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ITERATED LAGUERRE AND TURÁN INEQUALITIES

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ABSTRACT. New inequalities are investigated for real entire functions in the Laguerre-Pólya class. These are generalizations of the classical Turán and Laguerre inequalities. They provide necessary conditions for certain real entire functions to have only real zeros.

Key words and phrases: Laguerre-Pólya class, Multiplier sequence, Hankel matrix, Real zeros.

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1. INTRODUCTION AND NOTATION

Definition 1.1. A real entire function $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to be in the *Laguerre-Pólya class*, written $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, if $\varphi(x)$ can be expressed in the form

$$\varphi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \quad 0 \le \omega \le \infty, \tag{1.1}$$

where $c, \beta, x_k \in \mathbb{R}$, $\alpha \ge 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. If $\omega = 0$, then, by convention, the product is defined to be 1.

The significance of the Laguerre-Pólya class in the theory of entire functions stems from the fact that functions in this class, *and only these*, are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros. For various properties and algebraic and transcendental characterizations of functions in this class we refer the reader to Pólya and Schur [11, p. 100], [12] or [9, Kapitel II].

If $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}$, then the *Turán inequalities* $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \ge 0$ and the *Laguerre inequalities* $\varphi^{(k)}(x)^2 - \varphi(x)^{(k-1)}\varphi^{(k+1)}(x) \ge 0$ are known to hold for all $k = 1, 2, \ldots$ and for all real x (see [2] and the references contained therein). In this paper we consider

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generalizations of both of these inequalities. For some of these generalizations to hold, we must restrict our investigation to the following subclass of $\mathcal{L}$-$\mathcal{P}$.

Definition 1.2. A real entire function $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ in $\mathcal{L}$-$\mathcal{P}$ is said to be in $\mathcal{L}$-$\mathcal{P}^+$ if $\gamma_k \geq 0$ for all k. In particular, this means that all the zeros of φ lie in the interval $(-\infty, 0]$.

Now if $\varphi(x) \in \mathcal{L}$-$\mathcal{P}^+$, then φ can be expressed in the form

$$\varphi(x) = cx^n e^{\beta x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right), \quad 0 \leq \omega \leq \infty, \tag{1.2}$$

where $c, \beta \geq 0$, $x_k > 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} \frac{1}{x_k} < \infty$ [9, Section 9]. If $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}$-$\mathcal{P}$, then, following the usual convention, we call the sequence of coefficients, $\{\gamma_k\}_{k=0}^{\infty}$, a *multiplier sequence*.

In Section 2, the Laguerre inequality $\varphi'(x)^2 - \varphi(x)\varphi''(x) \geq 0$ is generalized to a system of inequalities $L_n(\varphi(x)) \geq 0$ for all $n = 0, 1, 2, \ldots$ and for all $x \in \mathbb{R}$, where $L_1(\varphi(x)) = \varphi'(x)^2 - \varphi(x)\varphi''(x)$ and $\varphi(x) \in \mathcal{L}$-$\mathcal{P}$ (cf. [10, Theorem 1]). This system of inequalities characterizes functions in $\mathcal{L}$-$\mathcal{P}$ (Theorem 2.2). We show that the (nonlinear) operators L_n satisfy a simple recursive relation (Theorem 2.1) and use this fact to give a different proof of a result of Patrick [10, Theorem 1]. This, together with the converse of Patrick's theorem (cf. [5, Theorem 2.9]), yields a necessary and sufficient condition for a real entire function (with appropriate restrictions on the order and type of the entire function) to belong to the Laguerre-Pólya class (Theorem 2.2).

In Section 3, we consider a different collection of inequalities based on the Laguerre inequality, namely an iterated form of them. Our original proof of the second iterated inequalities for functions in $\mathcal{L}$-$\mathcal{P}^+$ [2, Theorem 2.13] was based on the study of certain polynomial invariants. In Section 3, we give a shorter and a conceptually simpler proof of these inequalities (Proposition 3.2 and Theorem 3.3). Moreover, the proof of Proposition 3.2 leads to new necessary conditions for entire functions to belong to $\mathcal{L}$-$\mathcal{P}^+$ (Corollary 3.4). Evaluating these at $x = 0$ yields the classical Turán inequalities and iterated forms of them, considered in Section 4. In Section 4, we show that for multiplier sequences which decay sufficiently rapidly *all* the higher iterated Turán inequalities hold (Theorem 4.1). Our main result (Theorem 5.5) asserts that the third iterated Turán inequalities are valid for all functions of the form $\varphi(x) = x^2\psi(x)$, where $\psi(x) \in \mathcal{L}$-$\mathcal{P}^+$. An examination of the proof of Theorem 5.5 (see also Lemma 5.4) shows that the restriction that $\varphi(x)$ has a double zero at the origin is merely a ploy to render the, otherwise very lengthy and involved, computations tractable.

2. CHARACTERIZING $\mathcal{L}$-$\mathcal{P}$ VIA EXTENDED LAGUERRE INEQUALITIES

Let $\varphi(x)$ denote a real entire function; that is, an entire function with only real Taylor coefficients. Following Patrick [10], we define implicitly the action of the (nonlinear) operators $\{L_n\}_{n=0}^{\infty}$, taking $\varphi(x)$ to $L_n(\varphi(x))$, by the equation

$$|\varphi(x+iy)|^2 = \varphi(x+iy)\varphi(x-iy) = \sum_{n=0}^{\infty} L_n(\varphi(x))y^{2n}, \quad (x, y \in \mathbb{R})\,. \tag{2.1}$$

In the sequel, it will become clear that $L_n(\varphi(x))$ is also a real entire function (cf. Remark 2.4). In [10], Patrick shows that if $\varphi(x) \in \mathcal{L}$-$\mathcal{P}$, then $L_n(\varphi(x)) \geq 0$ for all $n = 0, 1, 2, \ldots$ and for all $x \in \mathbb{R}$. The novel aspect of our approach to these inequalities is based on the remarkable fact that the operators L_n satisfy a simple recursive relation (Theorem 2.1). By virtue of this recursion relation, we obtain a short proof of Patrick's theorem. This, when combined with the known converse result [5, Theorem 2.9], yields a complete characterization of functions in $\mathcal{L}$-$\mathcal{P}$ (Theorem 2.2).

