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SUFFICIENT CONDITIONS FOR STARLIKE FUNCTIONS OF ORDER α

Dedicated to the memory of Prof. K.S. Padmanabhan.

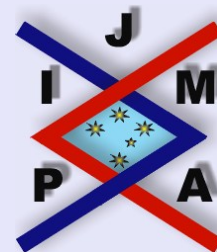
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Abstract

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Abstract

In this paper, we obtain some sufficient conditions for an analytic function $f(z)$, defined on the unit disk Δ , to be starlike of order α .

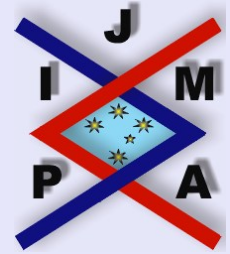
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1. Introduction

Let \mathcal{A}_n be the class of all functions $f(z) = z + a_{n+1}z^{n+1} + \dots$ which are analytic in $\Delta = \{z; |z| < 1\}$ and let $\mathcal{A}_1 = \mathcal{A}$. A function $f(z) \in \mathcal{A}$ is starlike of order α , if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad 0 \leq \alpha < 1,$$

for all $z \in \Delta$. The class of all starlike functions of order α is denoted by $S^*(\alpha)$. We write $S^*(0)$ simply as S^* . Recently, Li and Owa [3] proved the following:

Theorem 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2}, \quad z \in \Delta,$$

for some $\alpha (\alpha \geq 0)$, then $f(z) \in S^$.*

In fact, Lewandowski, Miller and Zlotkiewicz [1] and Ramesha, Kumar, and Padmanabhan [7] have proved a weaker form of the above theorem. If the number $-\alpha/2$ is replaced by $-\alpha^2(1-\alpha)/4$, ($0 \leq \alpha < 2$) in the above condition, Li and Owa [3] have proved that $f(z)$ is in $S^*(\alpha/2)$.

Li and Owa [3] have also proved the following:

Theorem 1.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho, \quad z \in \Delta,$$

where $\rho = 2.2443697$, then $f(z) \in S^$.*



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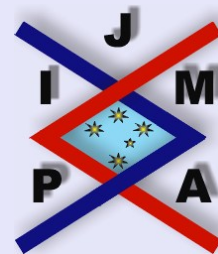
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The above theorem with $\rho = 3/2$ and $\rho = 1/6$ were earlier proved by Li and Owa [2] and Obradovic [6] respectively.

In this paper, we obtain some sufficient conditions for functions to be starlike of order β . To prove our result, we need the following:

Lemma 1.3. [4] *Let Ω be a set in the complex plane \mathcal{C} and suppose that Φ is a mapping from $\mathcal{C}^2 \times \Delta$ to \mathcal{C} which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \Delta$, and for all real x, y such that $y \leq -n(1+x^2)/2$. If the function $p(z) = 1 + c_n z^n + \dots$ is analytic in Δ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \Delta$, then $\operatorname{Re} p(z) > 0$.*



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2. Sufficient Conditions for Starlikeness

In this section, we prove some sufficient conditions for function to be starlike of order β .

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > \alpha\beta \left[\beta + \frac{n}{2} - 1 \right] + \left[\beta - \frac{\alpha n}{2} \right], \quad z \in \Delta, \quad 0 \leq \alpha, \beta \leq 1,$$

then $f(z) \in S^*(\beta)$.

Proof. Define $p(z)$ by

$$(1 - \beta)p(z) + \beta = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \dots$ and is analytic in Δ . A computation shows that

$$\frac{zf''(z)}{f'(z)} = \frac{(1 - \beta)zp'(z) + [(1 - \beta)p(z) + \beta]^2 - [(1 - \beta)p(z) + \beta]}{(1 - \beta)p(z) + \beta}$$

and hence

$$\begin{aligned} \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) &= \alpha(1 - \beta)zp'(z) + \alpha(1 - \beta)^2 p^2(z) \\ &\quad + (1 - \beta)(1 + 2\alpha\beta - \alpha)p(z) + \beta[\alpha\beta + 1 - \alpha] \\ &= \Phi(p(z), zp'(z); z), \end{aligned}$$



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where

$$\Phi(r, s; t) = \alpha(1 - \beta)s + \alpha(1 - \beta)^2 r^2 + (1 - \beta)(1 + 2\alpha\beta - \alpha)r + \beta[\alpha\beta + 1 - \alpha].$$

