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SOME INEQUALITIES AND BOUNDS FOR WEIGHTED RELIABILITY MEASURES

BRODERICK O. OLUYEDE

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
GEORGIA SOUTHERN UNIVERSITY
STATESBORO, GA 30460.
boluyede@gasou.edu

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ABSTRACT. Weighted distributions occur naturally in a wide variety of settings with applications in reliability, forestry, ecology, bio-medicine, and many other areas. In this note, bounds and stability results on the distance between weighted reliability functions, residual life distributions, equilibrum distributions with monotone weight functions and the exponential counterpart in the class of distribution functions with increasing or decreasing hazard rate and mean residual life functions are established. The problem of selection of experiments from the weighted distributions as opposed to the original distributions is addressed. The reliability inequalities are applied to repairable systems.

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1. INTRODUCTION

When data is unknowingly sampled from a weighted distribution as opposed to the parent distribution, the survival function, hazard function, and mean residual life function (MRLF) may be under or overestimated depending on the weight function. For size-biased sampling, the analyst will usually give an over optimistic estimate of the survival function and mean residual life functions. It is well known that the size-biased distribution of an increasing failure rate (IFR) distribution is always IFR. The converse is not true. Also, if the weight function is monotone increasing and concave, then the weighted distribution of an IFR distribution is an IFR distribution. Similarly, the size-biased distribution of a decreasing mean residual (DMRL) distribution has decreasing mean residual life. The residual life at age t, is a weighted distribution, with survival function given by

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \tag{1.1}$$

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for $x \geq 0$. The weight function is $W(x) = f(x+t)/f(x)$, where $f(u) = dF(u)/du$, the hazard function and mean residual life functions are $\lambda_{F_t}(x) = \lambda_F(x+t)$ and $\delta_{F_t}(x) = \delta_F(x+t)$. It is clear that if F is IFR, (DMRL) distribution, then F_t is IFR, (DMRL) distribution. The hazard function $\lambda_F(x)$ and mean residual life function $\delta_F(x)$ are given by $\lambda_F(x) = f(x)/\overline{F}(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ respectively. The functions $\lambda_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent ([6]). The purpose of this article is to establish bounds and stability results on the distance between weighted reliability functions, residual life distributions, equilibrium distributions with monotone weight functions and the exponential counterpart in the class of life distributions with increasing or decreasing hazard rate and mean residual life functions. We also present results on increasing hazard rate average (IHRA) weighted distributions and obtain inequalities for the weighted mean residual life function for large values of the response. In Section 2 some basic results and utility notions are presented. Section 3 contains stochastic inequalities for reliability measures under distributions with monotone weight functions as well as inequalities for IHRA and mean residual life weighted distributions. In Section 4 inequalities and stability results for repairable systems are established. Section 5 contains results on selection of experiments under weighted distributions as opposed to the parent distributions.

2. SOME BASIC RESULTS AND UTILITY NOTIONS

Let X be a nonnegative random variable with distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $W(x)$ be a positive weight function such that $0 < E(W(X)) < \infty$. The weighted distribution of X is given by

$$\overline{F}_W(x) = \frac{\overline{F}(x)\{W(x) + M_F(x)\}}{E(W(X))}, \tag{2.1}$$

where

$$M_F(x) = \frac{\int_x^\infty \overline{F}(t)W'(t)dt}{\overline{F}(x)},$$

assuming $W(x)\overline{F}(x) \longrightarrow 0$ as $x \longrightarrow \infty$. The corresponding pdf of the weighted random variable X_W is

$$f_W(x) = \frac{W(x)f(x)}{E(W(X))}, \tag{2.2}$$

$x \geq 0$, where $0 < E(W(X) < \infty$. We now give some basic and important definitions.

Definition 2.1. Let X and Y be two random variables with distribution functions F and G respectively. We say $F <_{st} G$, stochastically ordered, if $\overline{F}(x) \leq \overline{G}(x)$, for $x \geq 0$ or equivalently, for any increasing function $\Phi(x)$,

$$E(\Phi(X)) \leq E(\Phi(Y)). \tag{2.3}$$

Definition 2.2. A distribution function F is an increasing hazard rate (IHR) distribution if $\overline{F}(x+t)/\overline{F}(t)$ is decreasing in $0 < t < \infty$ for each $x \geq 0$. Similarly, a distribution function F is an decreasing hazard rate (DHR) distribution if $\overline{F}(x+t)/\overline{F}(t)$ is increasing in $0 < t < \infty$ for each $x \geq 0$. It is well known that IHR (DHR) implies DMRL (IMRL).