We remark that generalizations of the operators defined in (2.1) are given by Dilcher and Stolarsky in [6]. These authors study the distribution of zeros of $L_n^{(m)}(\varphi(x))$ for their generalized operators $L_n^{(m)}$ and certain functions $\varphi(x)$.

Theorem 2.1. *Let $\varphi(x)$ be any real entire function. Then the operators L_n satisfy the following:*

(1) $L_n((x+a)\varphi(x)) = (x+a)^2 L_n(\varphi(x)) + L_{n-1}(\varphi(x))$*, for $a \in \mathbb{R}$ and $n = 1, 2, \ldots$;*
(2) $L_0(\varphi) = \varphi^2$*;*
(3) $L_n(c) = 0$ *for any constant c and $n \geq 1$.*

Proof. Parts (2) and (3) are clear from the definition. To check (1), we compute as follows:

$$\begin{aligned}
|(x+a+iy)\varphi(x+iy)|^2 &= ((x+a)^2+y^2)\sum_{n=0}^{\infty} L_n(\varphi(x))y^{2n} \\
&= (x+a)^2\sum_{n=0}^{\infty} L_n(\varphi(x))y^{2n} + \sum_{n=0}^{\infty} L_n(\varphi(x))y^{2n+2} \\
&= (x+a)^2\sum_{n=0}^{\infty} L_n(\varphi(x))y^{2n} + \sum_{n=1}^{\infty} L_{n-1}(\varphi(x))y^{2n} \\
&= (x+a)^2 L_0(\varphi(x)) + \sum_{n=1}^{\infty}[(x+a)^2 L_n(\varphi(x)) + L_{n-1}(\varphi(x))]y^{2n},
\end{aligned}$$

from which (1) follows. □

Using the recursion of Theorem 2.1, we obtain the following characterization of functions in $\mathcal{L}$-$\mathcal{P}$.

Theorem 2.2. *Let $\varphi(x) \not\equiv 0$ be a real entire function of the form $e^{-\alpha x^2}\varphi_1(x)$, where $\alpha \geq 0$ and $\varphi_1(x)$ has genus 0 or 1. Then $\varphi(x) \in \mathcal{L}$-$\mathcal{P}$ if and only if $L_n(\varphi) \geq 0$ for all $n = 0, 1, 2, \ldots$.*

Proof. First assume that all $L_n(\varphi) \geq 0$. If $\varphi \notin \mathcal{L}$-$\mathcal{P}$, then φ has a nonreal zero $z_0 = x_0 + iy_0$ with $y_0 \neq 0$. Hence

$$0 = |\varphi(z_0)|^2 = \sum_{n=0}^{\infty} L_n(\varphi(x_0))y_0^{2n}\ .$$

Since all terms in the sum are nonnegative and $y_0 \neq 0$, we must have $L_n(\varphi(x_0)) = 0$ for all n. But this gives $|\varphi(x_0+iy)|^2 = \sum_{n=0}^{\infty} L_n(\varphi(x_0))y^{2n} = 0$ for any choice of $y \in \mathbb{R}$, whence φ itself must be identically zero.

Conversely, assume that $\varphi \in \mathcal{L}$-$\mathcal{P}$. Since φ can be uniformly approximated on compact sets by polynomials with only real zeros, it will suffice to prove that $L_n(\varphi) \geq 0$ for polynomials φ. For this we use induction on the degree of φ. From Theorem 2.1(2) and (3), we see that $L_n(\varphi) \geq 0$ for any n if φ has degree 0. If the degree of φ is greater than zero, we can write $\varphi(x) = (x+a)g(x)$, where $a \in \mathbb{R}$ and $g(x)$ is a polynomial. By the induction hypothesis, $L_n(g(x)) \geq 0$ for all $n \geq 0$ and all $x \in \mathbb{R}$. Hence, Theorem 2.1(1) gives the desired conclusion for φ. □

Next we show that the explicit form of $L_n(\varphi)$ given in [10] can also be obtained from Theorem 2.1.

Theorem 2.3. *For any $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, operators $L_n(\varphi)$ satisfying the recursion and initial conditions of Theorem 2.1 are uniquely determined and are given by*

$$L_n(\varphi(x)) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j}\varphi^{(j)}(x)\varphi^{(2n-j)}(x)\ . \tag{2.2}$$

Proof. As before, it will suffice to prove the result for polynomials in $\mathcal{L}\text{-}\mathcal{P}$. It is clear that the recursion formula of Theorem 2.1, together with the initial conditions given there, uniquely determine the value of L_n on any polynomial with only real zeros. Thus it will suffice to show that the formula given in (2.2) satisfies the conditions of Theorem 2.1. We do a double induction, beginning with an induction on n.

If $n = 0$, then $L_0(\varphi) = \varphi^2$ by Theorem 2.1 and this agrees with (2.2). Assume that $n > 0$ and that (2.2) holds for L_{n-1}. Now we begin an induction on the degree of φ.

