}(-1)^{n-k}$. We also have

$$\mathbf{Lah}[u+v] = \mathbf{Lah}[u] \cdot \mathbf{Lah}[v].$$

This follows since

$$\begin{aligned}
\mathbf{Lah}[u]\cdot\mathbf{Lah}[v] &= [1, \frac{x}{1-ux}][1, \frac{x}{1-vx}] \\
&= [1, \frac{\frac{x}{1-vx}}{1-\frac{ux}{1-vx}}] \\
&= [1, \frac{\frac{x}{1-vx}}{\frac{1-vx-ux}{1-vx}}] \\
&= [1, \frac{x}{1-(u+v)x}] \\
&= \mathbf{Lah}[u+v].
\end{aligned}$$

For integer m, it follows that

$$\mathbf{Lah}[mt] = (\mathbf{Lah}[t])^m.$$

4 Laguerre related transforms

In this section, we will define the Laguerre transform on integer sequences to be the transform with matrix given by

$$\mathbf{Lag} = [\frac{1}{1-x}, \frac{x}{1-x}].$$

We favour this denomination through analogy with the Binomial transform, whose matrix is given by

$$(\frac{1}{1-x}, \frac{x}{1-x}).$$

Numerically, we have

$$\mathbf{Lag} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The inverse of the Laguerre transform, as we understand it in this section, is given by

$$\mathbf{Lag}^{-1} = [\frac{1}{1+x}, \frac{x}{1+x}].$$

Clearly, the general term $\mathrm{Lag}(n,k)$ of the matrix $\mathbf{Lag}$ is given by

$$\mathrm{Lag}(n,k) = \frac{n!}{k!}\binom{n}{k}.$$

Thus if b_n is the Laguerre transform of the sequence a_n, we have

$$b_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} a_k.$$

The e.g.f. of b_n is given by $\frac{1}{1-x} f(\frac{x}{1-x})$ where $f(x)$ is the e.g.f. of a_n. The inverse matrix of **Lag** has general term given by $(-1)^{n-k} \frac{n!}{k!} \binom{n}{k}$. Thus

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \binom{n}{k} b_k.$$

The relationship between the Lah transform with matrix **Lah** and the Laguerre transform with matrix **Lag** is now clear:

$$\begin{aligned} \mathbf{Lag} &= [\frac{1}{1-x}, \frac{x}{1-x}] \\ &= [\frac{1}{1-x}, x][1, \frac{x}{1-x}] \\ &= \mathbf{P} \cdot \mathbf{Lah} \end{aligned}$$

We note that this implies that

$$\begin{aligned} \mathrm{Lag}(n,k) &= \frac{n!}{k!} \binom{n}{k} \\ &= \sum_{i=0}^{n} [i \le n] \frac{n!}{i!} \frac{i!}{k!} \binom{i-1}{i-k} \\ &= \sum_{i=0}^{n} [k \le n] \frac{n!}{k!} \binom{i-1}{i-k} \\ &= \frac{n!}{k!} \sum_{i=0}^{n} \binom{i-1}{i-k} \end{aligned}$$

which indeed is true since

$$\binom{n}{k} = \sum_{i=0}^{n} \binom{i-1}{i-k}.$$

Numerically, we have

$$\begin{aligned}
\mathbf{P}\cdot\mathbf{Lah} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 6 & 6 & 1 & 0 & 0 & \dots \\ 0 & 24 & 36 & 12 & 1 & 0 & \dots \\ 0 & 120 & 240 & 120 & 20 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \mathbf{Lag}
\end{aligned}$$

Similarly we have

$$\mathbf{Lag}^{-1} = \mathbf{Lah}^{-1}\cdot\mathbf{P}^{-1},$$

which implies that

$$\begin{aligned}
\mathrm{Lag}^{-1}(n,k) &= (-1)^{n-k}\frac{n!}{k!}\binom{n}{k} \\
&= \sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}\binom{n-1}{n-i}(\delta_{i-k}-(k+1)\delta_{i-k-1}).
\end{aligned}$$

It is of course possible to pass from **Lag** to **Lah** by:

$$\mathbf{Lah} = \mathbf{P}^{-1}\cdot\mathbf{Lag}.$$

Thus

$$\begin{aligned}
\mathrm{Lah}(n,k) &= \frac{n!}{k!}\binom{n-1}{n-k} \\
&= \sum_{i=0}^{n}(\delta_{n-i}-(i+1)\delta_{n-i-1})\frac{i!}{k!}\binom{i}{k}
\end{aligned}$$

