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AN INTEGRAL APPROXIMATION IN THREE VARIABLES

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Abstract

In this paper we will investigate a method of approximating an integral in three independent variables. The Ostrowski type inequality is established by the use of Peano kernels and provides a generalisation of a result given by Pachpatte.

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Key words: Ostrowski inequality, Three independent variables, Partial derivatives.

Contents

1. Introduction

The numerical estimation of the integral, or multiple integral of a function over some specified interval is important in many scientific applications. Generally speaking, the error bound for the midpoint rule is about one half of the trapezoidal rule and Stewart [14] has a nice geometrical explanation of this generality. The speed of convergence of an integral is also important and Weideman [15] has some pertinent examples illustrating perfect, algebraic, geometric, super-geometric and sub-geometric convergence for periodic functions.

In particular, we shall establish an Ostrowski type inequality for a triple integral which provides a generalisation or extension of a result given by Pachpatte [10].

In 1938 Ostrowski [7] obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The following definitions will be used in this exposition

$$\mathcal{M}\left(f\right) := \frac{1}{b-a}\int_a^b f\left(t\right)dt, \tag{1.1}$$

$$I_T\left(f\right) := \frac{f\left(b\right)+f\left(a\right)}{2} \tag{1.2}$$

and

$$I_M\left(f\right) := f\left(\frac{a+b}{2}\right). \tag{1.3}$$

The Ostrowski result is given by:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), that is,*

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty.$$

Then we have the inequality

$$|f(x) - \mathcal{M}(f)| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty \tag{1.4}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

Improvements of the result (1.4) has also been obtained by Dedić, Matić and Pearce [2], Pearce, Pečarić, Ujević and Varošanec [11], Dragomir [3] and Sofo [12]. For a symmetrical point $x \in \left[a, \frac{a+b}{2}\right]$, very recently Guessab and Schmeisser [4] studied the more general quadrature formula

$$\mathcal{M}(f) - \left[\frac{f(x) + f(a+b-x)}{2} \right] = E(f; x)$$

where $E(f; x)$ is the remainder.

For $x = \frac{a+b}{2}$ and f defined on $[a, b]$ with Lipschitz constant M, then

$$|\mathcal{M}(f) - I_M(f)| \leq \frac{M(b-a)}{4}.$$

For $x = a$, then

$$\left|\mathcal{M}(f) - I_T(f)\right| \le \frac{M(b-a)}{4}.$$

The following result, which is a generalisation of Theorem 1.1, was given by Milovanović [6, p. 468] in 1975 concerning a function, f, of several variables.

Theorem 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function defined on $D = \{(x_1, \dots, x_m) \,|a_i \le x_i \le b_i, \ (i = 1, \dots, m)\}$ and let $\left|\frac{\partial f}{\partial x_i}\right| \le M_i$ ($M_i > 0$, $i = 1, \dots, m$) in D. Furthermore, let $x \mapsto p(x)$ be integrable and $p(x) > 0$ for every $x \in D$. Then for every $x \in D$, we have the inequality:*

$$\left| f(x) - \frac{\int_D p(y) f(y) \, dy}{\int_D p(y) \, dy} \right| \le \frac{\sum_{i=1}^m M_i \int_D p(y) |x_i - y_i| \, dy}{\int_D p(y) \, dy}. \tag{1.5}$$

In 2001, Barnett and Dragomir [1] obtained the following Ostrowski type inequality for double integrals.

Theorem 1.3. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exist on $(a,b) \times (c,d)$ and is bounded, that is,*

$$\left\| f''_{s,t} \right\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty,$$

then we have the inequality:

$$\left| \int_a^b \int_c^d f(s,t) \, ds dt - (b-a) \int_c^d f(x,t) \, dt \right. \tag{1.6}$$

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$$- (d-c) \int_a^b f(s,y)\,ds + (d-c)(b-a) f(x,y) \Bigg|$$

$$\leq \left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{(d-c)^2}{4} + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty$$

for all $(x,y) \in [a,b] \times [c,d]$.

Pachpatte [8], obtained an inequality in the vein of (1.6) but used elementary analysis in his proof.

Pachpatte [9] also obtains a discrete version of an inequality with two independent variables. Hanna, Dragomir and Cerone [5] obtained a further complementary result to (1.6) and Sofo [13] further improved the result (1.6).

2. Triple Integrals

In three independent variables Pachpatte obtains several results. For discrete variables he obtains a result in [9] and in [10] for continuous variables he obtained the following.

