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ADDITIONS TO THE TELYAKOVSKIĬ'S CLASS $\mathbb{S}$

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Abstract

A sufficient condition of new type is given which implies that certain sequences belong to the Telyakovskiĭ's class $\mathbb{S}$. Furthermore the relations of two subclasses of the class $\mathbb{S}$ are analyzed.

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Contents

1. Introduction

In 1973, S.A. Telyakovskiĭ [3] defined the class $\mathbb{S}$ of number sequences which has become a very flourishing definition. Several mathematicians have wanted to extend this definition, but it has turned out that most of them are equivalent to the class $\mathbb{S}$. For some historical remarks, we refer to [2]. These intentions show that the class $\mathbb{S}$ plays a very important role in many problems.

The definition of the class $\mathbb{S}$ is the following: A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class $\mathbb{S}$, or briefly $\mathbf{a} \in \mathbb{S}$, if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta\, a_n| \leq A_n$ hold for all n.

We recall only one result of Telyakovskiĭ [3] to illustrate the usability of the class $\mathbb{S}$.

Theorem 1.1. *Let the coefficients of the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

belong to the class $\mathbb{S}$. Then the series (1.1) is a Fourier series and

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right| dx \leq \mathbb{C} \sum_{n=0}^{\infty} a_n,$$

where $\mathbb{C}$ is an absolute constant.

Recently Ž. Tomovski [4] defined certain subclasses of $\mathbb{S}$, and denoted them by $\mathbb{S}_r,\ r = 1, 2, \ldots$ as follows:

A null-sequence $\{a_n\}$ belongs to $\mathbb{S}_r$, if there exists a monotonically decreasing sequence $\left\{A_n^{(r)}\right\}$ such that $\sum_{n=1}^{\infty} n^r A_n^{(r)} < \infty$ and $|\Delta a_n| \leq A_n^{(r)}$.

In [5] Tomovski established, among others, a theorem which states that if $\{a_n\} \in \mathbb{S}_r$ then the r-th derivative of the series (1.1) is a Fourier series and the integral of the absolute value its sum function less than equal to $\mathbb{C}(r) \sum_{n=1}^{\infty} n^r A_n^{(r)}$, where $\mathbb{C}(r)$ is a constant.

His proof is a constructive one and follows along similar lines to that of Theorem 1.1.

In [1] we also defined a certain subclass of $\mathbb{S}$ as follows:

Let $\alpha := \{\alpha_n\}$ be a positive monotone sequence tending to infinity. A null-sequence $\{a_n\}$ belongs to the class $\mathbb{S}(\alpha)$, if there exists a monotonically decreasing sequence $\left\{A_n^{(\alpha)}\right\}$ such that

$$\sum_{n=1}^{\infty} \alpha_n A_n^{(\alpha)} < \infty \quad \text{and} \quad |\Delta a_n| \leq A_n^{(\alpha)}.$$

Clearly $\mathbb{S}(\alpha)$ with $\alpha_n = n^r$ includes $\mathbb{S}_r$.

In [2] we verified that if $\{a_n\} \in \mathbb{S}_r$, then $\{n^r a_n\} \in \mathbb{S}$, with a sequence $\{A_n\}$ that satisfies the inequality

$$\sum_{n=1}^{\infty} A_n \leq (r+1) \sum_{n=1}^{\infty} n^r A_n^{(r)}. \tag{1.2}$$

Thus, this result and Theorem 1.1 immediately imply the theorem of Tomovski mentioned above.

Our theorem which yields (1.2) reads as follows.

Theorem 1.2. *Let $\gamma \geq \beta > 0$ and $\mathbb{S}_\alpha := \mathbb{S}(\alpha)$ if $\alpha_n = n^\alpha$. If $\{a_n\} \in S_\gamma$ then $\{n^\beta a_n\} \in \mathbb{S}_{\gamma-\beta}$ and*

$$\sum_{n=1}^{\infty} n^{\gamma-\beta} A_n^{(\gamma-\beta)} \leq (\beta+1) \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)} \tag{1.3}$$

holds.

