Journal of Inequalities in Pure and Applied Mathematics

A VARIANT OF JENSEN'S INEQUALITY

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volume 4, issue 4, article 73,
2003.

Received 14 August, 2003;
accepted 27 August, 2003.

Communicated by: P.S. Bullen

Abstract

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ISSN (electronic): 1443-5756
116-03

Abstract

If f is a convex function the following variant of the classical Jensen's Inequality is proved

$$f\left(x_1 + x_n - \sum w_k k_k\right) \leq f(x_1) + f(x_n) - \sum w_k f(x_k).$$

2000 Mathematics Subject Classification: 26D15.
Key words: Jensen's inequality, Convex functions.

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1. Main Theorem

Let $0 < x_1 \le x_2 \le \cdots \le x_n$ and let w_k $(1 \le k \le n)$ be positive weights associated with these x_k and whose sum is unity. Then Jensen's inequality [2] reads :

Theorem 1.1. *If f is a convex function on an interval containing the x_k then*

$$f\left(\sum w_k x_k\right) \le \sum w_k f(x_k). \tag{1.1}$$

Note: Here and, in all that follows, $\sum$ means $\sum_1^n$.

Our purpose in this note is to prove the following variant of (1.1).

Theorem 1.2. *If f is a convex function on an interval containing the x_k then*

$$f\left(x_1 + x_n - \sum w_k x_k\right) \le f(x_1) + f(x_n) - \sum w_k f(x_k).$$

Towards proving this theorem we shall need the following lemma:

Lemma 1.3. *For f convex we have:*

$$f(x_1 + x_n - x_k) \le f(x_1) + f(x_n) - f(x_k), \quad (1 \le k \le n). \tag{1.2}$$

2. The Proofs

Proof of Lemma 1.3. Write $y_k = x_1 + x_n - x_k$. Then $x_1 + x_n = x_k + y_k$ so that the pairs x_1, x_n and x_k, y_k possess the same mid-point. Since that is the case there exists λ such that

$$\begin{aligned} x_k &= \lambda x_1 + (1-\lambda)x_n, \\ y_k &= (1-\lambda)x_1 + \lambda x_n, \end{aligned}$$

where $0 \leq \lambda \leq 1$ and $1 \leq k \leq n$.

Hence, applying (1.1) twice we get

$$\begin{aligned} f(y_k) &\leq (1-\lambda)f(x_1) + \lambda f(x_n) \\ &= f(x_1) + f(x_n) - [\lambda f(x_1) + (1-\lambda)f(x_n)] \\ &\leq f(x_1) + f(x_n) - f(\lambda x_1 + (1-\lambda)x_n) \\ &= f(x_1) + f(x_n) - f(x_k) \end{aligned}$$

and since $y_k = x_1 + x_n - x_k$ this concludes the proof of the lemma. □

Proof of Theorem 1.2. We have

$$\begin{aligned} f\left(x_1 + x_n - \sum w_k x_k\right) &= f\left(\sum w_k(x_1 + x_n - x_k)\right) \\ &\leq \sum w_k f(x_1 + x_n - x_k) && \text{by (1.1)} \\ &\leq \sum w_k [f(x_1) + f(x_n) - f(x_k)] && \text{by (1.2)} \\ &= f(x_1) + f(x_n) - \sum w f(x_k) \end{aligned}$$

and this concludes the proof. □

3. Two Examples

Let us write $\widetilde{A} = x_1 + x_n - A$ and $\widetilde{G} = \frac{x_1 x_n}{G}$, where A and G denote the usual arithmetic and geometric means of the x_k.

(a) Then taking $f(x)$ as the convex function $-\log x$, Theorem 1.2 gives:

$$\widetilde{A} \geq \widetilde{G}$$

(b) Taking $f(x)$ as the function $\log \frac{1-x}{x}$ which is convex if $0 < x \leq \frac{1}{2}$, Theorem 1.2 gives

$$\frac{\widetilde{A}(x)}{\widetilde{A}(1-x)} \geq \frac{\widetilde{G}(x)}{\widetilde{G}(1-x)}$$

provided that $x_k \in (0, \frac{1}{2}]$ for all k.

The example (a) is a special case of a family of inequalities found by a different method in [1]. The example (b) is, of course, an analogue of Ky-Fan's Inequality [2].

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