Theorem 2.1. *Let* $\Delta := [a,k] \times [b,m] \times [c,n]$ *for* $a,b,c,k,m,n \in \mathbb{R}^+$ *and* $f(r,s,t)$ *be differentiable on* Δ*. Denote the partial derivatives by* $D_1 f(r,s,t) = \frac{\partial}{\partial r} f(r,s,t)$; $D_2 f(r,s,t) = \frac{\partial}{\partial s}$, $D_3 f(r,s,t) = \frac{\partial}{\partial t}$ *and* $D_3 D_2 D_1 f = \frac{\partial^3 f}{\partial t \partial s \partial r}$*. Let* $F(\Delta)$ *be the clan of continuous functions* $f : \Delta \to \mathbb{R}$ *for which* $D_1 f$, $D_2 f$, $D_3 f$, $D_3 D_2 D_1 f$ *exist and are continuous on* Δ*. For* $f \in F(\Delta)$ *we have*

$$
\begin{aligned}
(2.1)\quad \Bigg| & \int_a^k \int_b^m \int_c^n f(r,s,t)\, dt ds dr \\
& - \frac{1}{8}(k-a)(m-b)(n-c)\left[f(a,b,c) + f(k,m,n)\right] \\
& + \frac{1}{4}(m-b)(n-c)\int_a^k [f(r,b,c)+f(r,m,n)+f(r,m,c)+f(r,b,n)]dr \\
& + \frac{1}{4}(k-a)(n-c)\int_b^m [f(a,s,c)+f(k,s,n)+f(a,s,n)+f(k,s,c)]ds \\
& + \frac{1}{4}(k-a)(m-b)\int_c^n [f(a,b,t)+f(k,m,t)+f(k,b,t)+f(a,m,t)]dt \\
& - \frac{1}{2}(k-a)\int_b^m \int_c^n [f(a,s,t)+f(k,s,t)]\, dt ds \\
& - \frac{1}{2}(m-b)\int_a^k \int_c^n [f(r,b,t)+f(r,m,t)]\, dt dr
\end{aligned}
$$

$$-\frac{1}{2}(n-c)\int_a^k\int_b^m [f(r,s,c)+f(r,s,n)]\,dsdr\Bigg|$$
$$\leq \int_a^k\int_b^m\int_c^n |D_3D_2D_1 f(r,s,t)|\,dtdsdr.$$

The following theorem establishes an Ostrowski type identity for an integral in three independent variables.

Theorem 2.2. *Let* $f:[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\to\mathbb{R}$ *be a continuous mapping such that the following partial derivatives* $\frac{\partial^{i+j+k}f(\cdot,\cdot,\cdot)}{\partial x^i\partial y^j\partial z^k}$; $i=0,\dots,n-1$, $j=0,\dots,m-1$; $k=0,\dots,p-1$ *exist and are continuous on* $[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$. *Also, let*

$$P_n(x,r):=\begin{cases}\frac{(r-a_1)^n}{n!}; & r\in[a_1,x),\\ \frac{(r-b_1)^n}{n!}; & r\in[x,b_1],\end{cases} \tag{2.2}$$

$$Q_m(y,s):=\begin{cases}\frac{(s-a_2)^m}{m!}; & s\in[a_2,y),\\ \frac{(s-b_2)^m}{m!}; & s\in[y,b_2],\end{cases} \tag{2.3}$$

and

$$S_p(z,t):=\begin{cases}\frac{(t-a_3)^p}{p!}; & t\in[a_3,z),\\ \frac{(t-b_3)^p}{p!}; & s\in[z,b_3],\end{cases} \tag{2.4}$$

then for all $(x,y,z)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$ *we have the identity*

$$
\begin{aligned}
(2.5)\quad V := &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr \\
&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i(x)\,Y_j(y)\,Z_k(z)\,\frac{\partial^{i+j+k} f(x,y,z)}{\partial x^i\partial y^j\partial z^k} \\
&+(-1)^p\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i(x)\,Y_j(y)\int_{a_3}^{b_3} S_p(z,t)\,\frac{\partial^{i+j+p} f(x,y,t)}{\partial x^i\partial y^j\partial t^p}dt \\
&+(-1)^m\sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i(x)\,Z_k(z)\int_{a_2}^{b_2} Q_m(y,s)\,\frac{\partial^{i+m+k} f(x,s,z)}{\partial x^i\partial s^m\partial z^k}ds \\
&+(-1)^n\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j(y)\,Z_k(z)\int_{a_1}^{b_1} P_n(x,r)\,\frac{\partial^{n+j+k} f(r,y,z)}{\partial r^n\partial y^j\partial z^k}dr \\
&-(-1)^{m+p}\sum_{i=0}^{n-1} X_i(x)\int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_m(y,s)\,S_p(z,t) \\
&\qquad\times\frac{\partial^{i+m+p} f(x,s,t)}{\partial x^i\partial s^m\partial t^p}dtds \\
&-(-1)^{n+p}\sum_{j=0}^{m-1} Y_j(y)\int_{a_1}^{b_1}\int_{a_3}^{b_3} P_n(x,r)\,S_p(z,t) \\
&\qquad\times\frac{\partial^{n+j+p} f(r,y,t)}{\partial r^n\partial y^j\partial t^p}dtdr
\end{aligned}
$$