Definition 2.3. Let F be a right-continous distribution such that $F(0+) = 0$. F is said to be an increasing hazard rate average (IHRA) distribution if and only if for all $0 \leq \alpha \leq 1$, and $x \geq 0$,

$$\overline{F}(\alpha x) \geq \overline{F}^\alpha(x). \tag{2.4}$$

It is well known that F is an IHRA distribution if and only if

$$\int g(x)dF(x) \leq \left\{ \int g^{\alpha}\left(\frac{x}{\alpha}\right) dF(x) \right\}^{1/\alpha}, \tag{2.5}$$

for all $0 < \alpha < 1$ and all nonnegative non-decreasing functions g.

Consider a renewal process with life distribution $F(x)$ and weighted distribution function $F_W(x)$, with weight function $W(x) > 0$. Let X_t denote the residual life of the unit functioning at time t. Then as $t \longrightarrow \infty$, X_t has the limiting reliability function given by

$$\overline{F}_e(x) = \mu^{-1} \int_x^{\infty} \overline{F}(y)dy, \tag{2.6}$$

$x \geq 0$. The corresponding limiting pdf is

$$f_e(x) = \frac{\overline{F}(x)}{\mu_F}, \tag{2.7}$$

$x \geq 0$. The weighted equilibrium reliability or survival function is

$$\overline{F}_{W_e}(x) = \mu_{F_W}^{-1} \int_x^{\infty} \overline{F}_W(y)dy, \tag{2.8}$$

where $F_W(x)$ is given by (2.1). Note that $\overline{F}_{W_e}(x)$ can be expressed as

$$\overline{F}_{W_e}(x) = \mu_{F_W}^{-1} \overline{F}_W(x) \delta_{F_W}(x), \tag{2.9}$$

$x \geq 0$, and the corresponding hazard function is

$$\lambda_{F_{W_e}}(x) = \frac{f_{W_e}(x)}{\overline{F}_{W_e}(x)} = (\delta_{F_{W_e}}(x))^{-1}, \tag{2.10}$$

$x \geq 0$, where

$$\delta_{F_{W_e}}(x) = (\delta_F(x)\overline{F}(x))^{-1} \int_x^{\infty} \overline{F}(y)\delta_F(y)dy, \tag{2.11}$$

$x \geq 0$.

3. INEQUALITIES FOR WEIGHTED RELIABILITY MEASURES

In this section, we derive inequalities for reliability measures for weighted distributions. Bounds and stability results on the distance between the equilibrium reliability functions, weighted reliability functions and the size-biased equilibrium exponential distributions are established. These results are given in the context of life distributions with monotone hazard and mean residual life functions.

Theorem 3.1. *([1]). If F has DMRL, then $S_k(x) \leq S_k(0)e^{-x/\mu}, k = 1, 2, \ldots,$ and $S_k(x) \geq \mu S_{k-1}(0)e^{-x/\mu} - \mu S_{k-1}(0) + S_k(0), k = 2, 3, \ldots,$ where*

$$S_k(x) = \begin{cases} \overline{F}(x) & \text{if } k = 0, \\ \dfrac{\int_0^{\infty} \overline{F}(x+t)t^{k-1}dt}{(k-1)!} & \text{if } k = 1, 2, \ldots, \end{cases}$$

is a sequence of decreasing functions for which F possess moments of order J, that is $\mu_k = E(X^k)$ exists, $k = 1, 2, \ldots, J$.

We let $S_{-1}(x) = f(x)$ be the pdf of F if it exists. Then $S_k(0) = \mu_k/k!$, and $S'_k(x) = -S_{k-1}(x)$, $k = 0, 1, 2, \ldots, J$. The ratio $S_{k-1}(x)/S_k(x)$ is a hazard function of a distribution function with survival function $S_k(x)/S_k(0)$. The inequalities in Theorem 3.1 are reversed if F has increasing mean residual life (IMRL).

Theorem 3.2. *Let $\overline{F}_{W_e}(x)$ be an IHR weighted equilibrium reliability function with increasing weight function. Then*

$$\int_0^\infty |\overline{F}_{W_e}(x) - xe^{-x/\mu}|dx \le 2\left|\mu^2 - \frac{\mu_2}{2\mu}\right|. \tag{3.1}$$

