

# Journal of Inequalities in Pure and Applied Mathematics

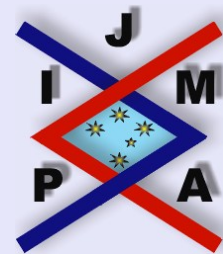
## AN INEQUALITY OF THE $1 - D$ KLEIN-GORDON EQUATION WITH A TIME-VARYING PARAMETER

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## Abstract

In this paper we present an inequality concerning the regularity of the solution of the Klein-Gordon equation with a time-varying parameter. In particular, we present an inequality comparing the norm of the initial state and the norm of the solution on some part of the boundary.

*2000 Mathematics Subject Classification:* 35L20, 93B05

*Key words:* Partial differential equation, Klein-Gordon, Inequality, Controllability

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# 1. Introduction

Let  $\Omega = (0, 1)$  and  $\alpha$  be a smooth function. We consider the following initial value problem

$$(1.1) \quad v_{tt} - v_{xx} + \alpha(t)v = 0 \text{ in } \Omega \times (0, T),$$

$$(1.2) \quad v(0, t) = 0 = v(1, t),$$

$$(1.3) \quad v(x, 0) = v_0(x) \in H_0^1(\Omega), \quad \text{and} \quad v_t(x, 0) = v_1(x) \in L^2(\Omega).$$

The smoothness of  $\alpha$  guarantees the existence of the above system. In this paper we are interested with the following function

$$(1.4) \quad z(t) = v_x(1, t).$$

In particular, we establish some conditions on  $\alpha$  that guarantee the existence of  $T$  and positive constants  $k_T, K_T$  that depend only on  $T > 0$  such that the  $L^2(0, T)$ -norm of  $z$  and the  $H_0^1(\Omega) \times L^2(\Omega)$ -norm of the initial state  $(v_0, v_1)$  are equivalent, i.e.

$$(1.5) \quad k_T \|(v_0, v_1)\|_{H_0^1 \times L^2}^2 \leq \int_0^T z^2(t) dt \leq K_T \|(v_0, v_1)\|_{H_0^1 \times L^2}^2$$



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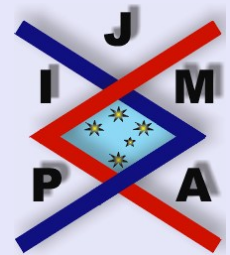
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for every  $(v_0, v_1) \in H_0^1 \times L^2$ . The upper estimate means that  $z$  belongs to  $L^2(0, T)$ , and the lower one means that the topology induced by the  $L^2(0, T)$ -norm of  $z$  is stronger than the  $H_0^1 \times L^2$ -norm.

We establish the inequality by the multiplier method, described by Komornik [3].

The particular inequality above is crucial in studying the exact controllability of distributed parameter systems. One can observe that by the inequality, one may regard the  $L^2(0, T)$ -norm of  $v_x(x, t)$  evaluated only at  $x = 1$  as the equivalent norm of the initial condition. This fact is fully utilized in solving the exact controllability of the systems. For detailed explanations on the exact controllability concepts and properties, readers should consult Lions [4].

In the higher dimensional case, it is proved in Pranoto [6] that the inequality is true. In the case  $\alpha \equiv 0$ , one can obtain the inequality and it is sharp in the sense that the time  $T$  must be  $> 2$  and cannot be smaller. For example, one can consult Bardos et al. in [2] for this result. They use micro local analysis to obtain the inequality. In the case  $\alpha \equiv 1$ , we are able to compute numerically the exact control. Please consult Pranoto [5] for its numerical scheme. It uses the Galerkin method. A more recent result on a different type of system is given in Avalos et al. [1].