$$
\begin{aligned}
&- (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(z) \int_{a_1}^{b_1} \int_{a_2}^{b_2} P_n(x,r) Q_m(y,s) \\
&\qquad \times \frac{\partial^{n+m+k} f(r,s,z)}{\partial r^n \partial s^m \partial z^k} ds dr \\
&= -(-1)^{n+m+p} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_n(x,r) Q_m(y,s) S_p(z,t) \\
&\qquad \times \frac{\partial^{n+m+p} f(r,s,t)}{\partial r^n \partial s^m \partial t^p} dt ds dr,
\end{aligned}
$$

where

$$X_i(x) := \frac{(b_1 - x)^{i+1} + (-1)^i (x - a_1)^{i+1}}{(i+1)!}, \tag{2.6}$$

$$Y_j(y) := \frac{(b_2 - y)^{j+1} + (-1)^j (y - a_2)^{j+1}}{(j+1)!}, \tag{2.7}$$

and

$$Z_k(z) := \frac{(b_3 - z)^{k+1} + (-1)^k (z - a_3)^{k+1}}{(k+1)!}. \tag{2.8}$$

Proof. We have an identity, see [5]

$$\int_{a_1}^{b_1} g(r) dr = \sum_{i=0}^{n-1} X_i(x) g^{(i)}(x) + (-1)^n \int_{a_1}^{b_1} P_n(x,r) g^{(n)}(r) dr. \tag{2.9}$$

Now for the partial mapping $f(\cdot, s, t)$, $s \in [a_2, b_2]$, we have

$$\text{(2.10)} \qquad \int_{a_1}^{b_1} f(r,s,t)\,dr = \sum_{i=0}^{n-1} X_i(x) \frac{\partial^i f}{\partial x^i} + (-1)^n \int_{a_1}^{b_1} P_n(x,r) \frac{\partial^n f}{\partial r^n} dr$$

for every $r \in [a_1, b_1]$, $s \in [a_2, b_2]$ and $t \in [a_3, b_3]$.

Now integrate over $s \in [a_2, b_2]$

$$\text{(2.11)} \qquad \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(r,s,t)\,ds dr$$
$$= \sum_{i=0}^{n-1} X_i(x) \int_{a_2}^{b_2} \frac{\partial^i f}{\partial x^i} ds + (-1)^n \int_{a_1}^{b_1} P_n(x,r) \left(\int_{a_2}^{b_2} \frac{\partial^n f}{\partial r^n} ds \right) dt$$

for all $x \in [a_1, b_1]$.

From (2.9) for the partial mapping $\frac{\partial^i f}{\partial x^i}$ on $[a_2, b_2]$ we have,

$$\text{(2.12)} \int_{a_2}^{b_2} \frac{\partial^i}{\partial x^i} f(x,s,t)\,ds$$
$$= \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^j}{\partial y^j} \left(\frac{\partial^i f}{\partial x^i} \right) + (-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^m}{\partial s^m} \left(\frac{\partial^i f}{\partial x^i} \right) ds$$
$$= \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds.$$

Also, from (2.8)

$$\int_{a_2}^{b_2}\frac{\partial^n f}{\partial r^n}ds=\sum_{j=0}^{m-1}Y_j(y)\frac{\partial^{j+n}f}{\partial y^j\partial r^n}+(-1)^m\int_{a_2}^{b_2}Q_m(y,s)\frac{\partial^m}{\partial s^m}\left(\frac{\partial^n f}{\partial r^n}\right)ds. \tag{2.13}$$

From (2.11) substitute (2.12) and (2.13), so that

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(r,s,t)\,dsdr \\
&=\sum_{i=0}^{n-1}X_i(x)\left[\sum_{j=0}^{m-1}Y_j(y)\frac{\partial^{i+j}f}{\partial x^i\partial y^j}+(-1)^m\int_{a_2}^{b_2}Q_m(y,s)\frac{\partial^{i+m}f}{\partial x^i\partial s^m}ds\right] \\
&\quad+(-1)^n\int_{a_1}^{b_1}P_n(x,r)\left[\sum_{j=0}^{m-1}Y_j(y)\frac{\partial^{j+n}f}{\partial y^j\partial r^n}\right. \\
&\qquad\left.+(-1)^m\int_{a_2}^{b_2}Q_m(y,s)\frac{\partial^m}{\partial s^m}\left(\frac{\partial^n f}{\partial r^n}\right)ds\right]dt \\
&=\sum_{i=0}^{n-1}X_i(x)\sum_{j=0}^{m-1}Y_j(y)\frac{\partial^{i+j}f}{\partial x^i\partial y^j} \\
&\quad+(-1)^m\sum_{i=0}^{n-1}X_i(x)\int_{a_2}^{b_2}Q_m(y,s)\frac{\partial^{i+m}f}{\partial x^i\partial s^m}ds \\
&\quad+(-1)^n\sum_{j=0}^{m-1}Y_j(y)\int_{a_1}^{b_1}P_n(x,r)\frac{\partial^{j+n}f}{\partial y^j\partial r^n}
\end{aligned} \tag{2.14}$$

$$+(-1)^{n+m}\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_n(x,r)\,Q_m(y,s)\,\frac{\partial^{n+m}f}{\partial s^m\partial r^n}dsdr$$