We note in passing that this gives us the identity

$$n!\binom{n-1}{n-k} = \sum_{i=0}^{n}(\delta_{n-i}-(i+1)\delta_{n-i-1})i!\binom{i}{k}.$$

Numerically, we have

$$\mathbf{P}^{-1}\cdot\mathbf{Lag} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -4 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 24 & 36 & 12 & 1 & 0 & \cdots \\ 0 & 120 & 240 & 120 & 20 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \mathbf{Lah}$$

Thus if the Laguerre transform of a_n has general term b_n, then the general term of the Lah transform of a_n will be given by

$$b_n - nb_{n-1}$$

(for $n > 0$).

Example 5. The row sums of the matrix **Lag**, that is, the transform of the sequence $1, 1, 1, \ldots$ with e.g.f. e^x, is the sequence $1, 2, 7, 34, 209, 1546, 13327, \ldots$ with e.g.f. $\frac{1}{1-x}e^{\frac{x}{1-x}}$. This is A002720. Among other things, it counts the number of matchings in the bipartite graph $K(n, n)$. It general term is $\sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k}$. This is equal to $L_n(-1)$ where $L_n(x)$ is the n-th Laguerre polynomial.

Example 6. The row sums of the matrix $\mathbf{Lag}^{-1}$ yield the sequence $1, 0, -1, 4, -15, 56, -185, 204, \ldots$ with e.g.f. $\frac{1}{1+x}e^{\frac{x}{1+x}}$. It has general term $\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$ which is equal to $(-1)^n L_n(1)$.

5 The Associated Laguerre transforms

The Lah and Laguerre transforms, as defined above, are elements of a one-parameter family of transforms, whose general element is given by

$$\mathbf{Lag}^{(\alpha)} = [\frac{1}{(1-x)^{\alpha+1}}, \frac{x}{1-x}].$$

We can calculate the general term of this matrix in the usual way:

$$\begin{aligned}
\mathrm{Lag}^{(\alpha)}(n,k) &= \frac{n!}{k!}[x^n](1-x)^{-(\alpha+1)}x^k(1-x)^{-k} \\
&= \frac{n!}{k!}[x^{n-k}](1-x)^{-(\alpha+k+1)} \\
&= \frac{n!}{k!}\sum_{j=0}^{\infty}\binom{\alpha+k+j}{j}x^j \\
&= \frac{n!}{k!}\binom{n+\alpha}{n-k}.
\end{aligned}$$

We note that $\mathbf{Lah} = \mathbf{Lag}^{(-1)}$ while $\mathbf{Lag} = \mathbf{Lag}^{(0)}$. We can factorize $\mathbf{Lag}^{(\alpha)}$ as follows:

$$\begin{aligned}
\mathbf{Lag}^{(\alpha)} &= [\frac{1}{(1-x)^{\alpha}}, x][\frac{1}{1-x}, \frac{x}{1-x}] \\
&= \mathbf{P}^{(\alpha)} \cdot \mathbf{Lag}
\end{aligned}$$

where $\mathbf{P}^{(\alpha)}$ has general term $\frac{n!}{k!}\binom{n+\alpha-k-1}{n-k}$. Clearly, $\mathbf{P}^{(1)} = \mathbf{P}$. In fact, we have $\mathbf{P}^{(\alpha)} = \mathbf{P}^{\alpha}$. The transform of the sequence a_n by the associated Laguerre transform for α is the sequence b_n with general term $b_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n+\alpha}{n-k}a_k$, which has e.g.f. $\frac{1}{(1-x)^{\alpha+1}}f(\frac{x}{1-x})$.

We note that in the literature of Riordan arrays, the subset of matrices of the form $(1, f(x))$ forms a sub-group, called *the associated group*. We trust that this double use of the term "associated" does not cause confusion.