If φ is a constant, then $L_n(\varphi) = 0$ by Theorem 2.1, which agrees with (2.2). Assume the formula holds for polynomials of degree less than $\deg\varphi$. Then we can write $\varphi(x) = (x+a)g(x)$, where $a \in \mathbb{R}$ and $L_{n-1}(g)$ and $L_n(g)$ are given by (2.2). The conclusion now follows from a computation using Theorem 2.1(1). Indeed,

$$\begin{aligned}
L_n(\varphi(x)) &= L_n((x+a)g(x)) \\
&= (x+a)^2 L_n(g(x)) + L_{n-1}(g(x)) \\
&= (x+a)^2 \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j} g^{(j)}(x)g^{(2n-j)}(x) \\
&\qquad + \sum_{j=0}^{2n-2} \frac{(-1)^{j+n-1}}{(2n-2)!}\binom{2n-2}{j} g^{(j)}(x)g^{(2n-2-j)}(x).
\end{aligned}$$

Also, using Leibniz's formula for higher derivatives of a product,

$$\begin{aligned}
&\sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j}\varphi^{(j)}(x)\varphi^{(2n-j)}(x) \\
&\quad = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j}\Big[j(2n-j)g^{(j-1)}g^{(2n-1-j)} + (x+a)jg^{(j-1)}g^{(2n-j)} \\
&\qquad + (x+a)(2n-j)g^{(j)}g^{(2n-1-j)} + (x+a)^2 g^{(j)}g^{(2n-j)}\Big] \\
&\quad = (x+a)^2 \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j} g^{(j)}(x)g^{(2n-j)}(x) \\
&\qquad + \sum_{j=0}^{2n-2} \frac{(-1)^{j+n-1}}{(2n-2)!}\binom{2n-2}{j} g^{(j)}(x)g^{(2n-2-j)}(x),
\end{aligned}$$

because the coefficient of $(x+a)$ is shown to be zero by

$$\begin{aligned}
&\sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j}\left[jg^{(j-1)}g^{(2n-j)} + (2n-j)g^{(j)}g^{(2n-1-j)}\right] \\
&\quad = \frac{(-1)^n}{(2n)!}\left[\sum_{j=1}^{2n}(-1)^j\binom{2n}{j}jg^{(j-1)}g^{(2n-j)} + \sum_{j=0}^{2n-1}(-1)^j\binom{2n}{j}(2n-j)g^{(j)}g^{(2n-j-1)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^n}{(2n)!}\left[-\sum_{j=0}^{2n-1}(-1)^j\binom{2n}{j+1}(j+1)g^{(j)}g^{(2n-j-1)}\right.\\
&\qquad\left.+\sum_{j=0}^{2n-1}(-1)^j\binom{2n}{j}(2n-j)g^{(j)}g^{(2n-j-1)}\right]\\
&= 0
\end{aligned}
$$

since $\binom{2n}{j+1}(j+1) = \binom{2n}{j}(2n-j)$. □

The main emphasis here has been that the result of Theorem 2.2 depends only on the recursive condition of Theorem 2.1, and this seems to be the easiest way to prove Theorem 2.2. However, the operators $L_n(\varphi)$ can be explicitly computed more easily than from the recursive condition as was done in Theorem 2.3, as well as in greater generality.

Remark 2.4. For any real entire function φ, the operators $L_n(\varphi)$ defined by equation (2.1) are given by the formula

$$L_n(\varphi(x)) = \sum_{j=0}^{2n}\frac{(-1)^{j+n}}{(2n)!}\binom{2n}{j}\varphi^{(j)}(x)\varphi^{(2n-j)}(x).$$

Proof. By Taylor's theorem, for each fixed $x \in \mathbb{R}$,

$$h(y) := |\varphi(x+iy)|^2 = \varphi(x+iy)\varphi(x-iy) = \sum_{n=0}^{\infty}\frac{h^{(2n)}(0)}{(2n)!}y^{2n},$$

where we have used the fact that $h(y)$ is an even function (of y). Let $D_y = d/dy$ denote differentiation with respect to y. Then by Leibniz's formula, for higher derivatives of a product, we have

$$
\begin{aligned}
h^{(2n)}(0) &= \sum_{k=0}^{2n}\binom{2n}{k}\left(D_y^k\varphi(x+iy)\right)_{y=0}\left(D_y^{2n-k}\varphi(x-iy)\right)_{y=0}\\
&= \sum_{k=0}^{2n}\binom{2n}{k}(-1)^{n+k}\varphi^{(k)}(x)\varphi^{(2n-k)}(x)\\
&= (2n)!L_n(\varphi(x)),
\end{aligned}
$$

by the uniqueness of the Taylor coefficients. □

3. ITERATED LAGUERRE INEQUALITIES

Definition 3.1. For any real entire function $\varphi(x)$, set

$$\mathcal{T}_k^{(1)}(\varphi(x)) := (\varphi^{(k)}(x))^2 - \varphi^{(k-1)}(x)\varphi^{(k+1)}(x) \quad \text{if} \quad k \geq 1,$$

and for $n \geq 2$, set

$$\mathcal{T}_k^{(n)}(\varphi(x)) := (\mathcal{T}_k^{(n-1)}(\varphi(x)))^2 - \mathcal{T}_{k-1}^{(n-1)}(\varphi(x))\,\mathcal{T}_{k+1}^{(n-1)}(\varphi(x)) \quad \text{if} \quad k \geq n \geq 2.$$

Remark 3.1. (a) Note that with the notation above, we have $\mathcal{T}_{k+j}^{(n)}(\varphi) = \mathcal{T}_k^{(n)}(\varphi^{(j)})$ for $k \geq n$ and $j = 0, 1, 2\ldots$.

(b) The authors' investigations of functions in the Laguerre-Pólya class ([2], [3]) have led to the following problem.

