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ON GRÜSS TYPE INEQUALITIES OF DRAGOMIR AND FEDOTOV

J.E. PEČARIĆ AND B. TEPEŠ

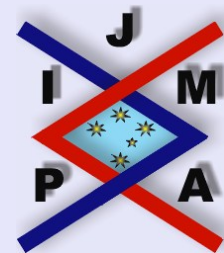
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Abstract

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Abstract

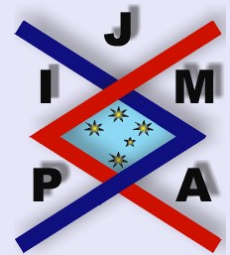
Weighted versions of Grüss type inequalities of Dragomir and Fedotov are given. Some related results are also obtained.

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1. Introduction

In 1935, G. Grüss proved the following inequality:

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma),$$

provided that f and g are two integrable functions on $[a, b]$ satisfying the condition

$$(1.2) \quad \varphi \leq f(x) \leq \Phi \text{ and } \gamma \leq g(x) \leq \Gamma \text{ for all } x \in [a, b].$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(x) = g(x) = \operatorname{sgn} \left(x - \frac{a+b}{2} \right).$$

The following result of Grüss type was proved by S.S. Dragomir and I. Fedotov [1]:

Theorem 1.1. *Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that u is L -Lipschitzian on $[a, b]$, i.e.,*

$$(1.3) \quad |u(x) - u(y)| \leq L|x - y| \text{ for all } x \in [a, b],$$

f is Riemann integrable on $[a, b]$ and there exist the real numbers m, M so that

$$(1.4) \quad m \leq f(x) \leq M \text{ for all } x \in [a, b].$$



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Then we have the inequality

$$(1.5) \quad \left| \int_a^b f(x) du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t) dt \right| \leq \frac{1}{2} L (M - m) (b - a),$$

and the constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

The following result of Grüss' type was proved by S.S. Dragomir and I. Fedotov [2]:

Theorem 1.2. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that u is L -lipschitzian on $[a, b]$, and f is a function of bounded variation on $[a, b]$. Denote by $\bigvee_a^b f$ the total variation of f on $[a, b]$. Then the following inequality holds:

$$(1.6) \quad \left| \int_a^b u(x) df(x) - \frac{f(b) - f(a)}{b - a} \cdot \int_a^b u(x) dx \right| \leq \frac{1}{2} L (b - a) \bigvee_a^b f.$$

The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

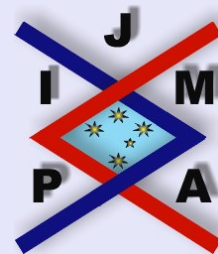
Remark 1.1. For other related results see [3].

Let us also state that the weighted version of (1.1) is well known, that is we have with condition (1.2) the following generalization of (1.1):

$$(1.7) \quad |D(f, g; w)| \leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma),$$

where

$$D(f, g; w) = A(f, g; w) - A(f; w) A(g; w),$$



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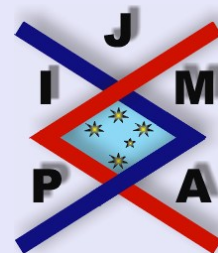
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and

$$A(f; w) = \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx}.$$

So, in this paper we shall show that corresponding weighted versions of (1.5) and (1.6) are also valid. Some related results will be also given.



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2. Results

Theorem 2.1. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that f is Riemann integrable on $[a, b]$ and u is L -Lipschitzian on $[a, b]$, i.e. (1.3) holds true. If $w : [a, b] \rightarrow \mathbb{R}$ is a positive weight function, then

$$(2.1) \quad |T(f, u; w)| \leq L \int_a^b w(x) |f(x) - A(f; w)| dx,$$

where

$$(2.2) \quad T(f, u; w) = \int_a^b w(x) f(x) du(x) - \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) du(x) \int_a^b w(x) f(x) dx.$$

Moreover, if there exist the real numbers m, M such that (1.4) is valid, then

$$(2.3) \quad |T(f, u; w)| \leq \frac{L}{2} (M - m) \int_a^b w(x) dx.$$

Proof. As in [1], we have

$$\begin{aligned} |T(f, u; w)| &= \left| \int_a^b w(x) [f(x) - A(f; w)] du(x) \right| \\ &\leq L \int_a^b w(x) |f(x) - A(f; w)| dx. \end{aligned}$$