6 The Generalized Laguerre transform

We define, for the parameter t, the generalized Laguerre matrix $\mathbf{Lag}[t]$ to be

$$\mathbf{Lag}[t] = [\frac{1}{1-tx}, \frac{x}{1-tx}].$$

We immediately have

$$\begin{aligned}
\mathbf{Lag}[t] &= [\frac{1}{1-tx}, \frac{x}{1-tx}] \\
&= [\frac{1}{1-tx}, x][1, \frac{x}{1-tx}] \\
&= \mathbf{P}[t] \cdot \mathbf{Lah}[t].
\end{aligned}$$

where the generalized permutation matrix $\mathbf{P}[t]$ has general term $[k \le n]\frac{n!}{k!}t^{n-k}$. It is clear that

$$\mathbf{Lag}[t]^{-1} = \mathbf{Lag}[-t] = [\frac{1}{1+tx}, \frac{x}{1+tx}].$$

It is possible to generalize the associated Laguerre transform matrices in similar fashion.

7 Transforming the expansion of $\frac{x}{1-\mu x-\nu x^2}$

The e.g.f. of the expansion of $\frac{x}{1-\mu x-\nu x^2}$ takes the form

$$f(x) = A(\mu,\nu)e^{r_1 x} + B(\mu,\nu)e^{r_2 x}$$

which follows immediately from the Binet form of the general term. Thus the transform of this sequence by $\mathbf{Lag}^{(\alpha)}$ will have e.g.f.

$$\frac{A(\mu,\nu)}{(1-x)^{\alpha+1}}e^{\frac{r_1 x}{1-x}} + \frac{B(\mu,\nu)}{(1-x)^{\alpha+1}}e^{\frac{r_2 x}{1-x}}.$$

In the case of the Lah transform ($\alpha = -1$), we get the simple form

$$Ae^{\frac{r_1 x}{1-x}} + Be^{\frac{r_2 x}{1-x}}$$

while in the Laguerre case ($\alpha = 0$) we get

$$A\frac{e^{\frac{r_1 x}{1-x}}}{1-x} + B\frac{e^{\frac{r_2 x}{1-x}}}{1-x}$$

But $\frac{e^{\frac{rx}{1-x}}}{1-x}$ is the e.g.f. of the sequence $n!L_n(-r)$. Thus is this case, the transformed sequence has general term $An!L_n(-r_1) + Bn!L_n(-r_2)$.

Example 7. The Laguerre transform of the Fibonacci numbers

$$F(n) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

is given by

$$\frac{1}{\sqrt{5}}n!L_n(-\frac{1+\sqrt{5}}{2}) - \frac{1}{\sqrt{5}}n!L_n(-\frac{1-\sqrt{5}}{2}).$$

This is A105277. It begins $0, 1, 5, 29, 203, 1680, 16058, \ldots$.

Example 8. The Laguerre transform of the Jacobsthal numbers [1], [15] (expansions of $\frac{x}{1-x-2x^2}$)

$$J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}$$

is given by

$$\frac{1}{3}n!L_n(-2) - \frac{1}{3}n!L_n(1).$$

This is A129695. It begins $0, 1, 5, 30, 221, 1936, 19587, \ldots$. We can use this result to express the Lah transform of the Jacobsthal numbers, since this is equal to $b_n - nb_{n-1}$ where b_n is the Laguerre transform of $J(n)$. We find

$$\frac{n!}{3}(L_n(-2) - L_{n-1}(-2) - (L_n(1) - L_{n-1}(1))).$$

Example 9. We calculate the $\mathbf{Lag}^{(1)}$ transform of the Jacobsthal numbers $J(n)$. Since $\mathbf{Lag}^{(1)} = \mathbf{P} \cdot \mathbf{Lag}$, we apply $\mathbf{P}$ to the Laguerre transform of $J(n)$. This gives us

$$\sum_{k=0}^{n} \frac{n!}{k!}(k!L_k(-2) - k!L_k(1))/3 = \frac{n!}{3}\sum_{k=0}^{n}(L_k(-2) - L_k(1)).$$

This sequence has e.g.f. $\frac{1}{(1-x)^2}\frac{e^{\frac{2x}{(1-x)}} - e^{\frac{-x}{1-x}}}{3}$.

8 The Lah and Laguerre transforms and Stirling numbers

In this section, we follow the notation of [6]. Thus the Stirling numbers of the first kind, denoted by $\left[{n \atop k}\right]$, are the elements of the matrix

$$\mathbf{s} = [1, \ln(\frac{1}{1-x})].$$

$\left[{n \atop k}\right]$ counts the number of ways to arrange n objects into k cycles.