Proof. Let $A = \{x|\overline{F}_{W_e} \le xe^{-x/\mu}\}$. Then we have for $x > 0$,

$$\begin{aligned}
\int_0^\infty |\overline{F}_{W_e}(x) - xe^{-x/\mu}|dx &\le 2\int_A (xe^{-x/\mu} - \overline{F}_{W_e}(x))dx \\
&\le 2\int_0^\infty (xe^{-x/\mu} - \overline{F}_e(x))dx \\
&= 2\int_0^\infty \left(xe^{-x/\mu} - \frac{S_1(x)}{\mu}\right)dx \\
&= 2\left(\mu^2 - \frac{S_2(0)}{\mu}\right) \\
&= 2\left(\mu^2 - \frac{\mu_2}{2\mu}\right). \qquad (3.2)
\end{aligned}$$

The first inequality is trivial and the second inequality is due to the stochastic order between $\overline{F}_{W_e}(x)$ and $\overline{F}_e(x)$ when $W(x)$ is increasing in $x \ge 0$. □

Theorem 3.3. *Let $\overline{F}_{W_e}$ be the weighted equilibrium reliability function with decreasing hazard rate(DHR). If the weight function $W(x)$ is increasing, then*

$$\int_0^\infty |\overline{F}_{W_e}(x) - xe^{-x/\mu}|dx \ge 2e^{-\eta/\mu}\max\{0, |\mu - \eta\mu - 1|\},$$

for $\eta \ge \mu$.

Proof. Let $\overline{F}_{W_e}$ be DHR reliability function, then there exist $\eta \ge \mu$ such that $\overline{F}_{W_e}(x) \le xe^{-x/\mu}$ or $\overline{F}_{W_e}(x) \ge xe^{-x/\mu}$ as $x \le \eta$ or $x \ge \eta$. Now,

$$\begin{aligned}
\int_0^\infty |\overline{F}_{W_e}(x) - xe^{-x/\mu}|dx &= 2\int_\eta^\infty (\overline{F}_{W_e}(x) - xe^{-x/\mu})dx \\
&\ge 2\int_\eta^\infty (\overline{F}_e(x) - xe^{-x/\mu})dx \\
&= \left(\frac{2}{\mu}\right)\int_\eta^\infty (S_1(x) - \mu e^{-x/\mu})dx \\
&= \left(\frac{2}{\mu}\right)(S_2(\eta) - \{\mu^2\eta e^{-\eta/\mu} + \mu e^{-\eta/\mu}\}) \\
&\ge 2S_1(\eta) - 2e^{-\eta/\mu}(\mu\eta + 1) \\
&\ge 2\mu S_0(\eta) - 2e^{-\eta/\mu}(\mu\eta + 1) \\
&= 2e^{-\eta/\mu}(\mu - \eta\mu - 1). \qquad (3.3)
\end{aligned}$$

The first inequality is due to the fact that $W(x)$ is increasing in x, so that $\overline{F}_{W_e}(x)$ and $\overline{F}_e(x)$ are stochastically ordered. The last two inequalities are due to the fact that $S_k(x) \geq \mu S_{k-1}(x)$ for all $x \geq 0$, $k = 1, 2, \dots$, where $S_k(x) = \int_x^\infty S_{k-1}(y)dy$. $\square$

Theorem 3.4. *Let $\overline{F}$ and $\overline{F}_W$ be the parent and weighted survival functions respectively. Suppose $W(x)$ is increasing and $\overline{F}$ has DMRL, then*

$$\overline{F}_W(x) \geq e^{-x/\mu} \max\left\{0, \left(\frac{2W(x)}{\alpha} + \frac{\mu_2}{\alpha\mu}\right) - \frac{2}{\alpha} - \frac{\mu_2}{2\mu}\right\}, \tag{3.4}$$

where $0 < \alpha = E(W(X)) < \infty$.

Proof. Note that

$$\begin{aligned}
\overline{F}_W(x) &= \alpha^{-1}\left\{W(x)\overline{F}(x) + \int_x^\infty \overline{F}(y)W'(y)dy\right\} \\
&\geq \alpha^{-1}\frac{W(x)S_2(x)}{S_2(0)} + \frac{\int_x^\infty S_2(y)dy}{S_2(0)} \\
&= (\alpha S_2(0))^{-1}\{W(x)S_2(0) + S_3(x)\} \\
&\geq (\alpha S_2(0))^{-1}\{W(x)[\mu S_1(0)e^{-x/\mu} - \mu S_1(x)] + \mu S_2(0)e^{-x/\mu} - \mu S_2(0)\} \\
&= (\alpha S_2(0))^{-1}\{\mu(e^{-x/\mu}[W(x)S_1(0) + S_2(0)] - S_1(0) - S_2(0))\} \\
&= 2(\alpha\mu)^{-1}\left\{e^{-x/\mu}\left(W(x)\mu + \frac{\mu_2}{2}\right) - \mu - \frac{\mu_2}{2}\right\} \\
&= e^{-x/\mu}\left\{\frac{2W(x)}{\alpha} + \frac{\mu_2}{\alpha\mu}\right\} - \frac{2}{\alpha} - \frac{\mu_2}{\alpha\mu}. \qquad (3.5)
\end{aligned}$$

