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# HADAMARD-TYPE INEQUALITIES FOR QUASICONVEX FUNCTIONS

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ABSTRACT. Recently Hadamard-type inequalities for nonnegative, evenly quasiconvex functions which attain their minimum have been established. We show that these inequalities remain valid for the larger class containing all nonnegative quasiconvex functions, and show equality of the corresponding Hadamard constants in case of a symmetric domain.

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## 1. INTRODUCTION

The well-known Hadamard inequality for convex functions has been recently generalized to include other types of functions. For instance, Pearce and Rubinov [2], generalized an earlier result of Dragomir and Pearce [1] by showing that for any nonnegative quasiconvex function defined on $[0,1]$ and any $u \in [0,1]$, the following inequality holds:

$$f(u) \leq \frac{1}{\min(u, 1-u)} \int_0^1 f(x)\, dx.$$

In a subsequent paper, Rubinov and Dutta [3] extended the result to the $n$-dimensional space, by imposing the restriction that the nonnegative function $f$ attains its minimum and is not just quasiconvex, but evenly quasiconvex. The purpose of this note is to establish the inequality without these restrictions, and to obtain a simpler expression of the "Hadamard constant" which appears multiplied to the integral. To be precise, given a convex subset $X$ of $\mathbb{R}^n$, a Borel measure $\mu$ on $X$ and an element $u \in X$, we show that any nonnegative quasiconvex function satisfies an inequality of the form $f(u) \leq \gamma \int_X f d\mu$ where $\gamma$ is a constant. An analogous inequality $f(u) \leq \gamma_* \int_X f d\mu$ is obtained for all quasiconvex nonnegative functions for which

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$f(0) = 0$ (under the assumption that $0 \in X$). We obtain simple expressions for the constants $\gamma$ and $\gamma_*$ and show that they are equal, under a symmetry assumption.

In what follows, $X$ is a convex, Borel subset of $\mathbb{R}^n$, $\mu$ is a finite Borel measure on $X$, and $\lambda$ is the Lebesgue measure. As usual, $\mu \ll \lambda$ means that $\mu$ is absolutely continuous with respect to $\lambda$. The open (closed) ball with center $u$ and radius $r$ will be denoted by $B(u,r)$ ($\overline{B}(u,r)$). We denote by $S$ the sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ and set, for each $v \in S$, $u \in \mathbb{R}$,

$$X_{v,u} = \{x \in X : \langle v, x-u \rangle > 0\}. \tag{1.1}$$

## 2. INEQUALITY FOR QUASICONVEX FUNCTIONS

The following proposition shows that the Hadamard-type inequality for nonnegative evenly quasiconvex functions that attain their minimum, established in [3], is true for all nonnegative quasiconvex, Borel measurable functions.

**Proposition 2.1.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a Borel measurable, nonnegative quasiconvex function. Then for every $u \in X$, the following inequality holds:*

$$\inf_{v \in S} \mu(X_{v,u}) f(u) \leq \int_X f d\mu. \tag{2.1}$$

*Proof.* Let $L = \{x \in X : f(x) < f(u)\}$. Then $L$ is convex and $u$ does not belong to the relative interior of $L$. We can thus separate $u$ and $L$ by a hyperplane, i.e., there exists $v \in S$ such that $\forall x \in L$, $\langle x, v \rangle \leq \langle u, v \rangle$. Hence, for every $x \in X_{v,u}$, $f(x) \geq f(u)$. Consequently,

$$\mu(X_{v,u}) f(u) \leq \int_{X_{v,u}} f d\mu \leq \int_X f d\mu$$

from which follows relation (2.1). □

Note that if $\mu = \lambda$, then we do not have to assume $f$ to be Borel measurable. Indeed, any convex subset of $\mathbb{R}^n$ is Lebesgue measurable since it can be written as the union of its interior and a subset of its boundary; the latter is a Lebesgue null set, thus is Lebesgue measurable. Consequently, every quasiconvex function is Lebesgue measurable since by definition its level sets are convex.