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## 2. Main Results

Let  $S = H_0^1(\Omega) \times L^2(\Omega)$  and  $|\cdot|$  be the  $L^2$ -norm. In  $S$ , we define

$$(2.1) \quad \|(v_0, v_1)\|_S = \sqrt{|\partial_x v_0|^2 + |v_1|^2}$$

for every  $(v_0, v_1) \in S$ . Here,  $\partial_x$  denotes the partial derivative on variable  $x$ . With this norm, we define *energy*  $E$  of the solution at time  $t$ , that is  $(v(\cdot, t), v_t(\cdot, t))$ , as

$$(2.2) \quad E(t) = \frac{1}{2} \left( \|(v, v_t)\|_S^2 + \alpha(t)|v|^2 \right).$$

**Proposition 2.1.** *If  $\alpha$  satisfies the following conditions:*

**A1** *There is an  $\epsilon \in (0, 1)$ , such that  $|\alpha(t)| \leq (1 - \epsilon)\pi^2$  for every  $t$ ;*

**A2**  *$\text{Var}(\alpha) < \infty$ , where  $\text{Var}(\alpha)$  denotes the variation of the function  $\alpha$  on  $(0, \infty)$ ,*

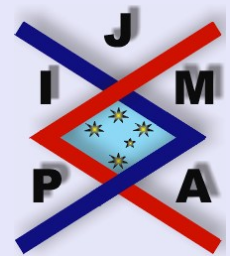
*then there exist  $T > 0$  and  $k_T, K_T > 0$  such that*

$$(2.3) \quad k_T \|(v_0, v_1)\|_{H_0^1 \times L^2}^2 \leq \int_0^T z^2(t) dt \leq K_T \|(v_0, v_1)\|_{H_0^1 \times L^2}^2$$

*for every  $(v_0, v_1) \in H_0^1 \times L^2$ .*

*Proof.* By the assumption (A1), the energy  $E(t)$  is non-negative. If we differentiate it, we obtain

$$(2.4) \quad E'(t) = \frac{\alpha'(t)}{2} |v|^2.$$



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One can show that  $|v|^2 \leq 2 \left[1 + \frac{1-\epsilon}{\pi^2}\right] E(t)$ , by (A1). It then implies that

$$|E'(t)| \leq 2|\alpha'(t)| \left[1 + \frac{1-\epsilon}{\pi^2}\right] E(t).$$

Thus, we obtain

$$(2.5) \quad \frac{E(0)}{M(T)} \leq E(t) \leq M(T) E(0),$$

where  $M(T) = \exp\left(2 \left[1 + \frac{1-\epsilon}{\pi^2}\right] \int_0^T |\alpha'(s)| ds\right)$ .

Because we assume (A2), the following inequality holds for every  $t > 0$

$$(2.6) \quad \frac{E(0)}{\mu} \leq E(t) \leq \mu E(0),$$

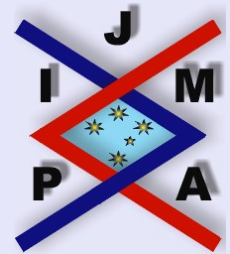
where  $\mu = \exp\left(2 \left[1 + \frac{1-\epsilon}{\pi^2}\right] \text{Var}(\alpha)\right)$ . If  $\alpha$  is constant,  $E(t) = E(0)$  for every  $t > 0$ . It means the energy is conserved.

After multiplying both sides of (1.1) by  $xv_x$  and integrating it over  $\Omega \times (0, T)$ , we obtain

$$(2.7) \quad \int_0^T \int_{\Omega} (v_{tt} - v_{xx} + \alpha(t)v) xv_x dx dt = 0.$$

Thus we have the following identity

$$(2.8) \quad \int_0^T \int_{\Omega} v_{tt} xv_x dx dt - \int_0^T \int_{\Omega} v_{xx} xv_x dx dt + \int_0^T \int_{\Omega} \alpha(t)v xv_x dx dt = 0.$$



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Next, we evaluate the first term on the left hand side, and we obtain

$$(2.9) \quad \int_0^T \int_{\Omega} v_{tt} x v_x dx dt = \rho(T) - \rho(0) - \left( \int_0^T \frac{x}{2} (v_t(x, t))^2 \Big|_{x=0}^1 dt - \frac{1}{2} \int_0^T \int_{\Omega} |v_t|^2 dx dt, \right).$$

By the boundary condition, the above equation becomes

$$(2.10) \quad \int_0^T \int_{\Omega} v_{tt} x v_x dx dt = \rho(T) - \rho(0) + \frac{1}{2} \int_0^T \int_{\Omega} |v_t|^2 dx dt,$$

where  $\rho(t) = \int_0^1 v_t(x, t) x v_x(x, t) dx$ . Next, we evaluate the second term on the left hand side of (2.8) and obtain

$$(2.11) \quad - \int_0^T \int_{\Omega} v_{xx} x v_x dx dt = \frac{1}{2} \int_0^T \int_{\Omega} |v_x|^2 dx dt - \frac{1}{2} \int_0^T |v_x(1, t)|^2 dt.$$

