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ON THE SEQUENCE $(p_n^2 - p_{n-1}p_{n+1})_{n\geq 2}$

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ABSTRACT. Let p_n be the n-th prime number and $x_n = p_n^2 - p_{n-1}p_{n+1}$. In this paper, we study sequences containing the terms of the sequence $(x_n)_{n\geq 1}$. The main result asserts that the series $\sum_{n=1}^{\infty} x_n/p_n^2$ is convergent, without being absolutely convergent.

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1. INTRODUCTION

We shall use the following notation:

$$\begin{aligned}
&p_n \text{ the } n\text{-th prime number}\\
&x_n = p_n^2 - p_{n-1}p_{n+1} \text{ for } n \geq 2,\\
&d_n = p_{n+1} - p_n \text{ for } n \geq 1,\\
&q_n = \frac{p_{n+1}}{p_n} \text{ for } n \geq 1,\\
&f(x) \asymp g(x) \text{ if there exist } c_1, c_2, M > 0 \text{ such that}\\
&\quad c_1 f(x) < g(x) < c_2 f(x) \text{ for every } x > M.
\end{aligned}$$

It will be our aim here to study the sequence $(x_n)_{n\geq 2}$ defined above.

It was proved in [1] that the sequence $(d_n)_{n\geq 1}$ is not nonotone. A similar result holds for the sequence $(q_n)_{n\geq 1}$ as well. This means that the sequence $(x_n)_{n\geq 2}$ has infinitely many positive terms, and infinitely many negative terms, hence it is not monotone.

In [1], the so-called method of the triple sieve (due to Vigo Brun) was used to prove that

$$\sum_{p_n \leq x} \left| \log \frac{q_n}{q_{n-1}} \right| \asymp \log x. \tag{1.1}$$

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This result plays an essential role in the following paragraph of the present paper.

Another useful result is proved in [4]:

(1.2) $$\text{the series } \sum_{n=1}^{\infty} \left(\frac{d_n}{p_n}\right)^n \text{ is convergent.}$$

2. THE SERIES $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$

Theorem 2.1. *The series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$ is convergent, but it is not absolutely convergent.*

In order to prove this fact, we need the following lemmas.

Lemma 2.2. *For $x \geq -\frac{1}{2}$, we have*

$$x^2 + |x| \geq |\log(1+x)| \geq |x| - \frac{x^2}{2}.$$

Proof. The inequalities are well known for $x > 0$. When $x \in \left[-\frac{1}{2}, 0\right]$, they take on the form

$$x^2 - x \geq -\log(1+x) \geq -x - \frac{x^2}{2}.$$

Let $f, g\colon \left[-\frac{1}{2}, 0\right] \to \mathbb{R}$ be defined by $f(x) = \log(1+x) - x - \frac{x^2}{2}$ and $g(x) = \log(1+x) - x + x^2$, respectively. We have $f'(x) = -\frac{x(x+2)}{1+x} \geq 0$, and $g'(x) = \frac{x(2x+1)}{1+x} \leq 0$. Since f is increasing and $f(0) = 0$, we get $f(x) \leq 0$. On the other hand, we have $g'(x) < 0$ and $g(0) = 0$, so that $g(x) \geq 0$. □

Lemma 2.3. *The series $\sum_{n=2}^{\infty} \frac{d_n - d_{n-1}}{p_n}$ is convergent.*

Proof. Denote $S_n = \sum_{k=2}^{n} \frac{d_k - d_{k-1}}{p_k}$, so that

$$S_n = \frac{d_n}{p_n} + \sum_{k=2}^{n} \frac{d_{k-1}^2}{p_k p_{k-1}} - \frac{1}{2}.$$

Since $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$, it suffices to prove that the series $\sum_{k=2}^{\infty} \frac{d_{k-1}^2}{p_k p_{k-1}}$ is convergent. Since $\frac{d_{k-1}^2}{p_k p_{k-1}} \sim \left(\frac{d_{k-1}}{p_{k-1}}\right)^2$ and the terms of the series are positive, it follows that the series $\sum_{k=2}^{\infty} \frac{d_{k-1}^2}{p_k p_{k-1}}$ and $\sum_{k=2}^{\infty} \left(\frac{d_{k-1}}{p_{k-1}}\right)^2$ are simultaneously convergent or not. Now just use (1.2) and the proof ends. □