It is possible that $\inf_{v \in S} \mu(X_{v,u}) = 0$. In this case relation (2.1) does not say much. We can avoid this if $u \in \operatorname{int} X$ and $\mu$ does not vanish on sets of nonzero Lebesgue measure:

**Proposition 2.2.** *Assumptions as in Proposition 2.1.*

$(i)$ *If $u \in \operatorname{int} X$ and $\lambda \ll \mu$, then $\inf_{v \in S} \mu(X_{v,u}) > 0$.*
$(ii)$ *If $u \notin \operatorname{int} X$ and $\mu \ll \lambda$, then $\inf_{v \in S} \mu(X_{v,u}) = 0$.*

*Proof.* $(i)$ Let $\varepsilon > 0$ be such that $B(u, \varepsilon) \subseteq X$. For each $v \in S$ and $x \in B\left(u + \frac{\varepsilon}{2}v, \frac{\varepsilon}{2}\right)$, the triangle inequality yields

$$\|x - u\| \leq \left\|x - \left(u + \tfrac{\varepsilon}{2}v\right)\right\| + \left\|\tfrac{\varepsilon}{2}v\right\| < \varepsilon$$

hence $x \in X$. Also,

$$\langle v, x-u \rangle = \left\langle v, x - \left(u + \tfrac{\varepsilon}{2}v\right)\right\rangle + \left\langle v, \tfrac{\varepsilon}{2}v \right\rangle \geq -\|v\| \left\|x - \left(u + \tfrac{\varepsilon}{2}v\right)\right\| + \tfrac{\varepsilon}{2} > 0.$$

Consequently, $x \in X_{v,u}$, i.e., $B\left(u + \frac{\varepsilon}{2}v, \frac{\varepsilon}{2}\right) \subseteq X_{v,u}$. Hence,

$$\inf_{v \in S} \lambda(X_{v,u}) \geq \lambda\left(B\left(u + \tfrac{\varepsilon}{2}v, \tfrac{\varepsilon}{2}\right)\right) = \lambda\left(B\left(0, \tfrac{\varepsilon}{2}\right)\right) > 0.$$

By absolute continuity, $\inf_{v \in S} \mu(X_{v,u}) > 0$.

$(ii)$ Since $u \notin \operatorname{int} X$, we can separate $u$ and $X$ by a hyperplane. It follows that for some $v \in S$, the set $X_{v,u}$ is a subset of this hyperplane, hence $\lambda\left(X_{v,u}\right) = 0$ which entails that $\mu\left(X_{v,u}\right) = 0$.

□

Let us set

$$\gamma = \frac{1}{\inf_{v \in S} \mu\left(X_{v,u}\right)}, \tag{2.2}$$

where we make the convention $\frac{1}{0} = +\infty$. Then we can write (2.1) in the form

$$f(u) \le \gamma \int_X f d\mu. \tag{2.3}$$

The following Lemma will be useful for obtaining alternative expressions of "Hadamard constants" such as $\gamma$ and showing their sharpness. In particular, it shows that $X_{v,u}$ could have been defined (see relation (1.1)) by using $\ge$ instead of $>$. Let

$$\overline{X}_{v,u} = \{x \in X : \langle v, x-u\rangle \ge 0\} \tag{2.4}$$

be the closure of $X_{v,u}$ in $X$.

**Lemma 2.3.** *If $\mu \ll \lambda$, then*

$(i)$ $\mu\left(X_{v,u}\right) = \mu\left(\overline{X}_{v,u}\right)$;

$(ii)$ *The function $v \to \mu\left(X_{v,u}\right)$ is continuous on $S$.*

*Proof.* $(i)$ We know that $\lambda\left(\{x \in \mathbb{R}^n : \langle v, x-u\rangle = 0\}\right) = 0$; consequently, $\lambda\left(\overline{X}_{v,u} \backslash X_{v,u}\right) = 0$ and this entails that $\mu\left(\overline{X}_{v,u} \backslash X_{v,u}\right) = 0$.