The inequalities follows from the fact that $W(x)$ is increasing and from the application of Theorem 3.1. $\square$

Theorem 3.5. *Let F_W be a weighted distribution function with increasing weight function $W(x) \geq 0$, then $\delta_F(x) \leq \delta_{F_W}(x) \leq (\lambda_{F_W}(x))^{-1}$ for all $x > 0$. Furthermore, if the hazard rate $\lambda_{F_W}(x)$ is such that*

$$\lambda_{F_W}(x) \geq \frac{c}{x} \tag{3.6}$$

for $x \geq x_0$, where c is a positive real number, then

$$\left(\frac{1}{t}\right)\int_0^t \{\lambda_F(x)\}^{-k}dx \geq c^* t^{-k} \tag{3.7}$$

for $t > 0$, $k > 1$, where c^* is a real number.

Proof. Let $W(x)$ be increasing in x, then $\lambda_{F_W}(x) \leq \lambda_F(x)$ and $\delta_F(x) \leq \delta_{F_W}(x)$ for all $x > 0$. Note that,

$$\delta_{F_W}(x) = \frac{\int_x^\infty \overline{F}_W(y)dy}{\overline{F}(x)} \leq \int_0^\infty e^{-\lambda_{F_W}(x)y}dy = (\lambda_{F_W}(x))^{-1}, \tag{3.8}$$

for all $x > 0$. Consequently,

$$\begin{aligned}\left(\frac{1}{t}\right)\int_0^t \{\delta_{F_W}(x)\}^{-k}dx &\geq \left(\frac{1}{t}\right)\int_0^t \{\lambda_{F_W}(x)\}^{k}dx \\ &\geq \left(\frac{1}{t}\right)\int_0^t \left(\frac{c}{x}\right)^k dx \\ &= c^* t^{-k} \end{aligned} \tag{3.9}$$

for $t > 0$. □

Theorem 3.6. *Let F_W be a weigthed distribution function with increasing weight function $W(x) \geq \delta^* > 0$. If F is an IHRA distribution, then F_W is an IHRA distribution.*

Proof. Let F be an IHR distribution, and $W(x) \geq \delta^* > 0$ be increasing. We show that $\overline{F}_W(\alpha x) \geq \overline{F}_W^{\alpha}(x)$ for all $0 < \alpha \leq 1$, and $x \geq 0$.

Clearly,

$$\overline{F}_W(\alpha x) \geq \overline{F}(\alpha x)$$

for all $0 \leq \alpha \leq 1$, and $x \geq 0$. This follows from the fact that F_W and F are stochastically ordered. For all $0 \leq \alpha \leq 1$, and $x \geq 0$, set

$$h(x) = \overline{F}^{\alpha}(x) - \overline{F}_W^{\alpha}(x),$$

and $\delta^* = E(W(X))$. Simple computation yeilds

$$\begin{aligned} h'(x) &= \alpha \overline{F}_W^{\alpha-1}(x)\left\{\frac{W(x)f(x)}{E(W(X))}\right\} - \alpha\overline{F}^{\alpha-1}(x)f(x) \\ &\geq \alpha\overline{F}^{\alpha-1}(x)f(x)\left\{\frac{W(x)}{E(W(X))} - 1\right\}. \end{aligned} \tag{3.10}$$

Since $\alpha\overline{F}^{\alpha-1}(x)f(x) \geq 0$, $W(x) \geq \delta^*$ and $h(0) = 0$, thus $h(x)$ is increasing and $h(x) \geq 0$. Using the fact that F is an IHRA distribution we get

$$\overline{F}_W(\alpha x) \geq \overline{F}(\alpha x) \geq \overline{F}^{\alpha}(x) \tag{3.11}$$

for all $0 \leq \alpha \leq 1$, and $x \geq 0$. Since $h(x) \geq 0$ for all $x \geq 0$, it follows therefore that

$$\overline{F}_W(\alpha x) \geq \overline{F}_W^{\alpha}(x),$$

for all $0 \leq \alpha \leq 1$, and $x \geq 0$. The proof is complete. □

Theorem 3.7. *Let G_W and H_W be two weighted distribution functions with increasing weight functions. Suppose the conditions of the previous theorem are satisfied, then the convolution of G_W and H_W, $G_W * H_W$, is an IHRA distribution.*

Proof. The proof follows directly from [2]. □

Theorem 3.8. *Let F_W be a weighted distribution function with increasing weight function $W(x)$ and pdf $f_W(x) > 0$ for $x \geq x_0$. Suppose the hazard function $\lambda_{F_W}(x)$ is such that $\lambda_{F_W}(x) \geq \frac{c}{x}$ for $x \geq x_0$, where c is a real positive number. If X is the original random variable then*

$$P(X - x \leq xt | X > t) \leq 1 - (1+t)^{-c}$$

for all $t > 0$ and $x \geq x_0$.