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That is, (2.1) is valid. Furthermore, from an application of Cauchy's inequality we have:

$$(2.4) \quad |T(f, u; w)| \leq L \left(\int_a^b w(x) dx \int_a^b w(x) (f(x) - A(f; w))^2 dx \right)^{\frac{1}{2}},$$

from where we obtain

$$(2.5) \quad |T(f, u; w)| \leq L \cdot (D(f, f; w))^{\frac{1}{2}} \cdot \int_a^b w(x) dx.$$

From (1.7) for $g \equiv f$ we get:

$$(2.6) \quad (D(f, f; w))^{\frac{1}{2}} \leq \frac{1}{2} (\Phi - \varphi).$$

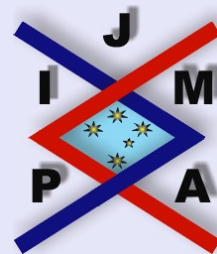
Now, (2.4) and (2.5) give (2.3). □

Now, we shall prove the following result.

Theorem 2.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be M -Lipschitzian on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{R}$ be L -Lipschitzian on $[a, b]$. If $w : [a, b] \rightarrow \mathbb{R}$ is a positive weight function, then*

$$(2.7) \quad |T(f, u; w)| \leq L \cdot M \cdot \frac{\int_a^b \int_a^b w(x) w(y) |x - y| dx dy}{\int_a^b w(y) dy}.$$

Proof. It follows from (2.1)



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$$\begin{aligned}
|T(f, u; w)| &\leq L \cdot \int_a^b w(x) \left| \frac{\int_a^b w(y) (f(x) - f(y)) dy}{\int_a^b w(y) dy} \right| dx \\
&\leq L \cdot \int_a^b w(x) \frac{\int_a^b w(y) |f(x) - f(y)| dy}{\int_a^b w(y) dy} dx \\
&\leq L \cdot M \cdot \frac{\int_a^b \int_a^b w(x) w(y) |x - y| dx dy}{\int_a^b w(y) dy}.
\end{aligned}$$

□

If in the previous result we set $w(x) \equiv 1$, then we can obtain the following corollary:

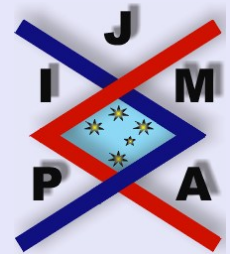
Corollary 2.3. *Let f and u be as in Theorem 2.2, then,*

$$\left| \int_a^b f(x) du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t) dt \right| \leq \frac{L \cdot M \cdot (b - a)^2}{3}.$$

Proof. The proof follows by the fact that

$$\begin{aligned}
\int_a^b \int_a^b |x - y| dx dy &= \int_a^b \left(\int_a^b |x - y| dx \right) dy \\
&= \int_a^b \left(\int_a^y (y - x) dx + \int_y^b (x - y) dx \right) dy \\
&= \frac{1}{2} \int_a^b ((y - a)^2 + (b - y)^2) dy = \frac{1}{3} (b - a)^3.
\end{aligned}$$

□



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Theorem 2.4. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that u is L -Lipschitzian on $[a, b]$, and f is a function of bounded variation on $[a, b]$. If $w : [a, b] \rightarrow \mathbb{R}$ is a positive weight function, then the following inequality holds:

$$|T(u, f; w)| \leq ML \bigvee_a^b g \leq WML \bigvee_a^b f,$$

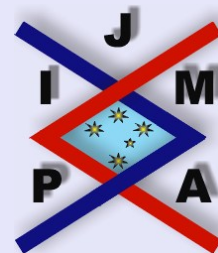
where $T(u, f; w)$ is defined by (2.2), $g : [a, b] \rightarrow \mathbb{R}$ is the function $g(x) = \int_a^x w(t) df(t)$,

$$W = \sup_{x \in [a, b]} w(x), \quad M = \max \left\{ \frac{\int_a^b w(t)(b-t) dt}{\int_a^b w(t) dt}, \frac{\int_a^b w(t)(t-a) dt}{\int_a^b w(t) dt} \right\},$$

and $\bigvee_a^b g$ and $\bigvee_a^b f$ denote the total variation of g and f on $[a, b]$, respectively.

