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ON CERTAIN NEW INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT. The aim of the present paper is to establish some variants integral inequalities in two independent variables. These integral inequalities given here can be applied as tools in the boundedness and uniquness of certain partial differential equations.

Key words and phrases: Integral inequality, Two independent variables, Partial differential equations, Nondecreasing, Nonincreasing.

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1. INTRODUCTION

The integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations (see [1]–[11]). In recent year, Pachpatte [11] discovered some new integral inequalities involving functions of two independent variables. These inequalities are applied to study the boundedness and uniqueness of the solutions of following terminal value problem for the hyperbolic partial differential equation (1.1) with the condition (1.2).

$$u_{xy}(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{1.1}$$

$$u(x,\infty) = \sigma_\infty(x), u(\infty,y) = \tau_\infty(y), u(\infty,\infty) = d. \tag{1.2}$$

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Our main objective here, motivated by Pachpatte's inequalities [11], is to establish additional new integral inequalities involving functions of two independent variables which can be used in the analysis of certain classes of partial differential equations.

2. MAIN RESULTS

Throughout the paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals are involved in existence on the domains of their definitions. We shall introduce some notation, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$ is the given subset of $\mathbb{R}$. The first order partial derivatives of a functions $z(x, y)$ defined for $x, y \in \mathbb{R}$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$ respectively.

We need the inequalities in the following Lemma 2.1 and Lemma 2.2, which are given in the book [1].

Lemma 2.1. *Let g be a monotone continuous function in an interval I, containing a point u_0, which vanishes in I. Let u and k be continuous functions in an interval $J = [\alpha, \beta]$ such that $u(J) \subset I$, and suppose that k is of fixed sign in J. Let $a \in I$.*

(i) *Assume that g is nondecreasing and k is nonnegative. If*

$$u(t) \leq a + \int_\alpha^t k(s) g\left(u(s)\right) ds, \quad t \in J,$$

then

$$u(t) \leq G^{-1}\left(G(a) + \int_\alpha^t k(s) ds\right), \quad \alpha \leq t \leq \beta_1,$$

where $G(u) = \int_{u_0}^u dx/g(x), u \in I$, and $\beta_1 = \min(v_1, v_2)$, with

$$v_1 = \sup\left\{v \in J : a + \int_\alpha^t k(s) g\left(u(s)\right) ds \in I, \alpha \leq t \leq v\right\},$$

$$v_2 = \sup\left\{v \in J : G(a) + \int_\alpha^t k(s) ds \in G(I), \alpha \leq t \leq v\right\}.$$

(ii) *Assume that $J = (\alpha, \beta]$. If*

$$u(t) \leq a + \int_t^\beta k(s) g\left(u(s)\right) ds, \quad t \in J,$$

then

$$u(t) \leq G^{-1}\left(G(a) + \int_t^\beta k(s) ds\right), \quad \alpha_1 < t \leq \beta,$$

where $\alpha_1 = \max(\mu_1, \mu_2)$, with

$$\mu_1 = \sup\left\{\mu_1 \in J : a + \int_t^\beta k(s) g\left(u(s)\right) ds \in I, \mu \leq t \leq \beta\right\},$$

$$\mu_2 = \sup\left\{\mu \in J : G(a) + \int_t^\beta k(s) ds \in G(I), \mu \leq t \leq \beta\right\}.$$

The proofs of the inequalities in (i), (ii) can be completed as in [1, p. 40–42]. Here we omit the details.

Let $u(x, y), a(x, y), b(x, y)$ be nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$.

Lemma 2.2. (i) *Assume that $a(x,y)$ is nondecreasing in x and nonincreasing in y for $x,y\in\mathbb{R}_+$. If*

$$u(x,y)\le a(x,y)+\int_0^x\int_y^\infty b(s,t)u(s,t)\,dtds$$

for all $x,y\in\mathbb{R}_+$, then

$$u(x,y)\le a(x,y)\exp\left(\int_0^x\int_y^\infty b(s,t)\,dtds\right).$$

(ii) *Assume that $a(x,y)$ is nonincreasing in each of the variables $x,y\in\mathbb{R}_+$. If*

$$u(x,y)\le a(x,y)+\int_x^\infty\int_y^\infty b(s,t)u(s,t)\,dtds$$

for all $x,y\in\mathbb{R}_+$, then

$$u(x,y)\le a(x,y)\exp\left(\int_x^\infty\int_y^\infty b(s,t)\,dtds\right).$$

The proofs of the inequalities in (i), (ii) can be completed as in [1, p. 109-110]. Here we omit the details.

To establish our results, we require the class of functions S as defined in [2]. A function $g:[0,\infty)\to[0,\infty)$ is said to belong to the class S if

(i) $g(u)$ is positive, nondecreasing and continuous for $u\ge 0$,
(ii) $(1/v)g(u)\le g(u/v), u>0, v\ge 1$.