$(ii)$ Suppose that $(v_n)$ is a sequence in $S$, converging to $v$. Let $\varepsilon > 0$ be given. Choose $r > 0$ large enough so that $\mu\left(X \backslash \overline{B}(u,r)\right) < \varepsilon/2$. Let us show that

$$\lim_{n\to\infty} \lambda\left(X_{v_n,u} \cap \overline{B}(u,r)\right) = \lambda\left(X_{v,u} \cap \overline{B}(u,r)\right).$$

For this it is sufficient to show that $\lim_{n\to\infty} \lambda(X_n) = 0$ where $X_n$ is the symmetric difference $\left(\left(X_{v,u} \backslash X_{v_n,u}\right) \cup \left(X_{v_n,u} \backslash X_{v,u}\right)\right) \cap \overline{B}(u,r)$. If $x \in X_n$ then $\|x-u\| \le r$ and

$$\langle v, x-u\rangle > 0 \ge \langle v_n, x-u\rangle \tag{2.5}$$

or

$$\langle v_n, x-u\rangle > 0 \ge \langle v, x-u\rangle. \tag{2.6}$$

If, say, (2.6) is true, then $\langle v, x-u\rangle \le 0 < \langle v_n - v, x-u\rangle + \langle v, x-u\rangle \le \|v_n - v\| r + \langle v, x-u\rangle$ thus $|\langle v, x-u\rangle| \le \|v_n - v\| r$. The same can be deduced if (2.5) is true. Thus the projection of $X_n$ on $v$ can be arbitrarily small; since $X_n$ is contained in $\overline{B}(u,r)$ this means that $\lim_{n\to\infty} \lambda(X_n) = 0$ as claimed.

By absolute continuity, $\lim_{n\to\infty} \mu\left(X_{v_n,u} \cap \overline{B}(u,r)\right) = \mu\left(X_{v,u} \cap \overline{B}(u,r)\right)$. Since

$$\begin{aligned}
\left|\mu\left(X_{v_n,u}\right) - \mu\left(X_{v,u}\right)\right| &\le \left|\mu\left(X_{v_n,u} \cap \overline{B}(u,r)\right) - \mu\left(X_{v,u} \cap \overline{B}(u,r)\right)\right| \\
&\quad + \left|\mu\left(X_{v_n,u} \backslash \overline{B}(u,r)\right)\right| + \left|\mu\left(X_{v,u} \backslash \overline{B}(u,r)\right)\right| \\
&\le \left|\mu\left(X_{v_n,u} \cap \overline{B}(u,r)\right) - \mu\left(X_{v,u} \cap \overline{B}(u,r)\right)\right| + \varepsilon,
\end{aligned}$$

it follows that $\lim_{n\to\infty} \left|\mu\left(X_{v_n,u}\right) - \mu\left(X_{v,u}\right)\right| \le \varepsilon$. This is true for all $\varepsilon > 0$, hence $\lim_{n\to\infty} \mu\left(X_{v_n,u}\right) = \mu\left(X_{v,u}\right)$.

□

We now obtain an alternative expression for the "Hadamard constant" $\gamma$, analogous to the one in [3]. For $u \in X$ define

$$A_u^+ = \{(v, x_0) \in \mathbb{R}^n \times X : \langle v, u - x_0 \rangle \geq 1\}.$$

Further, given $v \in \mathbb{R}^n$ and $x_0 \in X$ set[1]

$$X_{v,x_0}^+ = \{x \in X : \langle v, x - x_0 \rangle > 1\}.$$

**Proposition 2.4.** *The following equality holds for every $u \in \operatorname{int} X$:*

$$\gamma = \frac{1}{\inf_{(v,x_0) \in A_u^+} \mu\left(X_{v,x_0}^+\right)}.$$

*Proof.* For every $(v, x_0) \in A_u^+$, we set $v' = v/\|v\|$. For each $x \in X_{v',u}$, $\langle v, x - u \rangle > 0$ holds. Besides, $(v, x_0) \in A_u^+$ implies that $\langle v, u - x_0 \rangle \geq 1$. Hence, $\langle v, x - x_0 \rangle = \langle v, x - u \rangle + \langle v, u - x_0 \rangle > 1$ thus $x \in X_{v,x_0}^+$. It follows that $X_{v',u} \subseteq X_{v,x_0}^+$; consequently,

$$\inf_{(v,x_0) \in A_u^+} \mu\left(X_{v,x_0}^+\right) \geq \inf_{v \in S} \mu\left(X_{v,u}\right). \tag{2.7}$$