Open Problem If $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, are the iterated Laguerre inequalities valid for all $x \geq 0$? That is, is it true that

$$\mathcal{T}_k^{(n)}(\varphi(x)) \geq 0 \qquad \text{for all} \quad x \geq 0 \quad \text{and} \quad k \geq n? \tag{3.1}$$

(c) If we assume only that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, then the inequality $\mathcal{T}_k^{(n)}(\varphi(x)) \geq 0$, $x \geq 0$, need not hold in general, as the following example shows. Consider, for example, $\varphi(x) = (x-2)(x+1)^2 \in \mathcal{L}\text{-}\mathcal{P}$. Then $\mathcal{T}_2^{(2)}(\varphi(x)) = 216x(-2+3x+x^3)$ and so we see that $\mathcal{T}_2^{(2)}(\varphi(x))$ is negative for all sufficiently small positive values of x.

(d) There are, of course, certain easy situations for which the iterated Laguerre inequalities can be shown to always hold. For example, if $\varphi(x) = (x+a)e^x$, $a \geq 0$ or $\varphi(x) = (x+a)(x+b)e^x$, $a, b \geq 0$, this is true. Since the derivative of such a function again has the same form, the remarks above indicate that it suffices to show that $\mathcal{T}_k^{(k)}(\varphi(x)) \geq 0$ for $k = 1, 2, \ldots$ and all $x \geq 0$. For the quadratic case, we obtain

$$\begin{aligned}&\mathcal{T}_k^{(k)}(\varphi(x)) \\ &\quad = \begin{cases} 2^{2^k-2}e^{2^k x}\left((a+x)^2+(b+x)^2+2(k-1)(2x+a+b+k^2-k)\right), & \text{for } k \text{ odd} \\ 2^{2^k-1}e^{2^k x}\left((x+a)(x+b)+k(2x+a+b+k-1)\right), & \text{for } k \text{ even}\end{cases}\end{aligned}$$

and each expression is clearly nonnegative for all real x.

(e) A particularly intriguing open problem is the case of $\varphi(x) = x^m$ in (3.1). Special cases, such as the iterated Turán inequalities discussed in the next section, can be easily established (i.e. $\mathcal{T}_n^{(n)}(x^n) = (n!)^{2^n}$), but the general case of $\mathcal{T}_n^{(n)}(x^{n+k})$, $k = 0, 1, 2, \ldots$, seems surprisingly difficult.

In [2, Theorem 2.13] it is shown that (3.1) is true when $n = 2$; that is the double Laguerre inequalities are valid. Here we present a somewhat different and shorter proof (which still depends on Theorems 2.2 and 2.3) in the hope that it will shed light on the general case.

Proposition 3.2. *If $\varphi(x)$ is a polynomial with only real, nonpositive zeros and positive leading coefficient (so that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+ \cap \mathbb{R}[x]$), then*

$$\mathcal{T}_k^{(2)}(\varphi(x)) \geq 0 \qquad \textit{for all} \quad x \geq 0 \quad \textit{and} \quad k \geq 2. \tag{3.2}$$

Proof. First we prove (3.2) by induction, in the special case when $k = 2$. If $\deg \varphi = 0$ or 1, then $\mathcal{T}_2^{(2)}(\varphi) = 0$. Now suppose that (3.2) holds (with $k = 2$) for all polynomials $g \in \mathcal{L}\text{-}\mathcal{P}^+$ of degree at most n. Let $\varphi(x) := (x+a)g(x)$, where $a \geq 0$. For notational convenience, set $h(x) := \mathcal{T}_1^{(1)}(g(x)) = (g'(x))^2 - g(x)g''(x)$ and note that $h(x)$ is just $L_1(g(x))$ in Theorem 2.3. Then some elementary, albeit involved, calculations (which can be readily verified with the aid of a symbolic program) yield

$$\varphi(x)\mathcal{T}_2^{(2)}(\varphi(x)) = \varphi''(x)\left\{(x+a)^4\mathcal{T}_1^{(1)}(h(x)) + \varphi(x)\left[12\varphi(x)L_2(g(x)) + A(x)\right]\right\}, \tag{3.3}$$

where $L_2(g(x))$ is given by (2.2) and

$$A(x) = 8(g'(x))^3 - 12g(x)g'(x)g''(x) + 4g(x)^2g'''(x).$$

Since $\varphi(x), \varphi''(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, $\varphi(x) \geq 0$ and $\varphi''(x) \geq 0$ for all $x \geq 0$. Also, by Theorem 2.2, $L_2(g(x)) \geq 0$ for all $x \in \mathbb{R}$. Now, another calculation shows that

$$g''(x)\mathcal{T}_1^{(1)}(h(x)) = g(x)\mathcal{T}_2^{(2)}(g(x))$$

and so $\mathcal{T}_1^{(1)}(h(x)) \geq 0$ for $x \geq 0$, since by the induction assumption $\mathcal{T}_2^{(2)}(g(x)) \geq 0$ for $x \geq 0$. Therefore, it remains to show that $A(x) \geq 0$ for $x \geq 0$. Let $g(x) = c\prod_{j=1}^n(x+x_j)$, where

$c > 0$ and $x_j \geq 0$ for $1 \leq j \leq n$. Then using logarithmic differentiation and the product rule we obtain

$$A(x) = 4g(x)^3 \frac{d^2}{dx^2}\left(g'(x).\frac{1}{g(x)}\right) = 4g(x)^3 \sum_{j=1}^{n} \frac{2}{(x+x_j)^3} \geq 0 \quad \text{for all} \quad x > 0. \tag{3.4}$$