After that, we evaluate the last term on the left hand side of (2.8). By the boundary condition, we obtain

$$(2.12) \quad \int_0^T \int_{\Omega} \alpha(t) v x v_x dx dt = -\frac{1}{2} \int_0^T \int_{\Omega} \alpha(t) |v|^2 dx dt.$$

If one adds the equations (2.10-2.12) altogether, one obtains

$$(2.13) \quad 0 = \rho(T) - \rho(0) + \frac{1}{2} \int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2 - \alpha(t) |v|^2) dx dt - \frac{1}{2} \int_0^T |v_x(1, t)|^2 dt.$$



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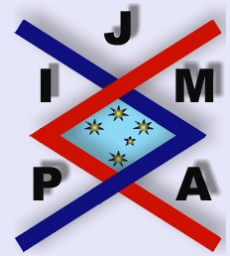
The value of the function  $\rho$  can be estimated by

$$(2.14) \quad |\rho(t)| \leq \left( \int_0^1 |v_t(x, t)|^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 |v_x(x, t)|^2 dx \right)^{\frac{1}{2}}.$$

Thus, we have

$$(2.15) \quad |\rho(t)| \leq \int_0^1 |v_t(x, t)|^2 dx + \int_0^1 |v_x(x, t)|^2 dx \leq D E(t),$$

where  $D = \frac{2}{\epsilon}$ .



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Next, we estimate the term

$$\int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2 - \alpha(t)|v|^2) \, dx \, dt$$

$$(2.16) \quad \geq \int_0^T \int_{\Omega} (|v_t|^2 + (1 - \epsilon)|v_x|^2 + \epsilon|v_x|^2 - (1 - \epsilon)\pi^2|v|^2) \, dx \, dt$$

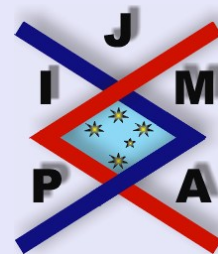
$$(2.17) \quad \geq \epsilon \int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2) \, dx \, dt$$

$$(2.18) \quad \geq \frac{\epsilon}{2} \int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2 + \pi^2|v|^2) \, dx \, dt$$

$$(2.19) \quad \geq \frac{\epsilon}{2} \int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2 + \alpha(t)|v|^2) \, dx \, dt.$$

We then apply the above estimates into (2.13), and we obtain

$$(2.20) \quad \frac{1}{2} \int_0^T |v_x(1, t)|^2 \, dt \geq -D(E(T) + E(0)) + \frac{\epsilon}{2} \int_0^T E(t) \, dt.$$



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By the estimate on the energy growth (2.5), this implies

$$(2.21) \quad \begin{aligned} \frac{1}{2} \int_0^T |v_x(1, t)|^2 dt &\geq -D(\mu + 1)E(0) + \frac{\epsilon}{2} \int_0^T \frac{E(0)}{\mu} dt \\ &= \left( -D(\mu + 1) + \frac{1}{D} \frac{T}{\mu} \right) E(0). \end{aligned}$$

We then choose any  $T > \mu(\mu + 1)D^2$ . So,  $\left( -D(\mu + 1) + \frac{1}{D} \frac{T}{\mu} \right) > 0$ . In order to simplify the notation, we write the same  $T$  to denote this particular  $T$ . We then let

$$(2.22) \quad c_T = \left( -D(\mu + 1) + \frac{1}{D} \frac{T}{\mu} \right).$$

Next, we know that

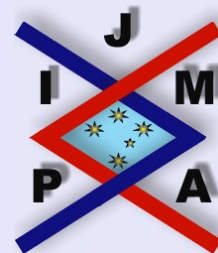
$$E(0) = \frac{1}{2} \|(v_0, v_1)\|_S^2 + \frac{\alpha(0)}{2} \int_{\Omega} |v_0(x)|^2 dx.$$

Moreover, by (A1), we have

$$(2.23) \quad \int_{\Omega} (|v_x(x, t)|^2 + \alpha(t)|v(x, t)|^2) dx \geq \epsilon \int_{\Omega} |v_x(x, t)|^2 dx.$$

Thus, we obtain

$$(2.24) \quad \int_0^T |z(t)|^2 dt \geq k_T \|(v_0, v_1)\|_S^2,$$



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where  $k_T = 2\epsilon c_T = -4(\mu + 1) + \epsilon^2 \frac{T}{\mu}$ .