Proof. We have

$$\begin{aligned} & T(u, f; w) \\ &= \int_a^b w(x) u(x) df(x) - \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) df(x) \int_a^b w(x) u(x) dx \\ &= \int_a^b w(x) \left(u(x) - \frac{\int_a^b w(t) u(t) dt}{\int_a^b w(t) dt} \right) df(x) \\ &= \int_a^b \left(\frac{\int_a^b w(t) (u(x) - u(t)) dt}{\int_a^b w(t) dt} \right) w(x) df(x). \end{aligned}$$



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Using the fact that u is L -Lipschitzian on $[a, b]$, we can state that:

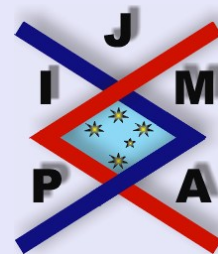
$$\begin{aligned}
 |T(u, f; w)| &= \left| \int_a^b \left(\frac{\int_a^b w(t) (u(x) - u(t)) dt}{\int_a^b w(t) dt} \right) w(x) df(x) \right| \\
 &= \left| \int_a^b \left(\frac{\int_a^b w(t) (u(x) - u(t)) dt}{\int_a^b w(t) dt} \right) d \left(\int_a^x w(t) df(t) \right) \right| \\
 &\leq L \sup_{x \in [a, b]} \left(\frac{\int_a^b w(t) |x - t| dt}{\int_a^b w(t) dt} \right) \bigvee_a^b \left(\int_a^x w(t) df(t) \right) \\
 &= ML \bigvee_a^b g.
 \end{aligned}$$

The constant M has the value

$$M = \sup_{x \in [a, b]} \left(\frac{\int_a^b w(t) |x - t| dt}{\int_a^b w(t) dt} \right).$$

If we denote a new function $y(x)$ as:

$$\begin{aligned}
 y(x) &= \int_a^b w(t) |x - t| dt \\
 &= \int_a^x w(t) (x - t) dt + \int_x^b w(t) (t - x) dt,
 \end{aligned}$$



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then the first derivative of this function is:

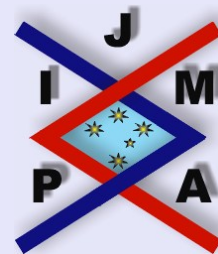
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(x \int_a^x w(t) t dt - \int_a^x t w(t) dt + \int_x^b w(t) t dt - x \int_x^b w(t) dt \right) \\ &= \int_a^x w(t) dt + w(x) x - w(x) x - w(x) x - \int_x^b w(t) dt + w(x) x \\ &= \int_a^x w(t) dt - \int_x^b w(t) dt; \end{aligned}$$

and the second derivative is:

$$\frac{d^2y}{dx^2} = w(x) + w(x) = 2w(x) > 0.$$

Obviously f is a convex function, so we have:

$$\begin{aligned} M &= \sup_{x \in [a,b]} \left(\frac{\int_a^b w(t) |x-t| dt}{\int_a^b w(t) dt} \right) \\ &= \sup_{x \in [a,b]} \left(\frac{y(x)}{\int_a^b w(t) dt} \right) \\ &= \max \left\{ \frac{\int_a^b w(t) (b-t) dt}{\int_a^b w(t) dt}, \frac{\int_a^b w(t) (t-a) dt}{\int_a^b w(t) dt} \right\}. \end{aligned}$$



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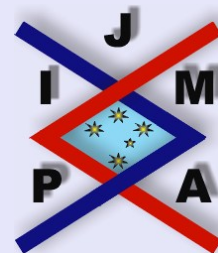
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That is:

$$\begin{aligned}
 |T(u, f; w)| &= \left| \int_a^b \left(\frac{\int_a^b w(t) (u(x) - u(t)) dt}{\int_a^b w(t) dt} \right) w(x) df(x) \right| \\
 &= \left| \int_a^b \left(\frac{\int_a^b w(t) (u(x) - u(t)) dt}{\int_a^b w(t) dt} \right) w(x) df(x) \right| \\
 &\leq \int_a^b \frac{\int_a^b w(t) |u(x) - u(t)| dt}{\int_a^b w(t) dt} w(x) |df(x)| \\
 &\leq \sup_{x \in [a, b]} w(x) L \sup_{x \in [a, b]} \left(\frac{\int_a^b w(t) |x - t| dt}{\int_a^b w(t) dt} \right) \bigvee_a^b f \\
 &= WML \bigvee_a^b f.
 \end{aligned}$$

□



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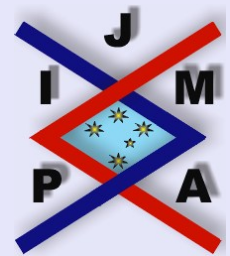
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