Theorem 2.3. *Let $u(x,y),a(x,y),b(x,y),c(x,y),d(x,y)$ be nonnegative continuous functions defined for $x,y\in\mathbb{R}_+$, let $g\in S$. Define a function $z(x,y)$ by*

$$z(x,y)=a(x,y)+c(x,y)\int_0^x\int_y^\infty d(s,t)u(s,t)\,dtds$$

with $z(x,y)$ is nondecreasing in x and $z(x,y)\ge 1$ for $x,y\in\mathbb{R}_+$. If

$$u(x,y)\le z(x,y)+\int_\alpha^x b(s,y)g\big(u(s,y)\big)\,ds, \tag{2.1}$$

for $\alpha,x,y\in\mathbb{R}_+$ and $\alpha\le x$, then

$$u(x,y)\le p(x,y)\left[a(x,y)+c(x,y)e(x,y)\exp\left(\int_0^x\int_y^\infty d(s,t)p(s,t)c(s,t)\,dtds\right)\right], \tag{2.2}$$

for $x,y\in\mathbb{R}_+$, where

$$p(x,y)=G^{-1}\left(G(1)+\int_\alpha^x b(s,y)\,ds\right), \tag{2.3}$$

$$e(x,y)=\int_0^x\int_y^\infty d(s,t)p(s,t)a(s,t)\,dtds, \tag{2.4}$$

$$G(u)=\int_{u_0}^u\frac{ds}{g(s)},\quad u\ge u_0>0, \tag{2.5}$$

G^{-1} is the inverse function of G, and

$$G(1)+\int_\alpha^x b(s,y)\,ds\in\text{Dom}\,(G^{-1}).$$

Proof. Let $z(x,y)$ is a nonnegative, continuous, nondecreasing and let $g \in S$. Then (2.1) can be restated as

$$\frac{u(x,y)}{z(x,y)} \leq 1 + \int_\alpha^x b(s,y) \frac{1}{z(x,y)} g\left(u(s,y)\right) \, ds. \tag{2.6}$$

Define a function $w(x,y)$ by the right side of (2.6), then $[u(x,y)/z(x,y)] \leq w(x,y)$ and

$$w(x,y) \leq 1 + \int_\alpha^x b(s,y) g\left(w(s,y)\right) \, ds. \tag{2.7}$$

Treating $y, y \in \mathbb{R}_+$ fixed in (2.7) and using (i) of Lemma 2.1 to (2.7), we get

$$w(x,y) \leq G^{-1}\left(G(1) + \int_\alpha^x b(s,y)\, ds\right). \tag{2.8}$$

Using (2.8) in $[u(x,y)/z(x,y)] \leq w(x,y)$, we obtain

$$u(x,y) \leq z(x,y)p(x,y),$$

where $p(x,y)$ is defined by (2.3). From the definition of $z(x,y)$ we have

$$u(x,y) \leq p(x,y)\left(a(x,y) + c(x,y)v(x,y)\right), \tag{2.9}$$

where $v(x,y)$ is defined by

$$v(x,y) = \int_0^x \int_y^\infty d(s,t)u(s,t)\, dtds.$$

From (2.9) we get

$$\begin{aligned} v(x,y) &\leq \int_0^x \int_y^\infty d(s,t)p(s,t)\left(a(s,t) + c(s,t)v(s,t)\right) dtds \\ &= e(x,y) + \int_0^x \int_y^\infty d(s,t)p(s,t)c(s,t)v(s,t)\, dtds, \end{aligned}$$

where $e(x,y)$ is defined by (2.4). Clearly, $e(x,y)$ is nonnegative, continuous, nondecreasing in $x, x \in \mathbb{R}_+$ and nonincreasing in $y, y \in \mathbb{R}_+$. Now, by (i) of Lemma 2.2, we obtain

$$v(x,y) \leq e(x,y) \exp\left(\int_0^x \int_y^\infty d(s,t)p(s,t)c(s,t)\, dtds\right). \tag{2.10}$$

Using (2.10) in (2.9) we get the required inequality in (2.2). □

Theorem 2.4. *Let $u(x,y), a(x,y), b(x,y), c(x,y), d(x,y)$ be nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$ and let $g \in S$. Define a function $z(x,y)$ by*

$$z(x,y) = a(x,y) + c(x,y) \int_x^\infty \int_y^\infty d(s,t)u(s,t)\, dtds$$

with $z(x,y)$ is nonincreasing in x and $z(x,y) \geq 1$ for $x, y \in \mathbb{R}_+$. If

$$u(x,y) \leq z(x,y) + \int_x^\beta b(s,y) g\left(u(s,y)\right) ds$$

for $\beta, x, y \in \mathbb{R}_+$ and $\beta \geq x$, then

$$u(x,y) \leq \overline{p}(x,y)\left[a(x,y) + c(x,y)\overline{e}(x,y) \exp\left(\int_x^\infty \int_y^\infty d(s,t)\overline{p}(s,t)c(s,t)\, dtds\right)\right]$$

for $x, y \in \mathbb{R}_+$, *where*

$$\overline{p}(x,y) = G^{-1}\left(G(1) + \int_x^\beta b(s,y)\,ds\right), \tag{2.11}$$

$$\overline{e}(x,y) = \int_x^\infty \int_y^\infty d(s,t)\overline{p}(s,t)a(s,t)\,dtds.$$