For all real x and y satisfying $y \leq -n(1 + x^2)/2$, we have

$$\begin{aligned} \operatorname{Re} \Phi(ix, y; z) &= \alpha(1 - \beta)y - \alpha(1 - \beta)^2 x^2 + \beta[\alpha\beta + 1 - \alpha] \\ &\leq -\frac{\alpha}{2}(1 - \beta)n - \left[\frac{n\alpha}{2}(1 - \beta) + \alpha(1 - \beta)^2 \right] x^2 + \beta[\alpha\beta + 1 - \alpha] \\ &= -\frac{\alpha}{2}(1 - \beta)n - \frac{\alpha(1 - \beta)}{2}(n + 2 - 2\beta)x^2 + \beta(\alpha\beta + 1 - \alpha) \\ &\leq \beta(\alpha\beta + 1 - \alpha) - \frac{\alpha}{2}(1 - \beta)n \\ &= \alpha\beta \left(\beta + \frac{n}{2} - 1 \right) + \left(\beta - \frac{n\alpha}{2} \right). \end{aligned}$$

Let $\Omega = \left\{ w; \operatorname{Re} w > \alpha\beta \left(\beta + \frac{n}{2} - 1 \right) + \left(\beta - \frac{n\alpha}{2} \right) \right\}$. Then $\Phi(p(z), zp'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \leq -n(1 + x^2)/2$, $z \in \Delta$. By an application of Lemma 1.3, the result follows. \square

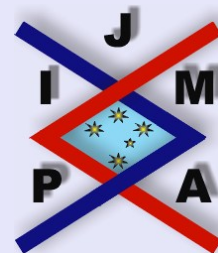
By taking $\beta = 0$ and $n = 1$ in the above theorem, we have the following:

Corollary 2.2. [3] If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2}, \quad z \in \Delta,$$

for some α ($\alpha \geq 0$), then $f(z) \in S^*$.

If we take $\beta = \alpha/2$ and $n = 1$, we get the following:



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Corollary 2.3. [3] If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha^2}{4}(1 - \alpha), \quad z \in \Delta,$$

for some α ($0 < \alpha \leq 2$), then $f(z) \in S^*(\alpha/2)$.

In fact, in the proof of the above theorem, we have proved the following: If $p(z) = 1 + c_n z^n + \dots$ is analytic in Δ and satisfies

$$\begin{aligned} \operatorname{Re}(\alpha(1 - \beta)zp'(z) + \alpha(1 - \beta)^2p^2(z) \\ + (1 - \beta)(1 + 2\alpha\beta - \alpha)p(z) + \beta[\alpha\beta + 1 - \alpha]) \\ > \alpha\beta \left[\beta + \frac{n}{2} - 1 \right] + \left(\beta - \frac{\alpha n}{2} \right), \end{aligned}$$

then $\operatorname{Re} p(z) > 0$. Using a method similar to the one used in the above theorem, we have the following:

Theorem 2.4. Let $\alpha \geq 0$, $0 \leq \beta < 1$. If $f(z) \in \mathcal{A}_n$ satisfies

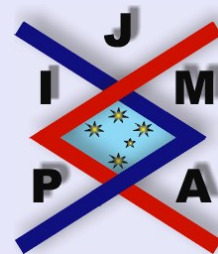
$$\operatorname{Re} \left\{ \frac{f(z)}{z} \left(\alpha \frac{zf'(z)}{f(z)} + 1 - \alpha \right) \right\} > -\frac{n}{2}\alpha(1 - \beta) + \beta, \quad z \in \Delta,$$

then

$$\operatorname{Re} \frac{f(z)}{z} > \beta.$$

As a special case, we get the following: If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re} \{f'(z) + \alpha z f''(z)\} > -\frac{\alpha}{2}, \quad z \in \Delta,$$



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$\alpha \geq 0$, then

$$\operatorname{Re} f'(z) > 0.$$

However, a sharp form of this result was proved by Nunokawa and Hoshino [5].

