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SUFFICIENT CONDITIONS FOR STARLIKE FUNCTIONS OF ORDER α

Dedicated to the memory of Prof. K.S. Padmanabhan.

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Abstract

In this paper, we obtain some sufficient conditions for an analytic function $f(z)$, defined on the unit disk $\triangle$, to be starlike of order α.

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1. Introduction

Let $\mathcal{A}_n$ be the class of all functions $f(z) = z + a_{n+1}z^{n+1} + \cdots$ which are analytic in $\triangle = \{z; |z| < 1\}$ and let $\mathcal{A}_1 = \mathcal{A}$. A function $f(z) \in \mathcal{A}$ is starlike of order α, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad 0 \leq \alpha < 1,$$

for all $z \in \triangle$. The class of all starlike functions of order α is denoted by $S^*(\alpha)$. We write $S^*(0)$ simply as S^*. Recently, Li and Owa [3] proved the following:

Theorem 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right\} > -\frac{\alpha}{2}, \quad z \in \triangle,$$

for some α ($\alpha \geq 0$), then $f(z) \in S^*$.

In fact, Lewandowski, Miller and Zlotkiewicz [1] and Ramesha, Kumar, and Padmanabhan [7] have proved a weaker form of the above theorem. If the number $-\alpha/2$ is replaced by $-\alpha^2(1-\alpha)/4$, $(0 \leq \alpha < 2)$ in the above condition, Li and Owa [3] have proved that $f(z)$ is in $S^*(\alpha/2)$.

Li and Owa [3] have also proved the following:

Theorem 1.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \rho, \quad z \in \triangle,$$

where $\rho = 2.2443697$, then $f(z) \in S^*$.

The above theorem with $\rho = 3/2$ and $\rho = 1/6$ were earlier proved by Li and Owa [2] and Obradovic [6] respectively.

In this paper, we obtain some sufficient conditions for functions to be starlike of order β. To prove our result, we need the following:

Lemma 1.3. *[4] Let Ω be a set in the complex plane $\mathcal{C}$ and suppose that Φ is a mapping from $\mathcal{C}^2 \times \triangle$ to $\mathcal{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \triangle$, and for all real x, y such that $y \leq -n(1+x^2)/2$. If the function $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\triangle$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \triangle$, then $\operatorname{Re} p(z) > 0$.*

2. Sufficient Conditions for Starlikeness

In this section, we prove some sufficient conditions for function to be starlike of order β.

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > \alpha\beta\left[\beta+\frac{n}{2}-1\right]+\left[\beta-\frac{\alpha n}{2}\right], \quad z\in\triangle,\ 0\le\alpha,\beta\le 1,$$

then $f(z) \in S^*(\beta)$.

Proof. Define $p(z)$ by

$$(1-\beta)p(z)+\beta = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \cdots$ and is analytic in $\triangle$. A computation shows that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\beta)zp'(z)+[(1-\beta)p(z)+\beta]^2-[(1-\beta)p(z)+\beta]}{(1-\beta)p(z)+\beta}$$

and hence

$$\begin{aligned}\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) &= \alpha(1-\beta)zp'(z)+\alpha(1-\beta)^2p^2(z)\\ &\quad+(1-\beta)(1+2\alpha\beta-\alpha)p(z)+\beta[\alpha\beta+1-\alpha]\\ &=\Phi(p(z),zp'(z);z),\end{aligned}$$

where

$$\Phi(r,s;t)=\alpha(1-\beta)s+\alpha(1-\beta)^2r^2+(1-\beta)(1+2\alpha\beta-\alpha)r+\beta[\alpha\beta+1-\alpha].$$

For all real x and y satisfying $y\le -n(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\Phi(ix,y;z)&=\alpha(1-\beta)y-\alpha(1-\beta)^2x^2+\beta[\alpha\beta+1-\alpha]\\&\le -\frac{\alpha}{2}(1-\beta)n-\left[\frac{n\alpha}{2}(1-\beta)+\alpha(1-\beta)^2\right]x^2+\beta[\alpha\beta+1-\alpha]\\&=-\frac{\alpha}{2}(1-\beta)n-\frac{\alpha(1-\beta)}{2}(n+2-2\beta)x^2+\beta(\alpha\beta+1-\alpha)\\&\le \beta(\alpha\beta+1-\alpha)-\frac{\alpha}{2}(1-\beta)n\\&=\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right).\end{aligned}$$

Let $\Omega=\left\{w;\operatorname{Re}w>\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right)\right\}$. Then $\Phi(p(z),zp'(z);z)\in\Omega$ and $\Phi(ix,y;z)\notin\Omega$ for all real x and $y\le -n(1+x^2)/2$, $z\in\triangle$. By an application of Lemma 1.3, the result follows. □

By taking $\beta=0$ and $n=1$ in the above theorem, we have the following:

Corollary 2.2. *[3] If $f(z)\in\mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\}>-\frac{\alpha}{2},\quad z\in\triangle,$$

for some α ($\alpha\ge 0$), then $f(z)\in S^*$.

If we take $\beta=\alpha/2$ and $n=1$, we get the following:

Corollary 2.3. *[3] If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{\alpha^2}{4}(1-\alpha), \quad z \in \triangle,$$

for some α $(0 < \alpha \leq 2)$, then $f(z) \in S^*(\alpha/2)$.