Lemma 2.4. *The series $\sum_{n=2}^{\infty} \frac{x_n^2}{p_n^4}$ is convergent.*

Proof. Since $x_n = d_n d_{n-1} + p_n(d_{n-1} - d_n)$, it follows that

(2.1) $$\frac{x_n}{p_n^2} = \frac{d_n d_{n-1}}{p_n^2} + \frac{d_{n-1} - d_n}{p_n}$$

hence

(2.2) $$\frac{x_n^2}{p_n^4} \leq 2\left(\frac{d_n^2 d_{n-1}^2}{p_n^4} + \frac{(d_{n-1} - d_n)^2}{p_n^2}\right).$$

Since the series $\sum_{n=1}^{\infty} \frac{d_n^2}{p_n^2}$ is convergent and $\frac{d_{n-1}^2}{p_{n-1}^2} \sim \frac{d_{n-1}^2}{p_n^2}$, it follows that the series $\sum_{n=2}^{\infty} \frac{d_{n-1}^2}{p_n^2}$ is convergent as well. This implies that the series $\sum_{n=2}^{\infty} \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2}$ is also convergent. Since

$$\frac{d_n^2 d_{n-1}^2}{p_n^4} < \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2} \quad \text{and} \quad \frac{(d_{n-1} - d_n)^2}{p_n^2} < \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2},$$

we deduce by (2.2) that the series $\sum_{n=2}^{\infty} \frac{x_n^2}{p_n^4}$ is convergent. □

Lemma 2.5. *For $x > 0$ we have*

$$\sum_{p_n \leq x} \left| \frac{q_n - q_{n-1}}{q_{n-1}} \right| \asymp \log x.$$

Proof. In view of Lemma 2.2, we have

$$\left(\frac{q_n - q_{n-1}}{q_{n-1}}\right)^2 + \left|\frac{q_n - q_{n-1}}{q_{n-1}}\right| \geq \left|\log \frac{q_n}{q_{n-1}}\right| \geq \left|\frac{q_n - q_{n-1}}{q_{n-1}}\right| - \frac{1}{2}\left(\frac{q_n - q_{n-1}}{q_{n-1}}\right)^2.$$

Since

$$\frac{q_n - q_{n-1}}{q_{n-1}} = \frac{\frac{p_n+1}{p_n} - \frac{p_n}{p_{n-1}}}{\frac{p_n}{p_{n-1}}} = -\frac{x_n}{p_n^2},$$

we have

$$\frac{1}{2} \cdot \frac{x_n^2}{p_n^4} + \left|\log \frac{q_n}{q_{n-1}}\right| > \left|\frac{q_n - q_{n-1}}{q_{n-1}}\right| \geq \left|\log \frac{q_n}{q_{n-1}}\right| - \frac{x_n^2}{p_n^4}. \tag{2.3}$$

Now the desired conclusion follows by (1.1) and Lemma 2.4. □

Proof of Theorem 2.1. By the relation (2.1) we have

$$S_n = \sum_{k=2}^{n} \frac{x_k}{p_k^2} = \sum_{k=2}^{n} \frac{d_k d_{k-1}}{p_k^2} + \sum_{k=2}^{n} \frac{d_{k-1} - d_k}{p_k}.$$

Since $d_k d_{k-1} \leq \max(d_k^2, d_{k-1}^2)$, and since the series $\sum_{n=2}^{\infty} \frac{\max(d_n^2, d_{n-1}^2)}{p_n^2}$ is convergent, see the proof of Lemma 2.4, it follows that the series $\sum_{n=2}^{\infty} \frac{d_n d_{n-1}}{p_n^2}$ is convergent too. Consequently the sequence $(S'_n)_{n\geq 1}$, defined by $S'_n = \sum_{k=2}^{n} \frac{d_k d_{k-1}}{p_k^2}$ is convergent. Lemma 2.3 implies that the sequence $(S''_n)_{n\geq 1}$, defined by $S''_n = \sum_{k=2}^{n} \frac{d_{k-1} - d_k}{p_k}$ is convergent as well. It then follows that the sequence $(S_n)_{n\geq 2}$ is convergent, that is, the series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$ is convergent.