The Stirling numbers of the second kind, denoted by $\left\{{n \atop k}\right\}$, count the number of ways to partition a set of n things into k nonempty subsets. $\left\{{n \atop k}\right\}$ are the elements of the matrix

$$\mathbf{S} = [1, e^x - 1].$$

We have

$$\left\{{n \atop k}\right\} = \sum_{i=0}^{k}\binom{k}{i}\frac{i^n}{k!}.$$

We note that the matrix $[1, e^x - 1]$ of Stirling numbers of the second kind is the inverse of the matrix with elements $(-1)^{n-k}\left[{n \atop k}\right]$, which is the matrix $[1, \ln(1+x)]$.

Related matrices include

$$[\frac{1}{1-x}, \ln(\frac{1}{1-x})]$$

whose elements are given by $\left[{n+1 \atop k+1}\right]$ and its signed version,

$$[\frac{1}{1+x}, \ln(\frac{1}{1+x})]$$

whose elements are given by $(-1)^{n-k}\left[{n+1 \atop k+1}\right]$, along with their inverses, given respectively by $[e^{-x}, 1-e^{-x}]$, with general element $(-1)^{n-k}\left\{{n+1 \atop k+1}\right\}$, and $[e^x, e^x - 1]$ with general element $\left\{{n+1 \atop k+1}\right\}$.

We can generalize a result linking the Lah matrix to the Stirling numbers [7] to the infinite matrix case as follows:

$$\mathbf{Lah} = \mathbf{s} \cdot \mathbf{S}.$$

This is because we have

$$\begin{aligned}
\mathbf{s}\cdot\mathbf{S} &= [1,\ln(\frac{1}{1-x})][1,e^x-1] \\
&= [1,e^{\ln(\frac{1}{1-x})}-1] \\
&= [1,\frac{1}{1-x}-1] \\
&= [1,\frac{x}{1-x}] = \mathbf{Lah}.
\end{aligned}$$

Thus we have

$$\mathbf{S}=\mathbf{s}^{-1}\cdot\mathbf{Lah}, \qquad \mathbf{s}=\mathbf{Lah}\cdot\mathbf{S}^{-1}.$$

We now observe that

$$\begin{aligned}
[\frac{1}{1-x},\frac{x}{1-x}][1,\ln(1+x)] &= [\frac{1}{1-x},\ln(1+\frac{x}{1-x})] \\
&= [\frac{1}{1-x},\ln(\frac{1}{1-x})]
\end{aligned}$$

or

$$\mathbf{Lag}\cdot\left((-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\right)=\left(\begin{bmatrix}n+1\\k+1\end{bmatrix}\right).$$

We deduce the identity

$$\begin{bmatrix}n+1\\k+1\end{bmatrix}=\sum_{j=0}^{n}\frac{n!}{j!}\binom{n}{j}(-1)^{j-k}\begin{bmatrix}j\\k\end{bmatrix}.$$

Taking the inverse of the matrix identity above, we obtain

$$\left(\begin{bmatrix}n+1\\k+1\end{bmatrix}\right)^{-1}=\left((-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\right)^{-1}\cdot\mathbf{Lag}^{-1}$$

which can be established alternatively by noting that

$$\begin{aligned}
[1,e^x-1][\frac{1}{1+x},\frac{x}{1+x}] &= [1.\frac{1}{1+e^x-1},\frac{e^x-1}{1+e^x-1}] \\
&= [e^{-x},1-e^{-x}].
\end{aligned}$$

This establishes the identity

$$(-1)^{n-k}\begin{Bmatrix}n+1\\k+1\end{Bmatrix}=\sum_{j=0}^{n}(-1)^{j-k}\binom{j}{k}\frac{j!}{k!}\begin{Bmatrix}n\\j\end{Bmatrix}.$$

Finally, from the matrix identity

$$\mathbf{Lag}\cdot\left((-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\right)=\left(\begin{bmatrix}n+1\\k+1\end{bmatrix}\right).$$

we deduce that

$$\begin{aligned}\mathbf{Lag} &= \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\right) \cdot \left((-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}\right)^{-1} \\ &= \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\right) \cdot \left(\begin{Bmatrix} n \\ k \end{Bmatrix}\right).\end{aligned}$$

Thus

$$\mathrm{Lag}(n,k) = \sum_{j=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix}.$$

This is equivalent to the factorization

$$\mathbf{Lag} = [\frac{1}{1-x}, \frac{x}{1-x}] = [\frac{1}{1-x}, \ln(\frac{1}{1-x})][1, e^x - 1].$$