Now integrate (2.14) for $t\in[a_3,b_3]$

$$\begin{aligned}(2.15)\quad &\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f(r,s,t)\,dtdsdr\\ &=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}X_i(x)\,Y_j(y)\int_{a_3}^{b_3}\frac{\partial^{i+j}f}{\partial x^i\partial y^j}dt\\ &\quad+(-1)^m\sum_{i=0}^{n-1}X_i(x)\int_{a_2}^{b_2}Q_m(y,s)\left(\int_{a_3}^{b_3}\frac{\partial^{i+m}f}{\partial x^i\partial s^m}dt\right)ds\\ &\quad+(-1)^n\sum_{j=0}^{m-1}Y_j(y)\int_{a_2}^{b_1}P_n(x,r)\left(\int_{a_3}^{b_3}\frac{\partial^{j+n}}{\partial y^j\partial r^n}dt\right)dr\\ &\quad+(-1)^{n+m}\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_n(x,r)\,Q_m(y,s)\left(\int_{a_3}^{b_3}\frac{\partial^{n+m}f}{\partial s^m\partial r^n}dt\right)dsdr.\end{aligned}$$

From (2.9),

$$\begin{aligned}(2.16)\quad \int_{a_3}^{b_3}\frac{\partial^{i+j}f}{\partial x^i\partial y^j}dt &=\sum_{k=0}^{p-1}Z_k(z)\,\frac{\partial^k}{\partial z^k}\left(\frac{\partial^{i+j}f}{\partial x^i\partial y^j}\right)\\ &\quad+(-1)^p\int_{a_3}^{b_3}S_p(z,t)\,\frac{\partial^p}{\partial t^p}\left(\frac{\partial^{i+j}f}{\partial x^i\partial y^j}\right)dt,\end{aligned}$$

$$\int_{a_3}^{b_3} \frac{\partial^{i+m} f}{\partial x^i \partial s^m} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{i+m} f}{\partial x^i \partial s^m} \right) + (-1)^p \int_{a_3}^{b_3} S_p(z,t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{i+m} f}{\partial x^i \partial s^m} \right) dt, \tag{2.17}$$

$$\int_{a_3}^{b_3} \frac{\partial^{j+n} f}{\partial y^j \partial r^n} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{j+n} f}{\partial y^j \partial r^n} \right) + (-1)^p \int_{a_3}^{b_3} S_p(z,t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{j+n} f}{\partial y^j \partial r^n} \right) dt,$$

and

$$\int_{a_3}^{b_3} \frac{\partial^{n+m} f}{\partial s^m \partial r^n} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k} \left(\frac{\partial^{n+m} f}{\partial r^n \partial s^m} \right) + (-1)^p \int_{a_3}^{b_3} S_p(z,t) \frac{\partial^p}{\partial t^p} \left(\frac{\partial^{n+m} f}{\partial r^n \partial s^m} \right) dt. \tag{2.18}$$

Putting (2.16), (2.17) and (2.18) into (2.15) we arrive at the identity (2.5). □

At the midpoint of the interval

$$\bar{x} = \frac{a_1 + b_1}{2}, \ \bar{y} = \frac{a_2 + b_2}{2}, \ \bar{z} = \frac{a_3 + b_3}{2}$$

we have the following corollary.

Corollary 2.3. *Under the assumptions of Theorem 2.2, we have the identity*

$$\begin{aligned}
\bar{V} := & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr \\
& - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i(\bar{x})\,Y_j(\bar{y})\,Z_k(\bar{z})\,\frac{\partial^{i+j+k} f(\bar{x},\bar{y},\bar{z})}{\partial x^i \partial y^j \partial z^k} \\
& + (-1)^p \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i(\bar{x})\,Y_j(\bar{y}) \int_{a_3}^{b_3} S_p(\bar{z},t)\,\frac{\partial^{i+j+p} f(\bar{x},\bar{y},t)}{\partial x^i \partial y^j \partial t^p}dt \\
& + (-1)^m \sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i(\bar{x})\,Z_k(\bar{z}) \int_{a_2}^{b_2} Q_m(\bar{y},s) \\
& \qquad \times \frac{\partial^{i+m+k} f(\bar{x},s,\bar{z})}{\partial x^i \partial s^m \partial z^k}ds \\
& + (-1)^n \sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j(\bar{y})\,Z_k(\bar{z}) \int_{a_1}^{b_1} P_n(\bar{x},r) \\
& \qquad \times \frac{\partial^{n+j+k} f(r,\bar{y},\bar{z})}{\partial r^n \partial y^j \partial z^k}dr \\
& - (-1)^{m+p} \sum_{i=0}^{n-1} X_i(\bar{x}) \int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_m(\bar{y},s)\,S_p(\bar{z},t) \\
& \qquad \times \frac{\partial^{i+m+p} f(\bar{x},s,t)}{\partial x^i \partial s^m \partial t^p}dtds
\end{aligned} \tag{2.19}$$