It is clear that if $\gamma = \beta = r$ then (1.3) gives (1.2) $\left(A_n^{(0)} = A_n\right)$.

In [2] we also verified that the statement of Theorem 1.2 is not reversible in general.

In [3] Telyakovskiĭ realized that in the definition of the class $\mathbb{S}$ we can take $A_n := \max_{k\geq n} |\Delta a_k|$, that is, $\{a_n\} \in \mathbb{S}$ if $a_n \to 0$ and $\sum_{n=1}^{\infty} \max_{k\geq n} |\Delta a_k| < \infty$.

This definition of $\mathbb{S}$ has not been used often, as I know.

The reason, perhaps, is the appearing of the inconvenient addends $\max\limits_{k\geq n} |\Delta a_k|$.

In the present note first we give a sufficient condition being of similar character as this definition of $\mathbb{S}$ but without $\max_{k\geq n} |\Delta a_k|$, which implies that $\{a_n\} \in \mathbb{S}$.

Second we show that with a certain additional assumption, the assertion of Theorem 1.2 is reversible and the additional condition to be given is necessary in general.

2. Results

Before formulating the first theorem we recall a definition.

A non-negative sequence $\mathbf{c} := \{c_n\}$ is called locally almost monotone if there exists a constant $K(\mathbf{c})$ depending only on the sequence $\mathbf{c}$, such that

$$c_n \leq K(\mathbf{c}) c_m$$

holds for any m and $m \leq n \leq 2m$. These sequences will be denoted by $\mathbf{c} \in LAMS$.

Theorem 2.1. *If $a := \{a_n\}$ is a null-sequence, $a \in LAMS$ and $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, then $a \in \mathbb{S}$.*

Theorem 2.2. *Let $\gamma \geq \beta > 0$. If $\{n^\beta a_n\} \in \mathbb{S}_{\gamma-\beta}$, and*

$$\sum_{n=1}^{\infty} n^\gamma |\Delta a_n| < \infty, \tag{2.1}$$

then $\{a_n\} \in \mathbb{S}_\gamma$.

Remark 2.1. *The condition (2.1) is not dispensable, moreover it cannot be weakened in general.*

The following lemma will be required in the proof of Theorem 2.1.

Lemma 2.3. *If $\mathbf{c} := \{c_n\} \in LAMS$ and $\alpha_n := \sup_{k \geq n} c_k$, then for any $\delta > -1$*

$$\sum_{n=1}^{\infty} n^\delta \alpha_n \leq K(K(\mathbf{c}), \delta) \sum_{n=1}^{\infty} n^\delta c_n. \tag{2.2}$$

Proof. Since $\mathbf{c} \in LAMS$ thus with $K := K(\mathbf{c})$

$$\alpha_{2^n} = \sup_{k \geq 2^n} c_k \leq \sup_{m \geq n} K\, c_{2^m} \leq K \sup_{m \geq n} c_{2^m}. \tag{2.3}$$

If $\sum n^\delta c_n < \infty$, then $c_n \to 0$, thus by (2.3) there exists an integer $p = p(n) \geq 0$ such that

$$\alpha_{2^n} \leq K\, c_{2^{n+p}}.$$

Then, by the monotonicity of the sequence $\{\alpha_n\}$,

$$\begin{aligned} \sum_{k=n}^{n+p} 2^{k(1+\delta)} \alpha_{2^k} &\leq K\, c_{2^{n+p}} \sum_{k=n}^{n+p} 2^{k(1+\delta)} \\ &\leq K\, 2^{(1+\delta)}\, 2^{(n+p)(1+\delta)}\, c_{2^{n+p}} \\ &\leq K^2\, 2^{(1+\delta)2} \sum_{\nu=2^{n+p-1}+1}^{2^{n+p}} \nu^\delta\, c_\nu \end{aligned}$$

clearly follows. If we start this arguing with $n = 0$, and repeat it with $n + p$ in place of n, if $p \geq 1$; and if $p = 0$ then with $n + 1$ in place of n, and make these blocks repeatedly, furthermore if we add all of these sums, we see that the sum $\sum_{k=3}^{\infty} 2^{k(1+\delta)} \alpha_{2^k}$ will be majorized by the sum $K^2\, 4^{(1+\delta)} \sum_{n=1}^{\infty} n^\delta\, c_n$, and this proves (2.2). □