To show the reverse inequality, let $v \in S$ be given. Since $u \in \operatorname{int} X$, we may find $x_0 \in X$ such that $\langle v, u - x_0 \rangle > 0$. Choose $t > 0$ so that for $v' = tv$ one has $\langle v', u - x_0 \rangle = 1$. The following equivalences hold:

$$\begin{aligned}
x \in X_{v',x_0}^+ &\Leftrightarrow \langle v', x - x_0 \rangle > 1 \\
&\Leftrightarrow \langle v', x - u \rangle + \langle v', u - x_0 \rangle > 1 \\
&\Leftrightarrow \langle v', x - u \rangle > 0 \\
&\Leftrightarrow \langle v, x - u \rangle > 0 \\
&\Leftrightarrow x \in X_{v,u}.
\end{aligned}$$

Thus, for every $v \in S$ there exists $(v', x_0) \in A_u^+$ such that $X_{v,u} = X_{v',x_0}^+$. Hence equality holds in (2.7). $\square$

## 3. INEQUALITY FOR QUASICONVEX FUNCTIONS SUCH THAT $f(0) = 0$.

Whenever $0 \in X$ and $f(0) = 0$, another Hadamard-type inequality has being obtained in [3], assuming that $f$ is nonnegative and evenly quasiconvex. We generalize this result to nonnegative quasiconvex functions and compare with the previous findings. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing with $h(c) > 0$ for all $c > 0$ and $\lambda_h := \sup_{c>0} \frac{c}{h(c)} < +\infty$ (we follow the notation of [3]).

**Proposition 3.1.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be Borel measurable, nonnegative and quasiconvex. If $0 \in X$ and $f(0) = 0$, then for every $u \in X$,*

$$\inf_{v \in S, \langle v, u \rangle \geq 0} \mu\left(X_{v,u}\right) f(u) \leq \lambda_h \int_X h(f(x))\, d\mu. \tag{3.1}$$

*Proof.* If $f(u) = 0$ we have nothing to prove. Suppose that $f(u) > 0$. Coming back to the proof of Proposition 2.1, we know that there exists $v \in S$ such that $\forall x \in X_{v,u}$, $f(x) \geq f(u)$; hence $h(f(x)) \geq h(f(u))$, from which it follows that

$$\mu\left(X_{v,u}\right) h(f(u)) \leq \int_{X_{v,u}} h(f(x))\, d\mu \leq \int_X h(f(x))\, d\mu.$$

[1] There is sometimes a change in notation with respect to [3].

Note that $0 \notin X_{v,u}$ because $f(0) < f(u)$; thus, $\langle v, u\rangle \geq 0$. Consequently,

$$\inf_{v \in S, \langle v,u\rangle \geq 0} \mu(X_{v,u})\, h(f(u)) \leq \int_X h(f(x))\, d\mu.$$

Finally, note that by definition of $\lambda_h$, $f(u) \leq \lambda_h h(f(u))$ from which follows (3.1). $\square$

Note that relation (3.1) is only interesting if $u \neq 0$ since otherwise it is trivially true. Let us define $\gamma_*$ by

$$\gamma_* = \begin{cases} \dfrac{1}{\inf_{v \in S, \langle v,u\rangle \geq 0} \mu(X_{v,u})} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0. \end{cases} \tag{3.2}$$

We obtain an alternative expression for $\gamma_*$, similar to that in [3]. Given $u \in X \setminus \{0\}$, set $B_u = \{v \in \mathbb{R}^n : \langle v, u\rangle \geq 1\}$, and for any $v \in \mathbb{R}^n$, set $X_v^+ = \{x \in X : \langle v, x\rangle > 1\}$.

