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THE INEQUALITIES $G \leq L \leq I \leq A$ IN n VARIABLES

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Abstract

In the short note, the inequalities $G \leq L \leq I \leq A$ for the geometric, logarithmic, identric, and arithmetic means in n variables are proved.
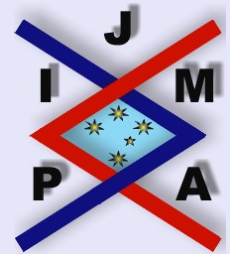
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1. Introduction

For positive numbers a_i, $1 \leq i \leq 2$, let

$$(1.1) \qquad A = A(a_1, a_2) = \frac{a_1 + a_2}{2};$$

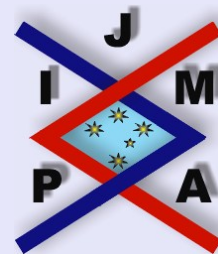$$(1.2) \qquad G = G(a_1, a_2) = \sqrt{a_1 a_2};$$

$$(1.3) \qquad I = I(a_1, a_2) = \begin{cases} \exp\left[\dfrac{a_2 \ln a_2 - a_1 \ln a_1}{a_2 - a_1} - 1\right], & a_1 < a_2, \\ a_1, & a_1 = a_2; \end{cases}$$

$$(1.4) \qquad L = L(a_1, a_2) = \begin{cases} \dfrac{a_2 - a_1}{\ln a_2 - \ln a_1}, & a_1 < a_2, \\ a_1, & a_1 = a_2. \end{cases}$$

These are respectively called the arithmetic, geometric, identric, and logarithmic means.

The logarithmic mean [1, 3], somewhat supprisingly, has applications to the economical index analysis. K.B. Stolarsky first introduced the identric mean I and proved $G \leq L \leq I \leq A$ in two variables in [4]. See [2] also.

The purpose of this short note is to prove the inequalities $G \leq L \leq I \leq A$ of the geometric, logarithmic, identric, and arithmetic means in n variables.

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2. Definitions and the Main Result

Let $a = (a_1, a_2, \ldots, a_n)$ and $a_i > 0$ for $1 \le i \le n$, then the arithmetic, geometric, identric, and logarithmic means in n variables are defined respectively as follows

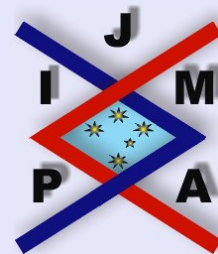$$(2.1) \qquad A = A(a) = A(a_1, \ldots, a_n) = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

$$(2.2) \qquad G = G(a) = G(a_1, \ldots, a_n) = \sqrt[n]{a_1 a_2 \cdots a_n},$$

$$(2.3) \qquad I = I(a) = I(a_1, \ldots, a_n)$$
$$= \exp\left[\frac{1}{V(a)} \sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1} V_i(a) \ln a_i - m\right],$$

$$(2.4) \qquad L = L(a) = L(a_1, \ldots, a_n) = \frac{(n-1)!}{V(\ln a)} \sum_{i=1}^{n} (-1)^{n+i} a_i V_i(\ln a),$$

where $\ln a = (\ln a_1, \ldots, \ln a_n)$, $a_i \ne a_j$ for $i \ne j$,

$$(2.5) \qquad V(a) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (a_i - a_j)$$

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is the determinant of Van der Monde's matrix of the n-th order,

$$
(2.6) \qquad V_i(a) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_{i-1} & a_{i+1} & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_{i-1}^2 & a_{i+1}^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_1^{n-2} & a_2^{n-2} & \dots & a_{i-1}^{n-2} & a_{i+1}^{n-2} & \dots & a_n^{n-2} \end{vmatrix},
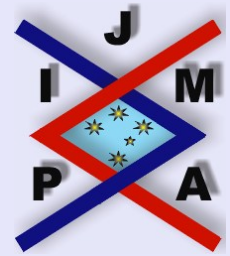$$

$m = \sum_{k=1}^{n-1} \frac{1}{k}$, and $1 \le i \le n$.