Proof. Let $W(x)$ be increasing in x, then the hazard function of the distribution function F of X satisfies the inequality

$$\lambda_F(x) \geq \lambda_{F_W}(x) \geq \frac{c}{x}$$

for $x \geq x_0$. Now for $t > 0$,

$$\int_x^{(1+t)x} \lambda_F(y)dy \geq \int_x^{(1+t)x} \lambda_{F_W}(y)dy \geq c\int_x^{(1+t)x} \left(\frac{1}{y}\right) dy \geq 1-(1+t)^{-1}, \tag{3.12}$$

for $t > 0$, using the fact that $ln(a) \geq 1 - a^{-1}$ for $a > 0$. □

The last result provides simple inequalities for the lower bound of the residual life time distributions for large values of x from the use of the information about the hazard function of the weighted distribution function.

4. INEQUALITIES FOR REPAIRABLE SYSTEMS

In this section we obtain useful inequalities for repairable systems under weighted distributions. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of operating times from a repairable system that start functioning at time $t = 0$. The sequence of times $\{X_i\}_{i=1}^{\infty}$ form a renewal-type stochastic point process. Following [4], if a system has virtual age $T_{m-1} = t$ immediately after the $(m-1)^{th}$ repair, then the length of the m^{th} cycle X_m has the distribution

$$F_t(x) = P(Xm \leq x | T_{m-1} = t) = \frac{\{F(x+t) - F(t)\}}{\overline{F}(t)}, \tag{4.1}$$

$x \geq 0$, where $\overline{F}(x) = 1 - F(x)$ is the reliability function of a new system. When $t = \left[\sum_{i=1}^{j} X_i\right]$, $j = 1, 2, \ldots, m-1$, minimal repair is performed, keeping the virtual age intact and when $t = 0$ we have perfect repair. The virtual age of the system is equal to its operating time for the case of minimal repair. The reliability function corresponding to (4.1) is given by

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \tag{4.2}$$

$x \geq 0$. The weighted reliability function corresponding to (4.2) is given by

$$\overline{F}_{W_t}(x) = \frac{\overline{F}_W(x+t)}{\overline{F}_W(t)}, \tag{4.3}$$

$x \geq 0$.

Theorem 4.1. *If $\overline{F}_{W_t}(x)$ is an IHR reliability function with increasing weight function. Then*

$$\int_0^{\infty} |\overline{F}_{W_t}(x) - e^{-x/\mu}| dx \leq 2\mu \left| 1 - \frac{\mu_2}{2\mu^2} \right|. \tag{4.4}$$

Proof. Let $A = \{x|\overline{F}_{W_t} \leq e^{-x/\mu}\}$. Then we have for $x \geq 0$,

$$
\begin{aligned}
\int_0^\infty |\overline{F}_{W_t}(x) - e^{-x/\mu}|dx &\leq 2\int_A (e^{-x/\mu} - \overline{F}_{W_t}(x))dx \\
&\leq 2\int_0^\infty (e^{-x/\mu} - \overline{F}_{W_t}(x))dx \\
&\leq 2\left\{\int_0^\infty (e^{-x/\mu} - \overline{F}_W(x+t))dx\right\} \\
&\leq 2\left\{\int_0^\infty (e^{-x/\mu} - \overline{F}(x+t))dx\right\} \\
&\leq 2\int_0^\infty \left(e^{-x/\mu} - \frac{S_1(x+t)}{\mu}\right)dx \\
&= 2\left(\mu - \frac{\mu^2}{2\mu}\right) \\
&= 2\mu\left(1 - \frac{\mu_2}{2\mu^2}\right). \qquad (4.5)
\end{aligned}
$$

The first two inequalities are straightforward,the third inequality follows from the fact that $W(x)$ is increasing, so that $\overline{F}_W$ and $\overline{F}$ are stochastically ordered. The fourth and fifth inequalities follow from Theorem 3.1. □

Theorem 4.2. *If $\overline{F}_{W_t}(x)$ is an DHR reliability function,*

$$\int_0^\infty |\overline{F}_{W_t}(x) - e^{-x/\mu}|dx \geq 2\mu e^{-\epsilon/\mu}|e^{-t/\mu} - 1|, \qquad (4.6)$$

provided $W(x)$ is an increasing weight function.