In fact, in the proof of the above theorem, we have proved the following: If $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\triangle$ and satisfies

$$\begin{aligned}\operatorname{Re}(\alpha(1-\beta)zp'(z) + \alpha(1-\beta)^2p^2(z)\\ + (1-\beta)(1+2\alpha\beta-\alpha)p(z) + \beta[\alpha\beta+1-\alpha])\\ > \alpha\beta\left[\beta+\frac{n}{2}-1\right]+\left(\beta-\frac{\alpha n}{2}\right),\end{aligned}$$

then $\operatorname{Re} p(z) > 0$. Using a method similar to the one used in the above theorem, we have the following:

Theorem 2.4. *Let $\alpha \geq 0$, $0 \leq \beta < 1$. If $f(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re}\left\{\frac{f(z)}{z}\left(\alpha\frac{zf'(z)}{f(z)}+1-\alpha\right)\right\} > -\frac{n}{2}\alpha(1-\beta)+\beta, \quad z \in \triangle,$$

then

$$\operatorname{Re}\frac{f(z)}{z} > \beta.$$

As a special case, we get the following: If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\{f'(z) + \alpha z f''(z)\} > -\frac{\alpha}{2}, \quad z \in \triangle,$$

$\alpha \geq 0$, then

$$\operatorname{Re} f'(z) > 0.$$

However, a sharp form of this result was proved by Nunokawa and Hoshino [5].

Theorem 2.5. *Let* $0 \leq \beta < 1$, $a = (n/2 + 1 - \beta)^2$ *and* $b = (n/2 + \beta)^2$ *satisfy* $(a + b)\beta^2 < b(1 - 2\beta)$. *Let* t_0 *be the positive real root of the equation*

$$2a(1-\beta)^2 t^2 + [3a\beta^2 + b(1-\beta)^2]t + [(a+2b)\beta^2 - (1-\beta)^2 b] = 0$$

and

$$\rho^2 = \frac{(1-\beta)^3(1+t_0)^2(at_0+b)}{\beta^2 + (1-\beta)^2 t_0}.$$

If $f(z) \in \mathcal{A}_n$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| \leq \rho, \quad z \in \triangle,$$

then $f(z) \in S^*(\beta)$.

Proof. Define $p(z)$ by

$$(1-\beta)p(z) + \beta = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \cdots$ and is analytic in $\triangle$. A computation shows that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - [(1-\beta)p(z)+\beta]}{(1-\beta)p(z)+\beta}$$

and hence

$$\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right) = \frac{(1-\beta)(p(z)-1)}{(1-\beta)p(z)+\beta}[(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - [(1-\beta)p(z)+\beta)]] \equiv \Phi(p(z), zp'(z); z).$$

Then, for all real x and y satisfying $y \le -n(1+x^2)/2$, we have

$$\begin{aligned} |\Phi(ix, y; z)|^2 &= \frac{(1-\beta)^2(1+x^2)}{\beta^2+(1-\beta)^2x^2}\left\{[(1-\beta)y - \beta + \beta^2 - (1-\beta)^2x^2]^2 + [2\beta(1-\beta)-(1-\beta)]^2x^2\right\} \\ &= \frac{(1-\beta)^2(1+t)}{\beta^2+(1-\beta)^2t}\{[(1-\beta)y - \beta + \beta^2 - (1-\beta)^2t]^2 + [2\beta(1-\beta)-(1-\beta)]^2t\} \\ &\equiv g(t,y), \end{aligned}$$

where $t = x^2 > 0$ and $y \le -n(1+t)/2$. Since

$$\frac{\partial g}{\partial y} = \frac{(1-\beta)^3(1+t)}{\beta^2+(1-\beta)^2t}[(1-\beta)y - \beta + \beta^2 - (1-\beta)^2t]^2 < 0,$$

we have

$$g(t,y) \ge g\left(t, -\frac{n}{2}(1+t)\right) \equiv h(t).$$

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Note that

$$h(t) = \frac{(1-\beta)^3(1+t)^2}{\beta^2 + (1-\beta)^2 t}\left[t\left(\frac{n}{2}+1-\beta\right)^2 + \left(\frac{n}{2}+\beta\right)^2\right].$$

Also it is clear that $h'(-1) = 0$ and the other two roots of $h'(t) = 0$ are given by

$$2a(1-\beta)^2 t^2 + [3a\beta^2 + b(1-\beta)^2]t + [(a+2b)\beta^2 - (1-\beta)^2 b] = 0,$$

where $a = (n/2+1-\beta)^2$ and $b = (n/2+\beta)^2$. Since t_0 is the positive root of this equation we have $h(t) \geq h(t_0)$ and hence

$$|\Phi(ix, y; z)|^2 \geq h(t_0).$$

Define $\Omega = \{w; |w| < \rho\}$. Then $\Phi(p(z), zp'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \leq -n(1+x^2)/2$, $z \in \triangle$. Therefore by an application of Lemma 1.3, the result follows. □

If we take $n = 1$, $\beta = 0$, we have $t_0 = \frac{\sqrt{73}-1}{36}$ and therefore we have the following:

Corollary 2.6. *[3] If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \rho, \quad z \in \triangle,$$

where $\rho^2 = \frac{827+73\sqrt{73}}{288}$, then $f(z) \in S^*$.

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