Remark 2.2. *Following the steps of the proof it is easy to see that with φ_n in place of n^δ, (2.2) also holds if $\{\varphi_n\} \in LAMS$ and $2^n \varphi_{2^n}$ is quasi geometrically increasing.*

3. Proofs

Proof of Theorem 2.1. Using Lemma 2.3 with $c_n = a_n$ and $\delta = 0$, we immediately get that

$$\sum_{n=1}^{\infty} \max_{k \geq n} |\Delta\, a_k| < \infty, \tag{3.1}$$

namely the assumption $a_n \to 0$ yields that $\sup |\Delta\, a_k| = \max |\Delta\, a_k|$, and thus (3.1) implies that $\{a_n\} \in \mathbb{S}$. $\square$

Proof of Theorem 2.2. With respect to the equality

$$|\Delta(n^\beta\, a_n)| = |n^\beta(a_n - a_{n+1}) - a_{n+1}((n+1)^\beta - n^\beta)|$$

it is clear that

$$n^\beta |\Delta\, a_n| \leq A_n^{(\gamma-\beta)} + K\, n^{\beta-1}|a_{n+1}|,$$

where K is a constant $K = K(\beta) > 0$ independent of n.

Hence, multiplying with $n^{-\beta}$, we get that

$$|\Delta\, a_n| \leq n^{-\beta} A_n^{(\gamma-\beta)} + K\, n^{-1} \sum_{k=n+1}^{\infty} |\Delta\, a_k|, \tag{3.2}$$

thus if we define

$$A_n^{(\gamma)} := n^{-\beta} A_n^{(\gamma-\beta)} + K\, n^{-1} \sum_{k=n+1}^{\infty} |\Delta\, a_k|,$$

then this sequence $A_n^{(\gamma)}$ is clearly monotonically decreasing, and $A_n^{(\gamma)} \geq |\Delta\, a_n|$, furthermore by the assumptions of Theorem 1.2 and (3.2)

$$\sum_{n=1}^{\infty} n^{\gamma}\, A_n^{(\gamma)} < \infty,$$

since

$$\sum_{n=1}^{\infty} n^{\gamma-1} \sum_{k=n+1}^{\infty} |\Delta\, a_k| \leq K(\gamma) \sum_{k=1}^{\infty} k^{\gamma} |\Delta\, a_k| < \infty.$$

Thus $\{a_n\} \in \mathbb{S}_{\gamma}$ is proved. The proof is complete. □

Proof of Remark 2.1. Let $a_n = n^{-\beta}$, then $|\Delta\, n^{\beta} a_n| = 0$, therefore $\{n^{\beta} a_n\} \in \mathbb{S}_{\gamma-\beta}$ holds e.g. with $A_n^{(\gamma-\beta)} = n^{\beta-\gamma-2}$. On the other hand $|\Delta\, a_n| \geq (n+1)^{-\beta-1}$, thus, by $\gamma \geq \beta$,

$$\sum_{n=1}^{\infty} n^{\gamma} |\Delta\, a_n| = \infty, \tag{3.3}$$

consequently, if $A_n^{(\gamma)} \geq |\Delta\, a_n|$, then

$$\sum_{n=1}^{\infty} n^{\gamma}\, A_n^{(\gamma)} = \infty$$

also holds, therefore $\{a_n\} \notin \mathbb{S}_{\gamma}$.

In this case, by (3.3), the additional condition (2.1) does not maintain.

Herewith, Remark 2.1 is verified, namely we can also see that the condition (2.1) cannot be weakened in general. □

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