$$-(-1)^{n+p}\sum_{j=0}^{m-1}Y_j(\bar{y})\int_{a_1}^{b_1}\int_{a_3}^{b_3}P_n(\bar{x},r)\,S_p(\bar{z},t)$$
$$\times\frac{\partial^{n+j+p}f(r,\bar{y},t)}{\partial r^n\partial y^j\partial t^p}dtdr$$
$$-(-1)^{n+m}\sum_{k=0}^{p-1}Z_k(\bar{z})\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_n(\bar{x},r)\,Q_m(\bar{y},s)$$
$$\times\frac{\partial^{n+m+k}f(r,s,\bar{z})}{\partial r^n\partial s^m\partial z^k}dsdr$$
$$=-(-1)^{n+m+p}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}P_n(\bar{x},r)\,Q_m(\bar{y},s)\,S_p(\bar{z},t)$$
$$\times\frac{\partial^{n+m+p}f(r,s,t)}{\partial r^n\partial s^m\partial t^p}dtdsdr.$$

The identity (2.5) will now be utilised to establish an inequality for a function of three independent variables which will furnish a refinement for the inequality (2.1) given by Pachpatte.

Theorem 2.4. *Let* $f:[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\to\mathbb{R}$ *be continuous on* $(a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)$ *and the conditions of Theorem 2.2 apply. Then we have the inequality*

$$|V|\leq$$

$$\leq \begin{cases} \left[\frac{(x-a_1)^{n+1}+(b_1-x)^{n+1}}{(n+1)!}\right]\left[\frac{(y-a_2)^{m+1}+(b_2-y)^{m+1}}{(m+1)!}\right] \\ \qquad \times\left[\frac{(z-a_3)^{p+1}+(b_3-z)^{p+1}}{(p+1)!}\right]\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\infty \\ \qquad \text{if } \frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\in L_\infty\left([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\right); \\ \\ \frac{1}{n!m!p!}\left[\frac{(x-a_1)^{n\beta+1}+(b_1-x)^{n\beta+1}}{n\beta+1}\right]^{\frac{1}{\beta}}\left[\frac{(y-a_2)^{m\beta+1}+(b_2-y)^{m\beta+1}}{m\beta+1}\right]^{\frac{1}{\beta}} \\ \qquad \times\left[\frac{(z-a_3)^{p\beta+1}+(b_3-z)^{p\beta+1}}{p\beta+1}\right]^{\frac{1}{\beta}}\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\alpha \\ \qquad \text{if } \frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\in L_\alpha\left([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\right), \\ \qquad \alpha>1,\ \alpha^{-1}+\beta^{-1}=1; \\ \\ \frac{1}{8n!m!p!}\left[(x-a_1)^n+(b_1-x)^n+\left|(x-a_1)^n-(b_1-x)^n\right|\right] \\ \qquad \times\left[(y-a_2)^m+(b_2-y)^m+\left|(y-a_2)^m-(b_2-y)^m\right|\right] \\ \qquad \times\left[(z-a_3)^p+(b_3-z)^p+\left|(z-a_3)^p-(b_3-z)^p\right|\right]\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_1 \\ \qquad \text{if } \frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\in L_1\left([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\right); \end{cases}$$

for all $(x,y,z)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$, *where*

$$\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\infty=\sup_{(r,s,t)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}\left|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right|<\infty,$$

and

$$\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\alpha=\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right|^\alpha dtdsdr\right)^{\frac{1}{\alpha}}<\infty. \tag{2.20}$$

Proof.

$$|V| = \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} P_n(x,r)\, Q_m(y,s)\, S_p(z,t)\, \frac{\partial^{n+m+p} f(r,s,t)}{\partial r^n \partial s^m \partial t^p} dt ds dr \right|$$
$$\leq \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)| \left| \frac{\partial^{n+m+p} f(r,s,t)}{\partial r^n \partial s^m \partial t^p} \right| dt ds dr.$$

Using Hölder's inequality and property of the modulus and integral, then we have that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)| \left| \frac{\partial^{n+m+p} f(r,s,t)}{\partial r^n \partial s^m \partial t^p} \right| dt ds dr \tag{2.21}$$
$$\leq \begin{cases} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)|\, dt ds dr, \\ \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)|^\beta\, dt ds dr \right)^{\frac{1}{\beta}}, \\ \qquad \alpha > 1,\ \alpha^{-1} + \beta^{-1} = 1; \\ \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1 \sup\limits_{(r,s,t) \in [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)|. \end{cases}$$