Thus, the right-hand side of (3.3) is nonnegative for all $x \geq 0$ and whence $\mathcal{T}_2^{(2)}(\varphi(x)) \geq 0$ if $x > 0$. But then continuity considerations show that $\mathcal{T}_2^{(2)}(\varphi(x)) \geq 0$ for all $x \geq 0$. Finally, since $\mathcal{L}$-$\mathcal{P}^+$ is closed under differentiation and since $\mathcal{T}_{k+j}^{(n)}(\varphi) = \mathcal{T}_k^{(n)}(\varphi^{(j)})$ for $k \geq n$ and $j = 0, 1, 2\ldots$ (see Remark 3.1(a)), we conclude that (3.2) holds. □

Recall from the introduction, that if $\varphi(x) \in \mathcal{L}$-$\mathcal{P}^+$, then $\varphi(x)$ can be expressed in the form

$$\varphi(x) = ce^{\sigma x} \prod_{j=1}^{\omega} \left(1 + \frac{x}{x_j}\right), \quad 0 \leq \omega \leq \infty, \tag{3.5}$$

where $c \geq 0$, $\sigma \geq 0$, $x_j > 0$ and $\sum 1/x_j < \infty$. Now set

$$\varphi_N(x) = c\left(1 + \frac{\sigma x}{N}\right)^N \prod_{j=1}^{\min(N,\omega)} \left(1 + \frac{x}{x_j}\right).$$

Then $\varphi_N(x) \to \varphi(x)$ as $N \to \infty$, uniformly on compact subsets of $\mathbb{C}$. Moreover, the class $\mathcal{L}$-$\mathcal{P}^+$ is closed under differentiation, and so the derivatives of $\varphi(x)$ can also be expressed in the form (3.5). Therefore, the following theorem is an immediate consequence of Proposition 3.2.

Theorem 3.3. *If $\varphi(x) \in \mathcal{L}$-$\mathcal{P}^+$, then for $j = 0, 1, 2\ldots$,*

$$\mathcal{T}_k^{(2)}(\varphi^{(j)}(x)) \geq 0 \qquad \textit{for all} \quad x \geq 0 \quad \textit{and} \quad k \geq 2.$$

In the course of the proof of Proposition 3.2, we have shown (see (3.4)) that for polynomials $g(x) \in \mathcal{L}$-$\mathcal{P}^+$, the following inequality holds

$$2(g'(x))^3 - 3g(x)g'(x)g''(x) + g(x)^2 g'''(x) \geq 0 \qquad \text{for all} \quad x \geq 0. \tag{3.6}$$

Next, we employ the foregoing limiting argument (see the paragraph preceding Theorem 3.3) and the fact that $\mathcal{L}$-$\mathcal{P}^+$ is closed under differentiation, to deduce from (3.6) the following corollary.

Corollary 3.4. *If*

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+,$$

then for $p = 0, 1, 2\ldots$ and for all $x \geq 0$,

$$2\left(\varphi^{(p+1)}(x)\right)^3 - 3\varphi^{(p)}(x)\varphi^{(p+1)}(x)\varphi^{(p+2)}(x) + \left(\varphi^{(p)}(x)\right)^2 \varphi^{(p+3)}(x) \geq 0. \tag{3.7}$$

The interest in inequality (3.7) stems, in part, from the fact that for $x = 0$ it provides a new necessary condition for a real entire function to belong to $\mathcal{L}$-$\mathcal{P}^+$. Indeed, for $x = 0$, inequality (3.7) may be expressed in the form

$$2\gamma_{p+1}\left(\gamma_{p+1}^2 - \gamma_p\gamma_{p+2}\right) \geq \gamma_p\left(\gamma_{p+1}\gamma_{p+2} - \gamma_p\gamma_{p+3}\right) \qquad (p = 0, 1, 2, \ldots). \tag{3.8}$$

The *Turán inequalities* $\gamma_{p+1}^2 - \gamma_p\gamma_{p+2} \geq 0$ imply that $\gamma_{p+1}\gamma_{p+2} - \gamma_p\gamma_{p+3} \geq 0$. Thus, if $\gamma_p > 0$ for all $p \geq 0$, then $\gamma_p\left(\gamma_{p+1}\gamma_{p+2} - \gamma_p\gamma_{p+3}\right)/(2\gamma_{p+1})$ is a nontrivial positive lower bound for the Turán expression $\gamma_{p+1}^2 - \gamma_p\gamma_{p+2}$.

4. ITERATED TURÁN INEQUALITIES

Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. We define the r-th iterated Turán sequence of Γ via $\gamma_k^{(0)} = \gamma_k$, $k = 0, \ldots$, and $\gamma_k^{(r)} = (\gamma_k^{(r-1)})^2 - \gamma_{k-1}^{(r-1)}\gamma_{k+1}^{(r-1)}$, $k = r, r+1, \ldots$. Thus, if we write $\varphi(x) = \sum \gamma_k x^k/k!$, then $\gamma_k^{(r)}$ is just $\mathcal{T}_k^{(r)}(\varphi(x))$ evaluated at $x = 0$. Under certain circumstances, we can show that *all* of the higher iterated Turán expressions are positive for a multiplier sequence. In Section 3 we mentioned some simple cases in which we could, in fact, show that all of the iterated Laguerre inequalities hold. In this section we establish the iterated Turán inequalities for a large class of interesting multiplier sequences.

Theorem 4.1. *Fix $c \geq 1$ and $d \geq 0$. Consider the set $\mathcal{M}_c$ of all sequences of positive numbers $\{\gamma_k\}_{k=0}^{\infty}$ satisfying*

$$\gamma_k^2 - c\,\gamma_{k-1}\gamma_{k+1} \geq 0, \tag{4.1}$$

for all k. Then

$$(\gamma_k^2 - \gamma_{k-1}\gamma_{k+1})^2 - (c+d)(\gamma_{k-1}^2 - \gamma_{k-2}\gamma_k)(\gamma_{k+1}^2 - \gamma_k\gamma_{k+2}) \geq 0 \tag{4.2}$$

for all k and all sequences in $\mathcal{M}_c$ if and only if $c \geq \frac{3+\sqrt{5+4d}}{2}$.

