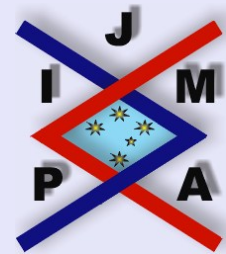


LAZHAR BOUGOFFA

King Khalid University
Faculty of Science
Department of Mathematics
P. O. Box 9004
Abha Saudi Arabia.
EMail: abogafah@kku.edu.sa



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Abstract

In this paper, using Leibnitz's formula and pre-Grüss inequality we prove some inequalities involving Taylor's remainder.

2000 Mathematics Subject Classification: 26D15

Key words: Taylor's remainder, Leibnitz's formula, Pre-Grüss inequality.

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1. Introduction

Recently, H. Gauchman ([1] – [2]) derived new types of inequalities involving Taylor’s remainder.

In this paper, we apply Leibnitz’s formula and pre-Grüss inequality [3] to create several integral inequalities involving Taylor’s remainder.

The present work may be considered as an continuation of the results obtained in [1] – [2].

Let $R_{n,f}(c, x)$ and $r_{n,f}(a, b)$ denote the n th Taylor’s remainder of function f with center c , and the integral Taylor’s remainder, respectively, i.e.

$$R_{n,f}(c, x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k,$$

and

$$r_{n,f}(a, b) = \int_a^b \frac{(b - x)^n}{n!} f^{(n+1)}(x) dx.$$

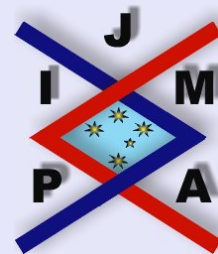
Lemma 1.1. *Let f be a function defined on $[a, b]$. Assume that $f \in C^{n+1}([a, b])$. Then,*

$$(1.1) \quad \int_a^b R_{n,f}(a, x) dx = \int_a^b \frac{(b - x)^{n+1}}{(n + 1)!} f^{(n+1)}(x) dx,$$

$$(1.2) \quad (-1)^{n+1} \int_a^b R_{n,f}(b, x) dx = \int_a^b \frac{(x - a)^{n+1}}{(n + 1)!} f^{(n+1)}(x) dx.$$

Proof. See [1].

□



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Lemma 1.2. Let f be a function defined on $[a, b]$. Assume that $f \in C^{n+1}([a, b])$.
Then

$$(1.3) \quad r_{n,f}(a, b) = f(b) - f(a) - (b - a)f^{(1)}(a) - \dots - \frac{(b - a)^n}{n!}f^{(n)}(a).$$



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2. Results Based on the Leibnitz's Formula

We prove the following theorem based on the Leibnitz's formula.

Theorem 2.1. Let f be a function defined on $[a, b]$. Assume that $f \in C^{n+1}([a, b])$.

Then

$$(2.1) \quad \left| \sum_{k=0}^p (-1)^k C_p^k R_{n-k, f}(a, x) \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(a)| \frac{(b-a)^{n-k+1}}{(n-k+1)!},$$

$$(2.2) \quad \left| \sum_{k=0}^p C_p^k R_{n-k, f}(b, x) \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(b)| \frac{(b-a)^{n-k+1}}{(n-k+1)!},$$

where $C_p^k = \frac{p!}{(p-k)!k!}$.

Proof. We apply the following Leibnitz's formula

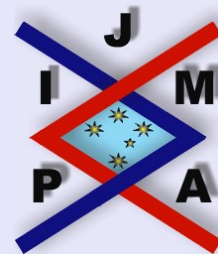
$$(FG)^{(p)} = F^{(p)}G + C_p^1 F^{(p-1)}G^{(1)} + \dots + C_p^{p-1} F^{(1)}G^{(p-1)} + FG^{(p)},$$

provided the functions $F, G \in C^p([a, b])$.

Let $F(x) = f^{(n-p+1)}(x)$, $G(x) = \frac{(b-x)^{n+1}}{(n+1)!}$. Then

$$\left(f^{(n-p+1)}(x) \frac{(b-x)^{n+1}}{(n+1)!} \right)^{(p)} = \sum_{k=0}^p (-1)^k C_p^k f^{(n-k+1)}(x) \frac{(b-x)^{n-k+1}}{(n-k+1)!}.$$

Integrating both sides of the preceding equation with respect to x from a to b



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gives us

$$\left[\left(f^{(n-p+1)}(x) \frac{(b-x)^{n+1}}{(n+1)!} \right)^{(p-1)} \right]_{x=a}^{x=b} = \sum_{k=0}^p (-1)^k C_p^k \int_a^b f^{(n-k+1)}(x) \frac{(b-x)^{n-k+1}}{(n-k+1)!} dx.$$