G *is defined in (2.5),* G^{-1} *is the inverse function of* G, *and*

$$G(1) + \int_x^\beta b(s,y)\,ds \in \text{Dom }(G^{-1}).$$

The details of the proof of Theorem 2.4 follows by an argument similar to that in the proofs of Theorem 2.3 with suitable changes. We omit the details.

Theorem 2.5. *Let* $u(x,y), a(x,y), b(x,y), c(x,y)$ *be nonnegative continuous functions defined for* $x, y \in \mathbb{R}_+$ *and* $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ *be a continuous function which satisfies the condition*

$$0 \le F(x,y,u) - F(x,y,v) \le K(x,y,v)(u-v) \tag{2.12}$$

for $u \ge v \ge 0$, *where* $K(x,y,v)$ *is a nonnegative continuous function defined for* $x, y, v \in \mathbb{R}_+$. *And let* $g \in S$. *Define a function* $z(x,y)$ *by*

$$z(x,y) = a(x,y) + c(x,y)\int_0^x \int_y^\infty F(s,t,u(s,t))\,dtds$$

with nondecreasing in x *and* $z(x,y) \ge 1$ *for* $x, y \in \mathbb{R}_+$. *If*

$$u(x,y) \le z(x,y) + \int_\alpha^x b(s,y)g\big(u(s,y)\big)\,ds \tag{2.13}$$

for $\alpha, x, y \in \mathbb{R}_+$ *and* $\alpha \le x$, *then*

$$u(x,y) \le p(x,y)\Bigg[a(x,y) + c(x,y)A(x,y) \times \exp\left(\int_0^x \int_y^\infty K\left(s,t,p(s,t)a(s,t)\right)p(s,t)c(s,t)\,dtds\right)\Bigg] \tag{2.14}$$

for $x, y \in \mathbb{R}_+$, *where* $p(x,y)$ *is defined by (2.3) and*

$$A(x,y) = \int_0^x \int_y^\infty F\left(s,t,p(s,t)a(s,t)\right)dtds, \tag{2.15}$$

$G(u) = \int_{u_0}^u (ds/g(s)), u \ge u_0 > 0$, G^{-1} *is the inverse function of* G, *and*

$$G(1) + \int_\alpha^x b(s,y)\,ds \in \text{Dom }(G^{-1}).$$

Proof. The proof of this theorem follows by argument similar to that given in the proof of Theorem 2.3. Let $z(x,y)$ is a nonnegative, continuous, nondecreasing and let $g \in S$, then, we observe that

$$u(x,y) \le z(x,y)p(x,y),$$

where $p(x,y)$ is defined by (2.3). From the definition of $z(x,y)$ we have

$$u(x,y) \le p(x,y)\big(a(x,y) + c(x,y)w(x,y)\big), \tag{2.16}$$

where $w(x,y)$ is defined by

$$w(x,y) = \int_0^x \int_y^\infty F(s,t,u(s,t))\,dtds.$$

From (2.12) and (2.16) we get

$$\begin{aligned} w(x,y) \le \int_0^x \int_y^\infty \Big[& F\left(s,t,p(s,t)\left(a(s,t)+c(s,t)w(s,t)\right)\right) \\ & + F\left(s,t,p(s,t)a(s,t)\right) - F\left(s,t,p(s,t)a(s,t)\right)\Big]\, dtds \\ \le A(x,y) + & \int_0^x \int_y^\infty K\left((s,t,p(s,t)a(s,t)\right) p(s,t)c(s,t)w(s,t)\, dtds, \end{aligned}$$

where $A(x,y)$ is defined by (2.15). Clearly, $A(x,y)$ is nonnegative, continuous, nondecreasing in $x, x \in \mathbb{R}_+$ and nonincreasing in $y, y \in \mathbb{R}_+$. Now, by (i) of Lemma 2.2, we obtain

$$w(x,y) \le A(x,y) \exp\left(\int_0^x \int_y^\infty K\left((s,t,p(s,t)a(s,t)\right) p(s,t)c(s,t)\, dtds \right). \tag{2.17}$$