From (2.21) and using (2.2), (2.3) and (2.4)

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)|\, dt ds dr$$
$$= \int_{a_1}^{b_1} |P_n(x,r)|\, dr \int_{a_2}^{b_2} |Q_m(y,s)|\, ds \int_{a_3}^{b_3} |S_p(z,t)|\, dt$$

$$= \left[\int_{a_1}^{x} \frac{(r-a_1)^n}{n!} dr + \int_{x}^{b_1} \frac{(b_1-r)^n}{n!} dr\right]$$
$$\times \left[\int_{a_2}^{y} \frac{(s-a_2)^m}{m!} ds + \int_{y}^{b_2} \frac{(b_2-s)^m}{m!} ds\right]$$
$$\times \left[\int_{a_3}^{z} \frac{(t-a_3)^p}{p!} dt + \int_{z}^{b_3} \frac{(b_3-t)^p}{p!} dt\right]$$
$$= \frac{\left[(x-a_1)^{n+1} + (b_1-x)^{n+1}\right]\left[(y-a_2)^{m+1} + (b_2-y)^{m+1}\right]}{(n+1)!\,(m+1)!}$$
$$\times \frac{\left[(z-a_3)^{p+1} + (b_3-z)^{p+1}\right]}{(p+1)!}$$

giving the first inequality in (2.20).

Now, if we again use (2.21) we have

$$\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \left|P_n(x,r)\, Q_m(y,s)\, S_p(z,t)\right|^{\beta} dt\,ds\,dr\right)^{\frac{1}{\beta}}$$
$$= \left(\int_{a_1}^{b_1} \left|P_n(x,r)\right|^{\beta} dr\right)^{\frac{1}{\beta}} \left(\int_{a_2}^{b_2} \left|Q_m(y,s)\right|^{\beta} ds\right)^{\frac{1}{\beta}} \left(\int_{a_3}^{b_3} \left|S_p(z,t)\right|^{\beta} dt\right)^{\frac{1}{\beta}}$$
$$= \frac{1}{n!m!p!} \left[\int_{a_1}^{x} (r-a_1)^{n\beta} dr + \int_{x}^{b_1} (b_1-r)^{n\beta} dr\right]^{\frac{1}{\beta}}$$
$$\times \left[\int_{a_2}^{y} (s-a_2)^{m\beta} ds + \int_{y}^{b_2} (b_2-s)^{m\beta} ds\right]^{\frac{1}{\beta}}$$

$$\times \left[\int_{a_3}^{z} (t - a_3)^{p\beta}\, dt + \int_{z}^{b_3} (b_3 - t)^{p\beta}\, dt \right]^{\frac{1}{\beta}}$$

$$= \frac{1}{n!m!p!} \left[\frac{(x - a_1)^{n\beta+1} + (b_1 - x)^{n\beta+1}}{n\beta + 1} \right]^{\frac{1}{\beta}}$$

$$\times \left[\frac{(y - a_2)^{m\beta+1} + (b_2 - y)^{m\beta+1}}{m\beta + 1} \right]^{\frac{1}{\beta}}$$

$$\times \left[\frac{(z - a_3)^{p\beta+1} + (b_3 - z)^{p\beta+1}}{p\beta + 1} \right]^{\frac{1}{\beta}}$$

producing the second inequality in (2.20).

Finally, we have

$$\sup_{(r,s,t)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} |P_n(x,r)\, Q_m(y,s)\, S_p(z,t)|$$

$$= \sup_{r\in[a_1,b_1]} |P_n(x,r)| \sup_{s\in[a_2,b_2]} |Q_m(y,s)| \sup_{t\in[a_3,b_3]} |S_p(z,t)|$$

$$= \max \left\{ \frac{(x-a_1)^n}{n!}, \frac{(b_1-x)^n}{n!} \right\} \max \left\{ \frac{(y-a_2)^m}{m!}, \frac{(b_2-y)^m}{m!} \right\}$$

$$\times \max \left\{ \frac{(z-a_3)^p}{p!}, \frac{(b_3-z)^p}{p!} \right\}$$

$$= \frac{1}{n!m!p!} \left[\frac{(x-a_1)^n + (b_1-x)^n}{2} + \left| \frac{(x-a_1)^n - (b_1-x)^n}{2} \right| \right]$$

$$\times \left[\frac{(y-a_2)^m + (b_2-y)^m}{2} + \left| \frac{(y-a_2)^m - (b_2-y)^m}{2} \right| \right]$$
$$\times \left[\frac{(z-a_3)^p + (b_3-z)^p}{2} + \left| \frac{(z-a_3)^p - (b_3-z)^p}{2} \right| \right],$$

giving us the third inequality in (2.20) and we have used the fact that for $A > 0$, $B > 0$ then

$$\max\{A, B\} = \frac{A+B}{2} + \left| \frac{A-B}{2} \right|.$$

Hence the theorem is completely solved. □

The following corollary is a consequence of Theorem 2.4.