Proof. Let $\overline{F}_{W_t}$ be a DHR survival function, then there exist $\epsilon \geq \mu$ such that $\overline{F}_{W_t} \leq e^{-x/\mu}$ or $\overline{F}_{W_t} \geq e^{-x/\mu}$ as $x \leq \epsilon$ or $x \geq \epsilon$. Now,

$$
\begin{aligned}
\int_0^\infty |\overline{F}_{W_t}(x) - e^{-x/\mu}|dx &= 2\int_\epsilon^\infty (\overline{F}_{W_t}(x) - e^{-x/\mu})dx \\
&\geq 2\int_\epsilon^\infty (\overline{F}_W(x+t) - e^{-x/\mu})dx \\
&\geq 2\int_\epsilon^\infty (\overline{F}(x+t) - e^{-x/\mu})dx \\
&= 2\{S_1(\epsilon+t) - \mu e^{-\epsilon/\mu}\} \\
&\geq 2\mu S_0(\epsilon+t) - \mu e^{-\epsilon/\mu} \\
&= 2\mu e^{-\epsilon/\mu}(e^{-\epsilon/\mu} - 1). \qquad (4.7)
\end{aligned}
$$

The first inequality is trivial. The second inequality follows from the fact that $W(x)$ is increasing, so that $\overline{F}_W(y) \geq \overline{F}(y)$ for all $y \geq 0$. The last inequality follow from Theorem 3.1. □

Theorem 4.3. *If $\overline{F}_{W_e}(x)$ is an IHR reliability function with increasing weight function. Then*

$$\int_0^\infty |\overline{F}_{W_e}(x) - e^{-x/\mu}|dx \leq 2\mu\left|1 - \frac{\mu_2}{2\mu^2}\right|. \qquad (4.8)$$

Proof. Let $D = \{x | \overline{F}_{W_e} \leq e^{-x/\mu}\}$. Then we have for $x \geq 0$,

$$\begin{aligned}
\int_0^\infty |\overline{F}_{W_e}(x) - e^{-x/\mu}| dx &\leq 2 \int_D (e^{-x/\mu} - \overline{F}_{W_e}(x)) dx \\
&\leq 2 \int_0^\infty (e^{-x/\mu} - \overline{F}_{W_e}(x)) dx \\
&= 2 \int_0^\infty \left\{ e^{-x/\mu} - \mu_{F_W}^{-1} \int_x^\infty \overline{F}_W(y)) dy \right\} dx \\
&\leq 2 \int_0^\infty \left\{ e^{-x/\mu} - \mu_{F_W}^{-1} \int_x^\infty \overline{F}(y)) dy \right\} dx \\
&= 2\mu - \frac{2S_2(0)}{\mu} \\
&= 2\mu - \frac{\mu_2}{\mu} \\
&= 2\mu \left(1 - \frac{\mu_2}{2\mu^2} \right). \qquad (4.9)
\end{aligned}$$

The first two inequalities are straightforward,the third inequality follows from the fact that $W(x)$ is increasing, so that $\overline{F}_{W_e}$ and $\overline{F}_e$ are stochastically ordered. □

Theorem 4.4. *If $\overline{F}_{W_e}(x)$ is an DHR reliability function,*

$$\int_0^\infty |\overline{F}_{W_e}(x) - e^{-x/\mu}| dx \geq 2\mu e^{-\epsilon/\mu} \left| \left(\frac{\mu^2}{\mu_2} \right) - \frac{\mu_2}{\mu} \right|, \qquad (4.10)$$

provided $W(x)$ is an increasing weight function.