This implies (see Appendix A) that

$$L_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \sum_{j=0}^{n} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix} (-x)^k.$$

It is natural in this context to define as *associated Stirling numbers of the first kind* the elements $\begin{bmatrix} n \\ k \end{bmatrix}_\alpha$ of the matrices

$$[\frac{1}{(1-x)^\alpha}, \ln(\frac{1}{1-x})].$$

For instance, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = \begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$. We note that signed versions of these numbers have been documented by Lang (see for instance A049444 and A049458). To calculate $\begin{bmatrix} n \\ k \end{bmatrix}_2$, we proceed as follows:

$$\begin{aligned}\left(\begin{bmatrix} n \\ k \end{bmatrix}_2\right) &= [\frac{1}{(1-x)^2}, \ln(\frac{1}{1-x})] \\ &= \mathbf{P} \cdot [\frac{1}{1-x}, \ln(\frac{1}{1-x})] \\ &= \left([k \le n]\frac{n!}{k!}\right) \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}\right).\end{aligned}$$

Thus

$$\begin{bmatrix} n \\ k \end{bmatrix}_2 = \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} j+1 \\ k+1 \end{bmatrix} = \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} n \\ k \end{bmatrix}_1.$$

More generally, since $[\frac{1}{(1-x)^\alpha}, \ln(\frac{1}{1-x})] = \mathbf{P}[\frac{1}{(1-x)^{\alpha-1}}, \ln(\frac{1}{1-x})]$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \sum_{j=0}^{n} \frac{n!}{j!} \begin{bmatrix} n \\ k \end{bmatrix}_{\alpha-1}.$$

$$\mathrm{Lag}^{(\alpha)}(n,k)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha+1}\begin{Bmatrix} n \\ k \end{Bmatrix}.$$

For example,

$$\begin{aligned}\mathbf{Lag}^{(1)} &= [\frac{1}{(1-x)^2},\ln(\frac{1}{1-x})][1,e^x-1]\\ &= \left(\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_2\begin{Bmatrix} j \\ k \end{Bmatrix}\right)\\ &= \left(\sum_{j=0}^{n}\sum_{i=0}^{n}\frac{n!}{i!}\begin{bmatrix} n+1 \\ j+1 \end{bmatrix}\begin{Bmatrix} j \\ k \end{Bmatrix}\right).\end{aligned}$$

In general, we have

$$\mathbf{Lag}^{(\alpha)}=[\frac{1}{(1-x)^{\alpha+1}},\frac{x}{1-x}]=[\frac{1}{(1-x)^{\alpha+1}},\ln(\frac{1}{1-x})][1,e^x-1].$$

This implies that

$$\mathrm{Lag}^{(\alpha)}(n,k)=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha+1}\begin{Bmatrix} n \\ k \end{Bmatrix}.$$

9 The generalized Lah, Laguerre and Stirling matrices

Given a parameter t define the generalized Stirling numbers of the first kind to be the elements of the matrix

$$\mathbf{s}[t]=[1,\frac{1}{t}\ln(\frac{1}{1-tx})].$$

Similarly, we define the generalized Stirling numbers of the second kind to be the elements of the matrix

$$\mathbf{S}[t]=[1,\frac{e^{tx}-1}{t}].$$

Then

$$\mathbf{S}[t]^{-1}=\mathbf{s}[-t].$$

For instance,

$$\begin{aligned}\mathbf{s}[-t]\cdot\mathbf{S}[t] &= [1,-\frac{1}{t}\ln(\frac{1}{1+tx})][1,\frac{e^{tx}-1}{t}]\\ &= [1,-\frac{1}{t}\ln(\frac{1}{1+t\frac{e^{tx}-1}{t}})]\\ &= [1,-\frac{1}{t}\ln(\frac{1}{e^{tx}})]\\ &= [1,\frac{1}{t}\ln(e^{tx})]\\ &= [1,x]=\mathbf{I}.\end{aligned}$$

The general term of $\mathbf{s}[t]$ is given by $t^{n-k}\left[{n \atop j}\right]$ and that of $\mathbf{S}[t]$ is given by $t^{n-k}\left\{{n \atop j}\right\}$. An easy calculation establishes that

$$\mathbf{Lah}[t] = \mathbf{s}[t]\mathbf{S}[t].$$

From this we immediately deduce that

$$\mathbf{Lag}[t] = \mathbf{P}[t]\mathbf{s}[t]\mathbf{S}[t].$$

Similarly results for the generalized associated Laguerre transform matrices can be derived.