Theorem 2.5. Let $0 \leq \beta < 1$, $a = (n/2 + 1 - \beta)^2$ and $b = (n/2 + \beta)^2$ satisfy $(a + b)\beta^2 < b(1 - 2\beta)$. Let t_0 be the positive real root of the equation

$$2a(1 - \beta)^2 t^2 + [3a\beta^2 + b(1 - \beta)^2]t + [(a + 2b)\beta^2 - (1 - \beta)^2 b] = 0$$

and

$$\rho^2 = \frac{(1 - \beta)^3 (1 + t_0)^2 (at_0 + b)}{\beta^2 + (1 - \beta)^2 t_0}.$$

If $f(z) \in \mathcal{A}_n$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| \leq \rho, \quad z \in \Delta,$$

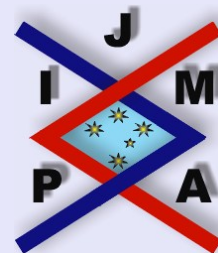
then $f(z) \in S^*(\beta)$.

Proof. Define $p(z)$ by

$$(1 - \beta)p(z) + \beta = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \dots$ and is analytic in Δ . A computation shows that

$$\frac{zf''(z)}{f'(z)} = \frac{(1 - \beta)zp'(z) + [(1 - \beta)p(z) + \beta]^2 - [(1 - \beta)p(z) + \beta]}{(1 - \beta)p(z) + \beta}$$



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and hence

$$\begin{aligned} \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) &= \frac{(1-\beta)(p(z)-1)}{(1-\beta)p(z)+\beta} [(1-\beta)zp'(z) \\ &\quad + [(1-\beta)p(z)+\beta]^2 - [(1-\beta)p(z)+\beta]] \\ &\equiv \Phi(p(z), zp'(z); z). \end{aligned}$$

Then, for all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

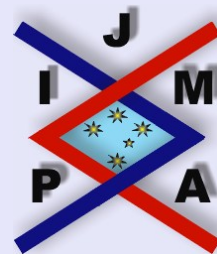
$$\begin{aligned} |\Phi(ix, y; z)|^2 &= \frac{(1-\beta)^2(1+x^2)}{\beta^2 + (1-\beta)^2x^2} \{ [(1-\beta)y - \beta + \beta^2 - (1-\beta)^2x^2]^2 \\ &\quad + [2\beta(1-\beta) - (1-\beta)]^2x^2 \} \\ &= \frac{(1-\beta)^2(1+t)}{\beta^2 + (1-\beta)^2t} \{ [(1-\beta)y - \beta + \beta^2 - (1-\beta)^2t]^2 \\ &\quad + [2\beta(1-\beta) - (1-\beta)]^2t \} \\ &\equiv g(t, y), \end{aligned}$$

where $t = x^2 > 0$ and $y \leq -n(1+t)/2$. Since

$$\frac{\partial g}{\partial y} = \frac{(1-\beta)^3(1+t)}{\beta^2 + (1-\beta)^2t} [(1-\beta)y - \beta + \beta^2 - (1-\beta)^2t]^2 < 0,$$

we have

$$g(t, y) \geq g\left(t, -\frac{n}{2}(1+t)\right) \equiv h(t).$$



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Note that

$$h(t) = \frac{(1 - \beta)^3(1 + t)^2}{\beta^2 + (1 - \beta)^2t} \left[t \left(\frac{n}{2} + 1 - \beta \right)^2 + \left(\frac{n}{2} + \beta \right)^2 \right].$$

Also it is clear that $h'(-1) = 0$ and the other two roots of $h'(t) = 0$ are given by

$$2a(1 - \beta)^2t^2 + [3a\beta^2 + b(1 - \beta)^2]t + [(a + 2b)\beta^2 - (1 - \beta)^2b] = 0,$$

where $a = (n/2 + 1 - \beta)^2$ and $b = (n/2 + \beta)^2$. Since t_0 is the positive root of this equation we have $h(t) \geq h(t_0)$ and hence

$$|\Phi(ix, y; z)|^2 \geq h(t_0).$$

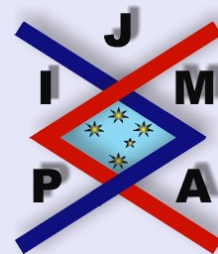
Define $\Omega = \{w; |w| < \rho\}$. Then $\Phi(p(z), zp'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \leq -n(1 + x^2)/2$, $z \in \Delta$. Therefore by an application of Lemma 1.3, the result follows. \square

If we take $n = 1$, $\beta = 0$, we have $t_0 = \frac{\sqrt{73}-1}{36}$ and therefore we have the following:

Corollary 2.6. [3] *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho, \quad z \in \Delta,$$

where $\rho^2 = \frac{827+73\sqrt{73}}{288}$, then $f(z) \in S^*$.



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