The main result of this short note can be stated as

Theorem 2.1. *Let* $a = (a_1, a_2, \ldots, a_n)$ *and* $a_i > 0$ *for* $1 \le i \le n$, *then the inequalities*

$$
(2.7) \qquad G(a) \le L(a) \le I(a) \le A(a)
$$

of the geometric, logarithmic, identric, and arithmetic means in n *variables hold. The equalities in* (2.7) *are valid if and only if* $a_1 = a_2 = \cdots = a_n$.

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3. Proof of Theorem 2.1

To prove inequalities in (2.7), we introduce the following means

$$\text{(3.1)} \qquad I_r(a) = \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n\geq 1}} \left[\sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k\right]^{\frac{1}{\binom{n+r-2}{r-1}}},$$

$$\text{(3.2)} \qquad I_r'(a) = \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n\geq 0}} \left[\sum_{k=1}^{n} \frac{i_k}{r} a_k\right]^{\frac{1}{\binom{n+r-1}{r}}},$$
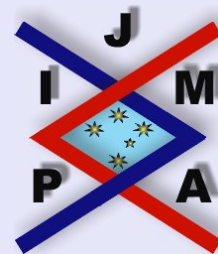
$$\text{(3.3)} \qquad L_r(a) = \frac{1}{\binom{n+r-1}{r}} \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n\geq 0}} \prod_{k=1}^{n} a_k^{i_k/r},$$

$$\text{(3.4)} \qquad L_r'(a) = \frac{1}{\binom{n+r-2}{r-1}} \sum_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n\geq 1}} \prod_{k=1}^{n} a_k^{i_k/(n+r-1)},$$

where $a = (a_1, a_2, \ldots, a_n)$ and $a_i > 0$ for $1 \leq i \leq n$.

Lemma 3.1. *Let* $a = (a_1, a_2, \ldots, a_n)$ *and* $a_i > 0$ *for* $1 \leq i \leq n$, *then we have*

1. $I_1(a) = L_1(a) = A(a)$;

2. $I_1'(a) = L_1'(a) = G(a)$;

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3. *For $1 \leq j \leq n-1$ and*

$$(3.5) \qquad \pi_j = \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_{k_1}=i_{k_2}=\cdots=i_{k_j}=0, \text{ for the rest } i_k \geq 1}} \sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k,$$

we have

$$(3.6) \qquad I'_{n+r-1}(a) = \prod_{j=1}^{n-1} \pi_j^{1 / \binom{2n+r-2}{n+r-1}} \left[I_r(a) \right]^{\binom{n+r-2}{r-1} / \binom{2n+r-2}{n+r-1}};$$

4. *For $1 \leq j \leq n-1$ and*

$$(3.7) \qquad \delta_j = \sum_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_{k_1}=i_{k_2}=\cdots=i_{k_j}=0, \text{ for the rest } i_k \geq 1}} \prod_{k=1}^{n} a_k^{i_k/(n+r-1)},$$
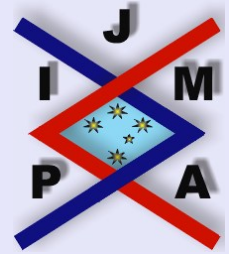
we have

$$(3.8) \qquad L_{n+r-1}(a) = \frac{1}{\binom{2n+r-2}{n+r-1}} \left[\sum_{j=1}^{n-1} \delta_j + \binom{n+r-2}{r-1} L'_r(a) \right];$$

5. *If $r \in \mathbb{N}$, then*

   (a) $I_r(a) \geq I_{r+1}(a)$,

   (b) $I'_r(a) \leq I'_{r+1}(a)$,

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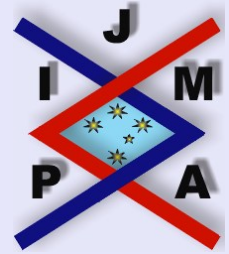
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(c) $L_r(a) \geq L_{r+1}(a)$,

(d) $L'_r(a) \leq L'_{r+1}(a)$,

(e) $I_r(a) \geq L_r(a)$,

(f) $I'_r(a) \geq L'_r(a)$,

where equalities above hold if and only if $a_1 = a_2 \cdots = a_n$.