Corollary 2.5. *Under the assumptions of Corollary 2.3, we have the inequality*

$$\left|\bar{V}\right| \leq \begin{cases} \left[\frac{(b_1-a_1)^{n+1}(b_2-a_2)^{m+1}(b_3-a_3)^{p+1}}{2^{n+m+p}(n+1)!(m+1)!(p+1)!} \right] \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty, \\ \\ \frac{1}{2^{n+m+p} n! m! p!} \left[\frac{(b_1-a_1)^{n\beta+1}(b_2-a_2)^{m\beta+1}(b_3-a_3)^{p\beta+1}}{(n\beta+1)(m\beta+1)(p\beta+1)} \right]^{\frac{1}{\beta}} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha, \\ \\ \frac{1}{2^{n+m+p} n! m! p!} (b_1-a_1)^n (b_2-a_2)^m (b_3-a_3)^p \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \end{cases}$$

where $\|\cdot\|_\alpha$ $(\alpha \in [1, \infty))$ *are the Lebesgue norms on* $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.

The following two corollaries concern the estimation of V at the end points.

Corollary 2.6. *Under the assumptions of Theorem 2.4 we have, for $x = a_1$, $y = a_2$ and $z = a_3$, the inequality*

$$
\begin{aligned}
&\left|V\left(a_1,a_2,a_3\right)\right| \\
&:= \left| \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f\left(r,s,t\right)dtdsdr \right. \\
&\qquad - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i\left(a_1\right)Y_j\left(a_2\right)Z_k\left(a_3\right)\frac{\partial^{i+j+k}f}{\partial x^i\partial y^j\partial z^k} \\
&\qquad + \left(-1\right)^p\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i\left(a_1\right)Y_j\left(a_2\right)\int_{a_3}^{b_3}\bar{S}_p\left(a_3,t\right)\frac{\partial^{i+j+p}f}{\partial x^i\partial y^j\partial t^p}dt \\
&\qquad + \left(-1\right)^m\sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i\left(a_1\right)Z_k\left(a_3\right)\int_{a_2}^{b_2}\bar{Q}_m\left(a_2,s\right)\frac{\partial^{i+m+k}f}{\partial x^i\partial s^m\partial z^k}ds \\
&\qquad + \left(-1\right)^n\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j\left(a_2\right)Z_k\left(a_3\right)\int_{a_1}^{b_1}\bar{P}_n\left(a_1,r\right)\frac{\partial^{n+j+k}f}{\partial r^n\partial y^j\partial z^k}dr \\
&\qquad - \left(-1\right)^{m+p}\sum_{i=0}^{n-1} X_i\left(a_1\right)\int_{a_2}^{b_2}\int_{a_3}^{b_3}\bar{Q}_m\left(a_2,s\right)\bar{S}_p\left(a_3,t\right)\frac{\partial^{i+m+p}f}{\partial x^i\partial s^m\partial t^p}dtds \\
&\qquad - \left(-1\right)^{n+p}\sum_{j=0}^{m-1} Y_j\left(a_2\right)\int_{a_1}^{b_1}\int_{a_3}^{b_3}\bar{P}_n\left(a_1,r\right)\bar{S}_p\left(a_3,t\right)\frac{\partial^{n+j+p}f}{\partial r^n\partial y^j\partial t^p}dtdr \\
&\qquad \left. - \left(-1\right)^{n+m}\sum_{k=0}^{p-1} Z_k\left(a_3\right)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\bar{P}_n\left(a_1,r\right)\bar{Q}_m\left(a_2,s\right)\frac{\partial^{n+m+k}f}{\partial r^n\partial s^m\partial z^k}dsdr \right|
\end{aligned}
$$

$$\leq \begin{cases} \frac{(b_1-a_1)^{n+1}(b_2-a_2)^{m+1}(b_3-a_3)^{p+1}}{(n+1)!(m+1)!(p+1)!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty, \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\infty \left([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \right); \\ \frac{(b_1-a_1)^{n+\frac{1}{\beta}}}{n!(n\beta+1)^{\frac{1}{\beta}}} \cdot \frac{(b_2-a_2)^{m+\frac{1}{\beta}}}{m!(m\beta+1)^{\frac{1}{\beta}}} \cdot \frac{(b_3-a_3)^{p+\frac{1}{\beta}}}{p!(p\beta+1)^{\frac{1}{\beta}}} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha, \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\alpha \left([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \right), \\ \qquad \alpha > 1, \ \alpha^{-1} + \beta^{-1} = 1; \\ \frac{(b_1-a_1)^{n}(b_2-a_2)^{m}(b_3-a_3)^{p}}{n!m!p!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \\ \quad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1 \left([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \right), \end{cases}$$

where

$$X_i(a_1) := \frac{(b_1 - a_1)^{i+1}}{(i+1)!}, \quad Y_j(a_2) := \frac{(b_2 - a_2)^{j+1}}{(j+1)!}, \quad Z_k(a_3) := \frac{(b_3 - a_3)^{k+1}}{(k+1)!}.$$