The integral on the right is $\int_a^b R_{n-k,f}(a,x)dx$, and to evaluate the term on the left hand side, we must again apply Leibnitz's formula, obtaining

$$- \sum_{k=0}^{p-1} (-1)^k C_{p-1}^k f^{(n-k)}(a) \frac{(b-a)^{n-k+1}}{(n-k+1)!} = \sum_{k=0}^p (-1)^k C_p^k \int_a^b R_{n-k,f}(a,x)dx.$$

Consequently,

$$\left| \sum_{k=0}^p (-1)^k C_p^k R_{n-k,f}(a,x) \right| \leq \sum_{k=0}^{p-1} C_{p-1}^k |f^{(n-k)}(a)| \frac{(b-a)^{n-k+1}}{(n-k+1)!},$$

which proves (2.1).

To prove (2.2), set $F(x) = f^{(n-p+1)}(x)$, $G(x) = \frac{(x-a)^{n+1}}{(n+1)!}$, and continue as in the proof of (2.1). \square



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3. Results based on the Grüss Type inequality

We prove the following theorem based on the pre-Grüss inequality.

Theorem 3.1. Let $f(x)$ be a function defined on $[a, b]$ such that $f \in C^{n+1}([a, b])$ and $m \leq f^{(n+1)}(x) \leq M$ for each $x \in [a, b]$, where m and M are constants. Then

$$(3.1) \quad \left| r_{n,f}(a, b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!} (b-a)^n \right| \leq \frac{M-m}{2} \cdot \frac{n}{(2n+1)^{\frac{1}{2}}} \cdot \frac{(b-a)^{n+1}}{(n+1)!}.$$

Proof. We apply the following pre-Grüss inequality [3]

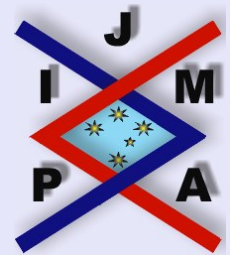
$$(3.2) \quad T(F, G)^2 \leq T(F, F) \cdot T(G, G),$$

where $F, G \in L_2(a, b)$ and $T(F, G)$ is the Chebyshev's functional:

$$T(F, G) = \frac{1}{b-a} \int_a^b F(x)G(x)dx - \frac{1}{b-a} \int_a^b F(x)dx \cdot \frac{1}{b-a} \int_a^b G(x)dx.$$

If there exists constants $m, M \in \mathbb{R}$ such that $m \leq F(x) \leq M$ on $[a, b]$, specially, we have [3]

$$T(F, F) \leq \frac{(M-m)^2}{4}$$



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and

$$(3.3) \quad \left| \frac{1}{b-a} \int_a^b F(x)G(x)dx - \frac{1}{b-a} \int_a^b F(x)dx \cdot \frac{1}{b-a} \int_a^b G(x)dx \right| \\ \leq \frac{1}{2}(M-m) \left[\frac{1}{b-a} \int_a^b G^2(x)dx - \left(\frac{1}{b-a} \int_a^b G(x)dx \right)^2 \right]^{\frac{1}{2}}.$$

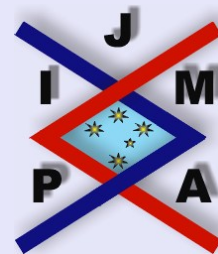
In formula (3.3) replacing $F(x)$ by $f^{(n+1)}(x)$, and $G(x)$ by $\frac{(b-x)^n}{n!}$, we obtain (3.1). \square

Remark 3.1. It is possible to define the similar expression $r'_{n,f}(a, b)$ by

$$r'_{n,f}(a, b) = \int_a^b \frac{(x-a)^n}{n!} f^{(n+1)}(x)dx.$$

In exactly the same way as inequality (3.1) was obtained, one can obtain the following inequality

$$(3.4) \quad \left| r'_{n,f}(a, b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!} (b-a)^n \right| \\ \leq \frac{M-m}{2} \cdot \frac{n}{(2n+1)^{\frac{1}{2}}} \cdot \frac{(b-a)^{n+1}}{(n+1)!}.$$



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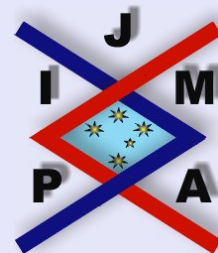
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