Proof. To see necessity, consider the specific sequence $\gamma_0 = 1,\ \gamma_1 = 1,\ \gamma_2 = \frac{1}{b},\ \gamma_3 = \frac{1}{cb^2},\ \gamma_k = 0$ for $k \geq 4$. This satisfies (4.1) for any $b \geq c$. But (4.2) yields

$$\left(\frac{1}{b^2} - \frac{1}{cb^2}\right)^2 - (c+d)\left(1 - \frac{1}{b}\right)\left(\frac{1}{c^2b^4}\right) = \frac{1}{c^2b^4}\left(c^2 - 3c + 1 + \frac{c}{b} - d + \frac{d}{b}\right).$$

Since b may be made as large as desired, this is only guaranteed to be nonnegative if $c \geq \frac{3+\sqrt{5+4d}}{2}$, the larger root of $c^2 - 3c + 1 - d$. The other alternative, $1 \leq c \leq \frac{3-\sqrt{5+4d}}{2}$ does not occur for $d > -1$ (and, in particular, for $d \geq 0$).

Conversely, assume (4.1) holds with $c \geq \frac{3+\sqrt{5+4d}}{2}$. An upper bound for $\gamma_k^{(1)}$ is γ_k^2. From (4.1), we obtain the lower bound

$$\gamma_k^{(1)} = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq (c-1)\gamma_{k-1}\gamma_{k+1}\,.$$

Estimating the expression in (4.2), we obtain

$$\begin{aligned}(\gamma_k^{(1)})^2 - (c+d)\gamma_{k-1}^{(1)}\gamma_{k+1}^{(1)} &\geq [(c-1)\gamma_{k-1}\gamma_{k+1}]^2 - (c+d)\gamma_{k-1}^2\gamma_{k+1}^2 \\ &= [c^2 - 3c + 1 - d]\gamma_{k-1}^2\gamma_{k+1}^2 \geq 0\end{aligned}$$

by the condition on c. ◻

The set $\mathcal{M}_4$ is of particular interest. Condition (4.1) forces the numbers γ_k to decrease rather quickly, leading us to term such sequences *rapidly decreasing sequences*. They are known to be multiplier sequences and were first investigated in some detail in [8]. These interesting sequences are discussed at some length in [3, Section 4] and [4, Section 4].

Corollary 4.2. *For a sequence as in (4.1) with $c > \frac{3+\sqrt{5}}{2} \approx 2.62$, the corresponding constant for the sequence of Turán expressions $\gamma_k^{(1)} = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1}$ is strictly greater than c by the amount $d = \frac{2c^2-3c+2}{2}$. If we then iterate this, forming the sequence $\{\gamma_k^{(2)}\}$, the corresponding constant again increases by more than d. After a finite number of steps, it will reach 4 (in the normalized case $\gamma_0 = \gamma_1 = 1$) and the sequence of higher Turán expressions $\{\gamma_k^{(r)}\}_{k=r}^{\infty}$, for r fixed and sufficiently large, will be a rapidly decreasing sequence. In particular, if the original sequence is a rapidly decreasing sequence, the sequence of Turán inequalities is again a rapidly decreasing sequence and we obtain an infinite sequence of multiplier sequences by iterating this process.*

Although the iterated Turán expressions seem to be positive for all multiplier sequences (an open question in general), it follows from the Theorem 4.1 that inequality (4.1) with $c = 1$ is not sufficient to achieve this since $\mathcal{M}_1$ contains sequences that fail to satisfy (4.2) for $d = 0$. But then, the specific sequence used in the proof is not a multiplier sequence if $c = 1$, as it violates condition (3.8) for $p = 1$.

5. THE THIRD ITERATED TURÁN INEQUALITY

In this section we establish the third iterated Turán inequality $\gamma_k^{(3)} \geq 0$ ($k = 3, 4, 5 \dots$) for multiplier sequences, $\{\gamma_k\}_{k=0}^{\infty}$, of the form $\gamma_k = k(k-1)\alpha_k$, $k = 1, 2, 3 \dots$, where $\{\alpha_k\}_{k=0}^{\infty}$ is an *arbitrary* multiplier sequence. With the notation adopted in Section 4, we have

$$\gamma_k^{(3)} = (\gamma_k^{(2)})^2 - \gamma_{k-1}^{(2)}\gamma_{k+1}^{(2)}, \quad k = 3, 4, 5, \dots, \tag{5.1}$$

or equivalently

$$\left(\mathcal{T}_k^{(3)}(\varphi(x))\right)_{x=0} = \left(\left(\mathcal{T}_k^{(2)}(\varphi(x)\right)^2 - \mathcal{T}_{k-1}^{(2)}(\varphi(x))\mathcal{T}_{k+1}^{(2)}(\varphi(x))\right)_{x=0}, \quad k = 3, 4, 5, \dots, \tag{5.2}$$

where

$$\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+. \tag{5.3}$$

Before embarking on the proof of the third iterated Turán inequality, we briefly discuss a representation of the third iterated Turán expression $\gamma_k^{(3)} = \left(\mathcal{T}_k^{(3)}(\varphi(x))\right)_{x=0}$ in terms of Wronskians and determinants of Hankel matrices (Proposition 5.1). We recall that the (n^{th} order) *Wronskian (determinant)* $W(\varphi(x), \varphi'(x), \dots, \varphi^{(n-1)}(x))$, where $\varphi(x)$ is an entire function, is defined as