Proof. Using the fact that $\overline{F}_{W_e}$ have DHR survival function, we have, for $\epsilon \geq \mu$

$$\begin{aligned}
\int_0^\infty |\overline{F}_{W_e}(x) - e^{-x/\mu}| dx &= 2 \int_\epsilon^\infty (\overline{F}_{W_e}(x) - e^{-x/\mu}) dx \\
&= 2 \int_\epsilon^\infty \left\{ \mu_{F_W}^{-1} \int_x^\infty \overline{F}(y) dy - e^{-x/\mu} \right\} dx \\
&\geq 2 \int_\epsilon^\infty \left\{ \mu_{F_W}^{-1} \int_x^\infty \overline{F}(y) dy - e^{-x/\mu} \right\} dx \\
&= \frac{2}{\mu_{F_W}} \int_\epsilon^\infty (S_1(x) - \mu_{F_W} e^{-x/\mu}) dx \\
&= \frac{2}{\mu_{F_W}} S_2(\epsilon) - 2\mu_{F_W}(\mu e^{-\epsilon/\mu}) \\
&\geq \left(\frac{2\mu}{\mu_{F_W}} \right) S_1(\epsilon) - 2\mu_{F_W}(\mu e^{-\epsilon/\mu}) \\
&\geq \left(\frac{2\mu^2}{\mu_{F_W}} \right) S_0(\epsilon) - 2\mu_{F_W}(\mu e^{-\epsilon/\mu}) \\
&= 2\mu^3 e^{-\epsilon/\mu} - 2 \left(\frac{\mu_2}{\mu} \right) e^{-\epsilon/\mu} \\
&= 2\mu e^{-\epsilon/\mu} \left\{ \frac{\mu^2}{\mu_2} - \frac{\mu_2}{\mu} \right\}, \qquad (4.11)
\end{aligned}$$

where $\mu_{F_W} = \int_0^\infty \overline{F}_W(x)dx$. The inequalities follows from the fact that $W(x)$ is increasing, so that $\overline{F}_W(y) \geq \overline{F}(y)$ for all $y \geq 0$, and $S_k(x) \geq S_{k-1}(x)$ for all $x \geq 0$, $k \geq 1$. □

5. SELECTION OF EXPERIMENTS

In this section, we deal with the problem of sampling and selection of experiments from weighted distribution as opposed to the original distribution. The results also extends to any general competing distribution functions F and G with probability density functions f and g respectively. For this purpose we consider two probability spaces (Ω, Ψ, ν_1) and (Ω, Ψ, ν_2) such that the probability measures ν_1 and ν_2 are absolutely continous with respect to each other. Let λ be a probability measure defined on Ψ and equivalent to ν_1 and ν_2. Suppose $f(x)$ and $g(x)$ are Radon-Nikodym derivatives of ν_1 and ν_2 with respect to λ.

Theorem 5.1. *Let*

(1) $B_1(x;\eta) = \min\left(\frac{f(x)}{g_W(x)} - \eta, 0\right)$,
and
(2) $B_2(x;c) = \min\left(\frac{g_W(x)}{f(x)} - \eta, 0\right)$,

and suppose $f(x)$ and $g_W(x)$ satisfy,

$$P_f(x : g_W(x) = 0) = P_{g_W}(x : f(x) = 0),$$

then

$$B(\eta f, g_W) \leq B(f, \eta g_W)$$

if and only if

$$E_{g_W}\{B_1(x,;\eta)\} \leq E_f\{B_2(x;\eta)\},$$

where E_f and E_{g_W} are expectations with respect to the probability density functions f and g_W, respectively, $g_W(x) = W(x)f(x)/E(W(X))$, $W(x)$ is increasing in x, and

$$B(f, g_W) = \int \max\{f(x), g_W(x)\} d\lambda(x).$$

Proof. Note that

$$B(f, \eta g_W) = \int_{\{x:\eta g_W(x) \leq f(x)\}} f(x)\, d\lambda(x) + \int_{\{x:\eta g_W(x) > f(x)\}} \eta g_W(x)\, d\lambda(x).$$

Similarly,

$$B(\eta f, g_W) = \int_{\{x:\eta f(x) > g_W(x)\}} \eta f(x)\, d\lambda(x) + \int_{\{x:\eta f(x) \leq g_W(x)\}} g_W(x)\, d\lambda(x).$$

Note that,

$$\begin{aligned} B(\eta f, g_W) - B(f, \eta g_W) &= \int_{\{x:\eta g_W(x) > f(x)\}} \{\eta g_W(x) - f(x)\}\, \lambda(x) \\ &\quad - \int_{\{x:\eta f(x) > g_W(x)\}} \{\eta f(x) - g_W(x)\}\, d\lambda(x) \\ &= \int_{\{x:\eta g_W(x) > f(x)\}} \{(f(x)/g_W(x)) - \eta\} g_W(x)\, d\lambda(x) \\ &\quad - \int_{\{x:\eta f(x) > g_W(x)\}} \{(g_W(x)/f(x)) - \eta\} f(x) d\lambda(x) \\ &= E_{g_W}\{B_1(x,\eta)\} - E_f\{B_2(x,\eta)\}. \end{aligned} \tag{5.1}$$

Consequently,

$$B(\eta f, g_W) \leq B(f, \eta g_W)$$

if and only if

$$E_{g_W}\{B_1(x,\eta)\} \leq E_f\{B_2(x,\eta)\}.$$

□

Now let us consider the case in which one of the two hypothesis is true. The hypothesis are $H_0 : X \sim f$ and $Y \sim g_W$, and $H_1 : X \sim g_W$ and $Y \sim f$.