Proof. The formula (3.6) follows from standard arguments and formulas (3.1) and (3.2).

If $r \in \mathbb{N}$ and $i_1 + i_2 + \cdots + i_n = n + r - 1$, then

$$\sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k = \sum_{k=1}^{n} \frac{i_k}{n+r} \cdot \frac{n+r}{n+r-1} a_k$$

$$= \frac{n+r}{n+r-1} \sum_{k=1}^{n} \frac{i_k}{n+r} a_k$$

$$= \frac{1}{n+r-1} \left[\sum_{j=1}^{n} i_j + 1 \right] \sum_{k=1}^{n} \frac{i_k}{n+r} a_k$$

$$= \frac{1}{n+r-1} \left[\sum_{j=1}^{n} i_j \sum_{k=1}^{n} \frac{i_k}{n+r} a_k + \sum_{k=1}^{n} \frac{i_k}{n+r} a_k \right]$$

$$= \frac{1}{n+r-1} \left[\sum_{j=1}^{n} i_j \sum_{k=1}^{n} \frac{i_k}{n+r} a_k + \sum_{j=1}^{n} \frac{i_j}{n+r} a_j \right]$$

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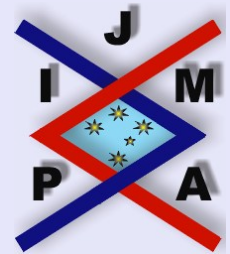◀◀   ▶▶

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$$= \sum_{j=1}^{n} \frac{i_j}{n+r-1} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right].$$

By using the weighted arithmetic-geometric mean inequality, we have

$$\sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k \geq \prod_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right]^{i_j/(n+r-1)},$$

and then

$$(3.9) \quad \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n \geq 1}} \sum_{k=1}^{n} \frac{i_k}{n+r-1} a_k$$

$$\geq \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n \geq 1}} \prod_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right]^{i_j/(n+r-1)}$$

$$= \prod_{j=1}^{n} \prod_{\substack{i_1+i_2+\cdots+i_n=n+r-1 \\ i_1,i_2,\ldots,i_n \geq 1}} \left[\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_j}{n+r} \right]^{i_j/(n+r-1)}$$

$$= \prod_{j=1}^{n} \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_{j-1},\nu_{j+1},\ldots,\nu_n \geq 1; \nu_j \geq 2}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{(\nu_j-1)/(n+r-1)}$$

$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n \geq 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{\sum_{j=1}^{n}(\nu_j-1)/(n+r-1)}$$

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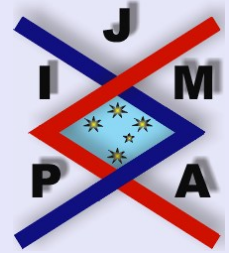
$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n\ge 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{\left(\sum_{j=1}^{n} \nu_j - n\right)\big/(n+r-1)}$$

$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n\ge 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{r/(n+r-1)}$$

$$= \prod_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n+r \\ \nu_1,\nu_2,\ldots,\nu_n\ge 1}} \left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{\binom{n+r-2}{r-1}\big/\binom{n+r-1}{r}},$$

notice that the result from line 4 to line 5 in (3.9) follows from a simple fact that

$$\left[\sum_{k=1}^{n} \frac{\nu_k}{n+r} a_k \right]^{(\nu_j-1)/(n+r-1)} = 1 \ \text{ for } \ \nu_j = 1.$$

The equalities above are valid if and only if

$$\sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_1}{n+r} = \sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_2}{n+r} = \cdots = \sum_{k=1}^{n} \frac{i_k a_k}{n+r} + \frac{a_n}{n+r},$$

which is equivalent to $a_1 = a_2 = \cdots = a_n$. This implies that $I_r(a) \ge I_{r+1}(a)$.