$$W(\varphi(x), \varphi'(x), \dots, \varphi^{(n-1)}(x)) := \begin{vmatrix} \varphi(x) & \varphi'(x) & \cdots & \varphi^{(n-1)}(x) \\ \varphi'(x) & \varphi^{(2)}(x) & \cdots & \varphi^{(n)}(x) \\ \vdots & \vdots & & \vdots \\ \varphi^{(n-1)}(x) & \varphi^{(n)}(x) & \cdots & \varphi^{(2n-2)}(x) \end{vmatrix}, \tag{5.4}$$

and that the (n^{th} order) *Hankel matrices*, associated with the sequence $\{\gamma_k\}_{k=0}^{\infty}$, are matrices of the form $H_k^{(n)} = (\gamma_{k+i+j-2})_{i,j=1}^{n}$, that is

$$H_k^{(n)} = \begin{pmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n-1} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n+1} \\ & & \cdots & \\ \gamma_{k+n-1} & \gamma_{k+n} & \cdots & \gamma_{k+2n-2} \end{pmatrix} \quad (n = 1, 2, 3, \dots, k = 0, 1, 2, \dots).$$

We note that if we set $A_k^{(n)} = \det H_k^{(n)}$, then $W(\varphi^{(k)}(0), \varphi^{(k+1)}(0), \dots, \varphi^{(k+n-1)}(0)) = A_k^{(n)}$ and for $n = 3$ the following relation holds

$$(-\gamma_{k+2})\, A_k^{(3)} = \gamma_{k+2}^{(2)} \quad k = 0, 1, 2 \dots. \tag{5.5}$$

Furthermore, if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ is given by (5.3) and if $\gamma_k > 0$, then by Theorem 3.3, $\left(\mathcal{T}_k^{(2)}(\varphi(x))\right)_{x=0} \geq 0$ for $x \geq 0$ ($k = 2, 3, 4 \dots$) and whence, in light of (5.5), $A_k^{(3)} \leq 0$ for $k = 0, 1, 2, \dots$. A straightforward, albeit lengthy, calculation yields the following representation of the third iterated Turán expression $\gamma_k^{(3)}$.

Proposition 5.1. *Let $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ be an entire function. Then for $x \in \mathbb{R}$,*

$$\begin{aligned} \mathcal{T}_k^{(3)}(\varphi(x)) \\ = \left(\mathcal{T}_k^{(1)}(\varphi(x))\right) \Bigg(W\left(\varphi^{(k-3)}(x), \varphi^{(k-2)}(x), \varphi^{(k-1)}(x), \varphi^{(k)}(x)\right) \varphi^{(k)}(x)^2 \\ + \frac{\mathcal{T}_{k-1}^{(2)}(\varphi(x)) \mathcal{T}_{k+1}^{(2)}(\varphi(x))}{\varphi^{(k-1)}(x)\varphi^{(k+1)}(x)} \Bigg) \end{aligned} \tag{5.6}$$

for $k = 3, 4, 5 \ldots$. In particular, if $x = 0$ and $k = 0, 1, 2 \ldots$, then

$$\gamma_{k+3}^{(3)} = \left(\mathcal{T}_{k+3}^{(3)}(\varphi(x))\right)_{x=0} = \left(\gamma_{k+3}^2 - \gamma_{k+2}\gamma_{k+4}\right) \left(A_k^{(4)} \gamma_{k+3}^2 + A_k^{(3)} A_{k+2}^{(3)}\right), \tag{5.7}$$

where $A_k^{(n)} = \det H_k^{(n)}$ denotes the determinant of the Hankel matrix $H_k^{(n)}$.

Remark 5.2.

(a) Since the equalities (5.6) and (5.7) are formal identities, the assumption that $\varphi(x)$ is an entire function is not needed.

(b) With the aid of some known identities (see, for example, [13, VII, Problem 19]), equation (5.7) can be recast in the following suggestive form

$$\gamma_{k+3}^{(3)} = \left(A_{k+1}^{(3)}\right)^2 \gamma_{k+3}^2 - A_k^{(3)} A_{k+2}^{(3)} \gamma_{k+2}\gamma_{k+4}. \tag{5.8}$$

Now, suppose that $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$. Then, by virtue of (5.8), $\gamma_{k+3}^{(3)} \geq 0$ whenever $\left(A_{k+1}^{(3)}\right)^2 - A_k^{(3)} A_{k+2}^{(3)} \geq 0$, $k = 0, 1, 2 \ldots$. However, this inequality is not valid, in general, as the following example shows. Let $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = x^2(x+1)^{11}$. Here, $\gamma_0 = \gamma_1 = 0, \gamma_2 = 2, \gamma_3 = 66, \gamma_4 = 1320, \gamma_5 = 19800$ and $\gamma_6 = 237600$. Then $\left(A_1^{(3)}\right)^2 - A_0^{(3)} A_2^{(3)} = -2718144$.

(c) Let $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ be given by (5.3). Since $A_k^{(3)} A_{k+2}^{(3)} \geq 0$ (cf. (5.5) and Theorem 3.3), (5.7) shows that $\gamma_{k+3}^{(3)} \geq 0$ whenever $A_k^{(4)} \geq 0$. However, $A_k^{(4)}$ may be negative, as may be readily verified using the function $\varphi(x)$ defined in part (b). Examples of this sort are subtle as they depict a heretofore inexplicable phenomenon. The technique used below sheds light on this and at the end of this paper we provide a sufficient condition which guarantees that $A_k^{(4)} < 0$. In connection with the investigations of a conjecture of S. Karlin, additional examples are considered in [7] and [1]. Furthermore, to highlight the intricate nature of Karlin's conjecture, it was pointed out in these papers, in particular, that $A_k^{(4)} \geq 0$ if $\gamma_k = \alpha_k/k!$, $k = 0, 1, 2 \ldots$, where $\{\alpha_k\}_{k=0}^{\infty}$ is any multiplier sequence.