Let $R_{X(Y)}(\pi_i)$ be the Bayes risk when $X(Y)$ is performed and π_i the prior probability that H_i is true, $i = 0, 1$. We obtain the following comparisons of g_W and f.

Theorem 5.2. *$B(\eta f, g_W) \leq B(f, \eta g_W)$ if and only if $R_X(\pi_0) \leq R_Y(\pi_0)$, where $\eta = \alpha_0\pi_0/\alpha_1\pi_1$, $\pi_1 > 0$, and $\alpha_i > 0$ is the loss if H_i is true and is not accepted.*

Proof. Let $D = \{x : (\alpha_0\pi_0/\alpha_1\pi_1)f(x) \leq g_W(x)\}$ and d_i the Bayes decision to accept H_i, then

$$\begin{aligned} R_X(\pi_0) &= \alpha_0\pi_0 P(d_1|H_0) + \alpha_1\pi_1 P(d_0|H_1) \\ &= \alpha_0\pi_0 \int_D g_W(x)d\lambda(x) + \alpha_1\pi_1 \int_{D^c} f(x)d\lambda(x) \\ &= B(\alpha_0\pi_0 g_W, \alpha_1\pi_1 f) \\ &= \alpha_1\pi_1 B(g_W, \eta f). \end{aligned} \tag{5.2}$$

Also,

$$R_Y(\pi_0) = \alpha_1\pi_1 B(f, \eta g_W),$$

so that

$$B(\eta f, g_W) \leq B(f, \eta g_W)$$

if and only if

$$R_X(\pi_0) \leq R_Y(\pi_0).$$

□

6. APPLICATIONS

Example 6.1. Let

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}\beta^{\alpha}}{\Gamma(\alpha)} e^{-x/\beta} & \text{if } x > 0,\ \alpha > 0,\ \beta > 0 \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Then $\mu = \alpha\beta$ and $\mu_2 = \alpha(\alpha+1)\beta^2$ and the hazard rate is increasing for $\alpha \geq 1$ and decreasing for $\alpha \leq 1$. If $W(x) = x$, then the weighted pdf is given by

$$f_W(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha}\beta^{\alpha+1}}{\Gamma(\alpha+1)} e^{-x/\beta} & \text{if } x > 0,\ \alpha > 0,\ \beta > 0 \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 4.1, for $\alpha > 1$, we have

$$\int_0^\infty |\overline{F}_{W_t}(x) - e^{-x/\alpha\beta}|dx \leq 2\alpha\beta \left| 1 - \left(\frac{\alpha(\alpha+1)}{2\alpha^2} \right) \right| = \beta|\alpha - 1|. \tag{6.1}$$

With $\beta = 1/2$,

$$C(\alpha) = \int_0^\infty |\overline{F}_{W_t}(x) - e^{-2x/\alpha}|dx \leq \frac{|\alpha - 1|}{2}. \tag{6.2}$$

Similarly, for $\alpha < 1$ and $\beta = 1/2$, we have

$$C(\alpha) = \int_0^\infty |\overline{F}_{W_e}(x) - e^{-2x/\alpha}|dx \leq \frac{|\alpha - 1|}{2}. \tag{6.3}$$

Note that $\mu_{F_W} = \beta(\alpha + 1)$. Consequently, for $\alpha < 1$ and $\beta = 1/2$, we get

$$\int_0^\infty |\overline{F}_{W_t}(x) - e^{-2x/\alpha}|dx \geq 2\alpha e^{-2\epsilon/\alpha} \left| \frac{1}{2(1+\alpha)} \right|.$$

Example 6.2. Let

$$f(x;\beta) = \begin{cases} 2\pi^{-1/2}\beta^{-1}e^{-x/\beta} & \text{if } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding weighted pdf with $W(x) = x$ is given by

$$g_W(x;\beta) = \begin{cases} 2x\beta^{-2}e^{-x/\beta} & \text{if } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

Note that, by Theorem 5.2 and for $\beta \geq \eta \geq 1$, $B(\eta f, g_W) \leq B(f, \eta g_W)$ and $R_X(\pi_0) \leq R_Y(\pi_0)$. Consequently, the experiment with lower Bayes risk is selected.

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