**Proposition 3.2.** *The following equality holds for every $u \in X \setminus \{0\}$:*

$$\inf_{v \in B_u} \mu\left(X_v^+\right) = \inf_{v \in S, \langle v,u\rangle > 0} \mu(X_{v,u}).$$

*Proof.* For every $v \in B_u$ we set $v' = v/\|v\|$ and show that $X_{v',u} \subseteq X_v^+$. Indeed, if $x \in X_{v',u}$, then we have $\langle v, x-u\rangle > 0$ hence $\langle v, x\rangle = \langle v, u\rangle + \langle v, x-u\rangle > 1$, i.e., $x \in X_v^+$. Since $\langle v', u\rangle > 0$, it follows that

$$\inf_{v \in B_u} \mu\left(X_v^+\right) \geq \inf_{v \in S, \langle v,u\rangle > 0} \mu(X_{v,u}). \tag{3.3}$$

To show equality, let $v \in S$ be such that $\langle v, u\rangle > 0$. Choose $t > 0$ such that $t\langle v, u\rangle = 1$ and set $v' = tv$. For every $x \in X_{v'}^+$ one has $\langle v', x\rangle > 1$, hence,

$$\langle v', x-u\rangle = \langle v', x\rangle - \langle v', u\rangle > 0.$$

It follows that $\langle v, x-u\rangle > 0$, i.e., $x \in X_{v,u}$. Thus, $X_{v'}^+ \subseteq X_{v,u}$ and $v' \in B_u$. This shows that in (3.3) equality holds. $\square$

**Proposition 3.3.** *If $\mu \ll \lambda$ then we also have the equalities*

$$\gamma_* = \frac{1}{\inf_{v \in S, \langle v,u\rangle > 0} \mu(X_{v,u})} = \frac{1}{\min_{v \in S, \langle v,u\rangle \geq 0} \mu\left(\overline{X}_{v,u}\right)} \text{ (if } u \neq 0\text{)}$$

$$\gamma = \frac{1}{\min_{v \in S} \mu\left(\overline{X}_{v,u}\right)}. \tag{3.4}$$

*Proof.* We first observe that, according to Lemma 2.3, $\mu\left(\overline{X}_{v,u}\right) = \mu(X_{v,u})$. The same lemma entails that $\inf_{v \in S, \langle v,u\rangle > 0} \mu(X_{v,u}) = \inf_{v \in S, \langle v,u\rangle \geq 0} \mu(X_{v,u})$ and that this infimum is attained, since the set $\{v \in S : \langle v, u\rangle \geq 0\}$ is compact. In the same way, the infimum in (2.2) is attained. $\square$

Whenever $\mu \ll \lambda$, the constant $\gamma$ is sharp, in the sense that given $u \in X$, there exists a nonnegative quasiconvex function $f$ such that $f(u) = \gamma \int_X f\, d\mu$. Indeed, since the minimum in (3.4) is attained for some $v_0 \in S$, it is sufficient to take $f$ to be the characteristic function of $\overline{X}_{v_0,u}$ (see Corollary 2 of [3]). Analogous considerations can be made for $\gamma_*$ (see Corollary 4 of [3]).

We now show the equality of $\gamma$ and $\gamma_*$ under a symmetry assumption:

**Corollary 3.4.** *Suppose that $X$ has $0$ as center of symmetry, $u \in \operatorname{int} X \setminus \{0\}$ and $\mu \ll \lambda$. If $\mu(A) = \mu(-A)$ for every Borel $A \subseteq X$, then $\gamma = \gamma_*$.*

*Proof.* For every $v \in S$ such that $\langle v, u\rangle < 0$, set $v' = -v$ and $Y = \{x \in X : \langle v, x + u\rangle > 0\}$. Since $0$ is a center of symmetry, one can check that $Y = -X_{v',u}$.

If $x \in Y$ then $\langle v, x - u\rangle = \langle v, x + u\rangle - 2\langle v, u\rangle > 0$. Thus, $Y \subseteq X_{v,u}$ and $\mu(X_{v,u}) \geq \mu(Y) = \mu(X_{v',u})$. It follows that the minimum in (3.4) can be restricted to $v \in S$ such that $\langle v, u\rangle \geq 0$. Thus, $\gamma = \gamma_*$. □

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