$$\bar{P}_n(a_1, r) = \frac{(r - b_1)^n}{n!}, \quad r \in [a_1, b_1];$$
$$\bar{Q}_m(a_2, s) = \frac{(s - b_2)^m}{m!}, \quad s \in [a_2, b_2]$$

and

$$\bar{S}_p(a_3, t) = \frac{(t - b_3)^p}{p!}; \quad t \in [a_3, b_3].$$

Corollary 2.7. *Under the assumptions of Theorem 2.4 we have, for $x = b_1$, $y = b_2$ and $z = b_3$, the inequality*

$$\begin{aligned}
&\left|V\left(b_1, b_2, b_3\right)\right| \\
&\quad := \Bigg| \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f\left(r,s,t\right) dtdsdr \\
&\qquad - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i\left(a_1\right) Y_j\left(a_2\right) Z_k\left(a_3\right) \frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} \\
&\qquad + \left(-1\right)^p \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i\left(b_1\right) Y_j\left(b_2\right) \int_{a_3}^{b_3} \bar{S}_p\left(b_3, t\right) \frac{\partial^{i+j+p} f}{\partial x^i \partial y^j \partial t^p} dt \\
&\qquad + \left(-1\right)^m \sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i\left(b_1\right) Z_k\left(b_3\right) \int_{a_2}^{b_2} \bar{Q}_m\left(b_2, s\right) \frac{\partial^{i+m+k} f}{\partial x^i \partial s^m \partial z^k} ds \\
&\qquad + \left(-1\right)^n \sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j\left(b_2\right) Z_k\left(b_3\right) \int_{a_1}^{b_1} \bar{P}_n\left(b_1, r\right) \frac{\partial^{n+j+k} f}{\partial r^n \partial y^j \partial z^k} dr \\
&\qquad - \left(-1\right)^{m+p} \sum_{i=0}^{n-1} X_i\left(b_1\right) \int_{a_2}^{b_2}\int_{a_3}^{b_3} \bar{Q}_m\left(b_2, s\right) \bar{S}_p\left(b_3, t\right) \frac{\partial^{i+m+p} f}{\partial x^i \partial s^m \partial t^p} dtds \\
&\qquad - \left(-1\right)^{n+p} \sum_{j=0}^{m-1} Y_j\left(b_2\right) \int_{a_1}^{b_1}\int_{a_3}^{b_3} \bar{P}_n\left(b_1, r\right) \bar{S}_p\left(b_3, t\right) \frac{\partial^{n+j+p} f}{\partial r^n \partial y^j \partial t^p} dtdr \\
&\qquad - \left(-1\right)^{n+m} \sum_{k=0}^{p-1} Z_k\left(b_3\right) \int_{a_1}^{b_1}\int_{a_2}^{b_2} \bar{P}_n\left(b_1, r\right) \bar{Q}_m\left(b_2, s\right) \frac{\partial^{n+m+k} f}{\partial r^n \partial s^m \partial z^k} dsdr \Bigg|
\end{aligned}$$

$$\leq \begin{cases} \frac{(b_1-a_1)^{n+1}(b_2-a_2)^{m+1}(b_3-a_3)^{p+1}}{(n+1)!(m+1)!(p+1)!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty, \\ \qquad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\infty ([a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]); \\ \frac{(b_1-a_1)^{n+\frac{1}{\beta}}}{n!(n\beta+1)^{\frac{1}{\beta}}} \cdot \frac{(b_2-a_2)^{m+\frac{1}{\beta}}}{m!(m\beta+1)^{\frac{1}{\beta}}} \cdot \frac{(b_3-a_3)^{p+\frac{1}{\beta}}}{p!(p\beta+1)^{\frac{1}{\beta}}} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha, \\ \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\alpha ([a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]), \quad \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{(b_1-a_1)^n (b_2-a_2)^m (b_3-a_3)^p}{n!m!p!} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \\ \qquad \text{if } \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1 ([a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]), \end{cases}$$

where

$$X_i(b_1) := \frac{(-1)^i (b_1-a_1)^{i+1}}{(i+1)!}, \quad Y_j(b_2) := \frac{(-1)^j (b_2-a_2)^{j+1}}{(j+1)!},$$
$$Z_k(b_3) := \frac{(-1)^k (b_3-a_3)^{k+1}}{(k+1)!}.$$

$$\bar{P}_n(b_1,r) = \frac{(r-a_1)^n}{n!}; \quad r \in [a_1,b_1],$$
$$\bar{Q}_m(b_2,s) = \frac{(s-a_2)^m}{m!}; \quad s \in [a_2,b_2]$$

and

$$\bar{S}_p(b_3,t) = \frac{(t-a_3)^p}{p!}; \quad t \in [a_3,b_3].$$

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