Next, we need to prove the upper estimate of (2.3). In order to do this, we start from the identity (2.13). Using (A1), one then obtains the following inequality

$$(2.25) \quad \frac{1}{2} \int_0^T |v_x(1, t)|^2 dt \leq |\rho(T)| + |\rho(0)| + \frac{1}{2} \left| \int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2 - \alpha(t)|v|^2) dx dt \right| \leq D(\mu + 1)E(0) + \frac{1}{2} \left| \int_0^T \int_{\Omega} (|v_t|^2 + |v_x|^2 + \alpha(t)|v|^2 + 2\pi^2|v|^2) dx dt \right|$$

$$(2.26) \quad \leq D(\mu + 1)E(0) + C \int_0^T E(t) dt$$

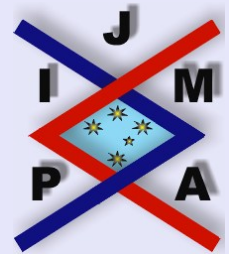
$$(2.27) \quad \leq (D(\mu + 1) + C\mu T) E(0),$$

where  $C = 1 + \frac{2}{\epsilon}$ . The inequalities (2.25) and (2.26) are obtained by the fact that

$$(2.28) \quad \|v_x\|_{L^2}^2 - \alpha(t)\|v\|_{L^2}^2 \leq \|v_x\|_{L^2}^2 + 2\pi^2\|v\|_{L^2}^2 + \alpha(t)\|v\|_{L^2}^2 \leq C (\|v_x\|_{L^2}^2 + \alpha(t)\|v\|_{L^2}^2).$$

Since  $E(0) \leq \frac{2-\epsilon}{2} \|(v_0, v_1)\|_S^2$ , we then obtain

$$(2.29) \quad \int_0^T |z(t)|^2 dt \leq K_T \|(v_0, v_1)\|_S^2,$$



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where the constant  $K_T$  equals  $\frac{4-2\epsilon}{\epsilon}(\mu + 1) + \left(\frac{4}{\epsilon} - \epsilon\right) \mu T$ . This completes the proof.  $\square$

The above proposition implies that the norm  $\|(v_0, v_1)\|_S$  of the initial state is equivalent to the  $L^2(0, T)$ -norm of the function  $z$ . It is interesting to note that  $z(t)$  which is equal to  $v_x(1, t)$  is evaluated at  $x = 1$  only. Roughly speaking, it means that one can observe the dynamic of the system via the information of  $z$  only. Thus, if we have two initial conditions that give the same  $z$ 's, then those two initial conditions are equal. This property is related to the Holmgren Uniqueness Theorem in the study of PDE's.



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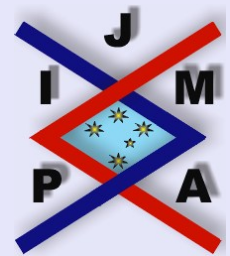
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## References

- [1] G. AVALOS AND I. LASIECKA, Exact controllability of structural acoustic interactions, *Preprint*, May 25, 2001.
- [2] C. BARDOS, G. LEBEAU AND J. RAUCH, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, *SIAM J. Control and Optimization*, **30**(5) (1992), 1024–1065.
- [3] V. KOMORNIK, Exact Controllability and Stabilization: The Multiplier Method, *Research in Applied Mathematics*, Masson, 1994.
- [4] J.L. LIONS, *Contrôlabilité Exacte Perturbations et Stabilisation de Systèmes Distribués*, Masson, Paris, 1988.
- [5] I. PRANOTO, A numerical scheme for computing the  $L^2$  optimal control of the Klein-Gordon system, presented at the *National Mathematics Seminar*, Malang - Indonesia, August 6th, 2001.
- [6] I. PRANOTO, Exact controllability of Klein-Gordon systems with a time-varying parameter, in *Topics in Applied and Theoretical Mathematics and Computer Science*, V.V. Kluev and N.E. Mastorakis (ed.), WSEAS Press, 2001, 76 – 80.



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