The inequality $I_r(a) \ge L_r(a)$ follows easily from the generalized Hölder inequality

$$(3.10) \qquad \prod_{\substack{i_1+i_2+\cdots+i_n=r+1 \\ i_1\ge 1,i_2\ge 1,\ldots,i_n\ge 1}} \left[\sum_{j=1}^{n} i_j a_j \right]^{\frac{1}{r}} \ge \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\ge 0,i_2\ge 0,\ldots,i_n\ge 0}} \left[\prod_{j=1}^{n} a_j^{i_j} \right]^{\frac{1}{r}}.$$

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The proofs of other formulas and inequalities will be left to the readers. $\quad\square$

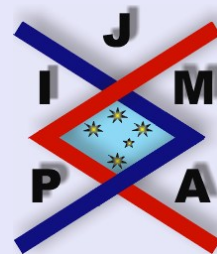**Lemma 3.2.** *Let* $a = (a_1, a_2, \ldots, a_n)$ *and* $a_i > 0$ *for* $1 \le i \le n$, *then we have*

1. $\lim_{r\to\infty} I_r(a) = \lim_{r\to\infty} I'_r(a) = I(a)$,

2. $\lim_{r\to\infty} L_r(a) = \lim_{r\to\infty} L'_r(a) = L(a)$.

Proof. It is easy to see that $\lim_{r\to\infty} I_r(a) = \lim_{r\to\infty} I'_r(a)$ and $\lim_{r\to\infty} L_r(a) = \lim_{r\to\infty} L'_r(a)$, since

$$\lim_{r\to\infty} \pi_j^{1/\binom{2n+r-2}{n+r-1}} = 1, \quad \lim_{r\to\infty} \frac{1}{\binom{2n+r-2}{n+r-1}} \sum_{j=1}^{n-1} \delta_j = 0, \quad \lim_{r\to\infty} \frac{\binom{n+r-2}{r-1}}{\binom{2n+r-2}{n+r-1}} = 1.$$

Straightforward computation yields

$$\ln \lim_{r\to\infty} I'_r(a)$$
$$= \lim_{r\to\infty} \ln I'_r(a)$$
$$= \lim_{r\to\infty} \frac{1}{\binom{n+r-1}{r}} \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n \ge 0}} \ln \sum_{k=1}^{n} \frac{i_k}{r} a_k$$
$$= (n-1)! \int \cdots \int_{\substack{x_1+x_2+\cdots+x_{n-1} \le 1 \\ x_\ell \ge 0, 1 \le \ell \le n-1}} \ln \left[\left(1 - \sum_{i=1}^{n-1} x_i \right) a_1 + \sum_{j=2}^{n} x_{j-1} a_j \right] dx_1 dx_2 \ldots dx_{n-1}$$
$$= \frac{(n-1)!}{V(a)} \sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1} V_i(a)(\ln a_i - m) = \ln I(a)$$

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and

$$(3.11) \quad \lim_{r \to \infty} L_r(a)$$

$$= \lim_{r \to \infty} \frac{1}{\binom{n+r-1}{r}} \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\ldots,i_n \geq 0}} \prod_{k=1}^{n} a_k^{i_k/r}$$

$$= (n-1)! \int \cdots \int_{\substack{x_1+x_2+\cdots+x_{n-1} \leq 1 \\ x_\ell \geq 0, 1 \leq \ell \leq n-1}} a_1^{1-\sum_{i=1}^{n-1} x_i} a_2^{x_1} \cdots a_n^{x_{n-1}} dx_1 dx_2 \cdots dx_{n-1}$$

$$= \frac{(n-1)!}{V(\ln a)} \sum_{i=1}^{n} (-1)^{n+i} a_i V_i(\ln a) = L(a).$$

The proof is complete. $\qquad \square$

Proof of Theorem 2.1. This follows from combination of Lemma 3.1 and Lemma 3.2.
$\qquad \square$
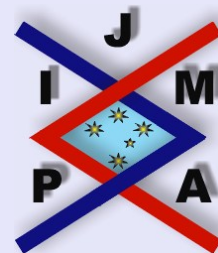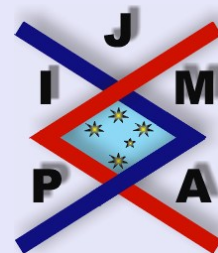
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