Using (2.16) in (2.17) we get the required inequality in (2.14). □

Theorem 2.6. *Let the assumptions of Theorem 2.5 be fulfilled. Define a function $z(x,y)$ by*

$$z(x,y) = a(x,y) + c(x,y) \int_x^\infty \int_y^\infty F(s,t,u(s,t))\, dtds,$$

with nonincreasing in x and $z(x,y) \ge 1$ for $x, y \in \mathbb{R}_+$. If

$$u(x,y) \le z(x,y) + \int_x^\beta b(s,y) g\left(u(s,y)\right)\, ds$$

for $\beta, x, y \in \mathbb{R}_+$ and $\beta \ge x$, then

$$\begin{aligned} u(x,y) \le \overline{p}(x,y) \Big[& a(x,y) + c(x,y)\overline{A}(x,y) \\ & \times \exp\left(\int_x^\infty \int_y^\infty K\left(s,t,\overline{p}(s,t)a(s,t)\right) \overline{p}(s,t)c(s,t)\, dtds \right) \Big] \end{aligned}$$

for $x, y \in \mathbb{R}_+$, where $\overline{p}(x,y)$ is defined by (2.11) and

$$\overline{A}(x,y) = \int_x^\infty \int_y^\infty F\left(s,t,\overline{p}(s,t)a(s,t)\right)\, dtds.$$

G is defined in (2.5), G^{-1} is the inverse function of G, and

$$G(1) + \int_x^\beta b(s,y)\, ds \in \text{Dom}\,(G^{-1}).$$

The details of the proof of Theorem 2.6 follows by an argument similar to that in the proofs of Theorem 2.5 with suitable changes. We omit the details.

3. Some Applications

In this section we present some immediate applications of Theorem 2.3 to study certain properties of solutions of the following terminal value problem for the hyperbolic partial differential equation

$$u_{xy}(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{3.1}$$

$$u(x,\infty) = \sigma_\infty(x), u(0,y) = \tau(y), u(0,\infty) = k, \tag{3.2}$$

where $h : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}, r : \mathbb{R}_+^2 \to \mathbb{R}, \sigma_\infty, \tau(y) : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and k is a real constant.

The following example deals with the estimate on the solution of the partial differential equation (3.1) with the conditions (3.2).

Example 3.1. Let $c(x,y)$ continuous, nonnegative, nondecreasing in x and nonincreasing in y for $x, y \in \mathbb{R}_+$, and let

$$|h(x,y,u)| \leq c(x,y)d(x,y)\,|u|\,, \tag{3.3}$$

$$\left|\sigma_\infty(x) + \tau(y) - k - \int_0^x \int_y^\infty r(s,t)\,dtds\right| \leq a(x,y) + \int_\alpha^x b(s,y)g(|u|)\,ds, \tag{3.4}$$

where $a(x,y), b(x,y), d(x,y), g$ are as defined in Theorem 2.3. If $u(x,y)$ is a solution of (3.1) with the conditions (3.2), then it can be written as (see [1, p. 80])

$$u(x,y) = \sigma_\infty(x) + \tau(y) - k - \int_0^x \int_y^\infty \left(h(s,t,u(s,t)) + r(s,t)\right)\,dtds \tag{3.5}$$

for $x, y \in \mathbb{R}$. From (3.3), (3.4), (3.5) we get

$$|u(x,y)| \leq a(x,y) + \int_\alpha^x b(s,y)g(|u|)\,ds + c(x,y)\int_0^x \int_y^\infty d(s,t)|u|\,dtds. \tag{3.6}$$

Now, a suitable application of Theorem 2.3 to (3.6) yields the required estimate following

$$|u(x,y)| \leq p(x,y)\left[a(x,y) + c(x,y)e(x,y)\exp\left(\int_0^x \int_y^\infty d(s,t)p(s,t)c(s,t)\,dtds\right)\right]$$

for $x, y \in \mathbb{R}_+$, where $e(x,y), p(x,y)$ are define in Theorem 2.3.

Our next result deals with the uniqueness of the solution of the partial differential equation (3.1) with the conditions (3.2).

Example 3.2. Suppose that the function h in (3.1) satisfies the condition

$$|h(x,y,u) - h(x,y,v)| \leq c(x,y)d(x,y)\,|u-v|\,, \tag{3.7}$$

where $c(x,y), d(x,y)$ is as defined in Theorem 2.3 with $c(x,y)$ is nonincreasing in y. Let $u(x,y), v(x,y)$ be two solutions of equation (3.1) with the conditions (3.2). From (3.5), (3.7) we have

$$|u(x,y) - v(x,y)| \leq c(x,y)\int_0^x \int_y^\infty d(s,t)|u(s,t) - v(s,t)|\,dtds. \tag{3.8}$$

Now a suitable application of Theorem 2.3 yields $u(x,y) = v(x,y)$, that is, there is at most one solution to the problem (3.1) with the conditions (3.2).

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