10 Appendix A - the Laguerre and associated Laguerre functions

The associated Laguerre polynomials [16] are defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^n \frac{n!}{k!}\binom{n+\alpha}{n-k}(-x)^k.$$

Their exponential generating function is

$$\frac{e^{\frac{xz}{1-z}}}{(1-z)^{\alpha+1}}$$

The Laguerre polynomials are given by $L_n(x) = L_n^{(0)}(x)$. The associated Laguerre polynomials are orthogonal on the interval $[0, \infty)$ for the weight $e^{-x}x^\alpha$.

Using the notation developed above, we have

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{n!}\sum_{k=0}^n \mathrm{Lag}^{(\alpha)}(n,k)(-x)^k \\ &= \frac{1}{n!}\sum_{k=0}^n \sum_{i=0}^n \left[{n \atop j}\right]_{\alpha+1} \left\{{j \atop k}\right\}(-x)^k. \end{aligned}$$

In particular,

$$L_n(x) = \frac{1}{n!}\sum_{k=0}^n \sum_{i=0}^n \left[{n+1 \atop j+1}\right]\left\{{j \atop k}\right\}(-x)^k.$$

11 Acknowledgements

The author would like to thank an anonymous reviewer for their careful reading of this manuscript and their constructive remarks.

12 Appendix B - Lah and Laguerre transforms in the OEIS

Table 1. Table of Lah transforms

a_n	Lah transform b_n
A000290	A103194
A104600	A121020
A000262	A025168
A000079	A052897
A000110	A084357
A000085	A049376
A000670	A084358

Table 2. Table of Laguerre transforms

a_n	Laguerre transform b_n
A000007	A000142
A000012	A002720
A000045	A105277
A000079	A087912
A001045	A129695

References

[1] P. Barry, Triangle Geometry and Jacobsthal Numbers, *Irish Math. Soc. Bulletin* **51** (2003), 45–57.

[2] P. Barry, On a family of generalized Pascal triangles defined by exponential Riordan arrays, *J. Integer Sequences* **10**, Article 07.3.5.

[3] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, arXiv:math.CO/0702638v1, 2007.

[4] E. Deutsch, L. Shapiro, Exponential Riordan Arrays, Lecture Notes, Nankai University, 2004, available electronically at `http://www.combinatorics.net/ppt2004/Louis%20W.%20 Shapiro/shapiro.pdf`, 2007.

[5] S. Getu, L. Shapiro, W. Woan, and L. Woodson, The Riordan group, *Discr. Appl. Math.* **34** (1991), 229–239.

[6] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[7] Tan Mingshu, Wang Tianming, Lah Matrix and its algebraic properties, *Ars Combinatoria* **70** (2004), 97–108.

[8] J. Riordan, *An Introduction to Combinatorial Analysis*, Dover, 2002.

[9] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, 1968.

[10] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Available at http://www.research.att.com/~njas/sequences/.

[11] N. J. A. Sloane, The on-line encyclopedia of integer sequences, *Notices of the AMS*, **50** (2003), 912–915.

[12] R. Sprugnoli, *Riordan Array Proofs of Identities in Gould's Book*. Published electronically at http://www.dsi.unifi.it/~resp/GouldBK.pdf, 2007.

[13] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.*,**132** (1994), 267–290.

[14] E. W. Weisstein, http://mathworld.wolfram.com/BinomialTransform.html/, 2007.

[15] E. W. Weisstein, http://mathworld.wolfram.com/JacobsthalNumber.html/, 2007.

[16] E. W. Weisstein, http://mathworld.wolfram.com/LaguerrePolynomial.html/, 2007.

2000 *Mathematics Subject Classification*: Primary 11B83; Secondary 05A19, 15A30, 33C45.
Keywords: Lah transform, Laguerre polynomials, Stirling numbers, exponential Riordan array.

(Concerned with sequences A000079, A000085, A000110, A000142, A000262, A000290, A000670, A002720, A007318, A025168, A049376, A052897, A084357, A084358, A087912, A103194, A104600, A105277, A121020, and A129695.)

Received April 18 2007; revised version received May 1 2007. Published in *Journal of Integer Sequences*, May 2 2007.

Return to Journal of Integer Sequences home page.