On the other hand, Lemma 2.5 and the relation $\left|\frac{q_n - q_{n-1}}{q_{n-1}}\right| = \frac{|x_n|}{p_n^2}$ imply that

$$\sum_{p_n \leq x} \frac{|x_n|}{p_n^2} \asymp \log x, \tag{2.4}$$

hence the series $\sum_{n=2}^{\infty} \frac{x_n}{p_n^2}$ is not absolutely convergent. □

3. THE SERIES $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^{\alpha} n}$

Since the series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2}$ is divergent, it is natural to study what "correction" does it need to become convergent. In this connection, we prove the following fact.

Theorem 3.1. *The series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^{\alpha} n}$ is convergent if and only if $\alpha > 1$.*

Proof. We are going to put to use a technique from [3].

To begin with, we recall an inequality due to Abel: Let $a_k, b_k \in \mathbb{R}$, $k \in \overline{1,n}$ such that, if $S_i = \sum_{k=1}^{i} b_k$, then $S_i \geq 0$ for $i \in \overline{1,n}$. Then $\sum_{i=1}^{n} a_i b_i = S_1(a_1 - a_2) + S_2(a_2 - a_3) + \cdots + S_{n-1}(a_{n-1} - a_n) + S_n a_n$, which implies the inequalities

$$\sum_{i=1}^{n} a_i b_i \geq a_n S_n \text{ provided } a_1 \geq \cdots \geq a_n, \tag{3.1}$$

and

$$\sum_{i=1}^{n} a_i b_i \le a_n S_n \quad \text{when} \quad a_1 \le \cdots \le a_n. \tag{3.2}$$

It follows by (2.4) that there exist c_1 and c_2 such that $0 < c_1 < c_2$ and

$$c_1 \log x < \sum_{p_n \le x} \frac{|x_n|}{p_n^2} < c_2 \log x \quad \text{for all} \quad x \ge 2. \tag{3.3}$$

For $\alpha > 0$ and $n \ge 1$, we denote $a_1 = 1, b_1 = 0$ and for $i \ge 2$ $a_i = \frac{1}{\log^\alpha i}$ and $b_i = \frac{c'}{i} \cdot \frac{|x_i|}{p_i^2}$, where $c' > 0$ is chosen such that $S_1, S_2, \ldots, S_n \ge 0$. Such a choice is possible because $\sum_{2 \le i \le x} \frac{1}{i} \sim \log x$ and (3.3) holds.

It now follows by (3.1) that $\sum_{i=2}^{n} \frac{1}{\log^\alpha i} \left(\frac{c'}{i} - \frac{|x_i|}{p_i^2} \right) \ge 0$, that is, $\sum_{i=2}^{n} \frac{|x_i|^2}{p_i^2 \log^\alpha i} < c' \sum_{i=2}^{n} \frac{1}{i \log^\alpha i}$. Since the series $\sum_{i=2}^{\infty} \frac{1}{i \log^\alpha i}$ is convergent for $\alpha > 1$, we deduce that the series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^\alpha n}$ is convergent as well.

One can similarly show that there exists $c'' > 0$ such that

$$\sum_{i=2}^{n} \frac{|x_i|^2}{p_i^2 \log^\alpha i} > c'' \sum_{i=2}^{n} \frac{1}{i \log^\alpha i}.$$

Since the series $\sum_{i=2}^{\infty} \frac{1}{i \log^\alpha i}$ is divergent for $\alpha \le 1$, it follows that in this case the series $\sum_{n=2}^{\infty} \frac{|x_n|}{p_n^2 \log^\alpha n}$ is in turn divergent. □

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