(d) In the sequel we will prove that if

$$\varphi(x) := \sum_{k=0}^{\infty} \frac{k(k-1)\alpha_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+,$$

where $\{\alpha_k\}_{k=0}^{\infty}$ is any multiplier sequence, then $\gamma_3^{(3)} \geq 0$. It is not hard to see that this is equivalent to proving the result only for multiplier sequences that begin with two zeros. However, the assumption that $\{\alpha_k\}_{k=0}^{\infty}$ is a multiplier sequence is not necessarily required to make the inequalities hold. To see this, we consider once again the example in part (b). Let $\alpha_0 = \alpha_1 = 0$ and for $k \geq 2$, set $\alpha_k = \frac{\gamma_k}{k(k-1)}$. We claim that $\{\alpha_k\}_{k=0}^{\infty}$ is *not* a multiplier sequence. Indeed, consider the fourth Jensen polynomial (defined, for example, in [2]) associated with the sequence $\{\alpha_k\}_{k=0}^{\infty}$, that is,

$$g_4(x) = \sum_{k=0}^{4} \binom{4}{k} \alpha_k x^k = 2x^2(3 + 22x + 55x^2).$$

Since $g_4(x)$ has two nonreal zeros, $\{\alpha_k\}_{k=0}^{\infty}$ is not a multiplier sequence, though our main theorem will establish the third iteration of the Turán inequalities for $\{\gamma_k\}_{k=0}^{\infty}$.

The proof of the main theorem requires that we express $\left(\mathcal{T}_3^{(3)}(\varphi(x))\right)_{x=0}$ in terms of sums of powers of the logarithmic derivatives of $\varphi(x)$. Accordingly, we proceed to establish the following preparatory result.

Lemma 5.3. *Let* $\varphi(x) = \prod_{j=1}^{n}(x + x_j)$, $x_j > 0$, $j = 1, 2, \ldots, n$, *be a polynomial in* $\mathcal{L}$-$\mathcal{P}^+$. *For fixed* $x \geq 0$ *and* $j = 1, 2, \ldots, n$, *set* $a_j := \frac{1}{x+x_j}$ *and let*

$$A := \sum_{j=1}^{n} a_j, \quad B := \sum_{j=1}^{n} a_j^2, \quad C := \sum_{j=1}^{n} a_j^3, \quad \textit{and} \quad D := \sum_{j=1}^{n} a_j^4. \tag{5.9}$$

Then

$$\frac{\varphi'(x)}{\varphi(x)} = A, \quad \frac{\varphi''(x)}{\varphi(x)} = A^2 - B, \quad \frac{\varphi'''(x)}{\varphi(x)} = A^3 - 3AB + 2C \tag{5.10}$$

and

$$\frac{\varphi^{(4)}(x)}{\varphi(x)} = A^4 - 6A^2B + 3B^2 + 8AC - 6D.$$

Proof. Logarithmic differentiation yields

$$\frac{\varphi'(x)}{\varphi(x)} = \sum_{j=1}^{n} \frac{1}{x + x_j} \quad \text{and} \quad \varphi''(x) = \varphi'(x)\left(\sum_{j=1}^{n} \frac{1}{x + x_j}\right) - \varphi(x) \sum_{j=1}^{n} \frac{1}{(x + x_j)^2}.$$

Hence,

$$\begin{aligned}\frac{\varphi''(x)}{\varphi(x)} &= \left(\frac{\varphi'(x)}{\varphi(x)}\right)\left(\sum_{j=1}^{n} \frac{1}{x + x_j}\right) - \sum_{j=1}^{n} \frac{1}{(x + x_j)^2} \\ &= \left(\sum_{j=1}^{n} \frac{1}{x + x_j}\right)^2 - \sum_{j=1}^{n} \frac{1}{(x + x_j)^2} = A^2 - B.\end{aligned}$$

Continuing in this manner, similar calculations yield

$$\begin{aligned}\frac{\varphi'''(x)}{\varphi(x)} &= \left(\sum_{j=1}^{n} \frac{1}{x + x_j}\right)^3 - 3\left(\sum_{j=1}^{n} \frac{1}{x + x_j}\right)\left(\sum_{j=1}^{n} \frac{1}{(x + x_j)^2}\right) + 2\left(\sum_{j=1}^{n} \frac{1}{(x + x_j)^3}\right) \\ &= A^3 - 3AB + 2C\end{aligned}$$

and

$$\begin{aligned}\frac{\varphi^{(4)}(x)}{\varphi(x)} &= \left(\sum_{j=1}^{n}\frac{1}{x+x_j}\right)^4 - 6\left(\sum_{j=1}^{n}\frac{1}{x+x_j}\right)^2\left(\sum_{j=1}^{n}\frac{1}{(x+x_j)^2}\right) + 3\left(\sum_{j=1}^{n}\frac{1}{(x+x_j)^2}\right)^2\\ &\quad + 8\left(\sum_{j=1}^{n}\frac{1}{x+x_j}\right)\left(\sum_{j=1}^{n}\frac{1}{(x+x_j)^3}\right) - 6\left(\sum_{j=1}^{n}\frac{1}{x+x_j}\right)\left(\sum_{j=1}^{n}\frac{1}{(x+x_j)^4}\right)\\ &= A^4 - 6A^2B + 3B^2 + 8AC - 6D.\end{aligned}$$

□

The next lemma gives an explicit expression for $\gamma_3^{(3)} = \left(\mathcal{T}_3^{(3)}(\varphi(x))\right)_{x=0}$, where $\varphi(x)$ is of the form $\varphi(x) = x^2\psi(x)$. While the verification involves only simple algebraic manipulations, the expression obtained is sufficiently involved to warrant the use of a computer.

Lemma 5.4. *Let $\psi(x) := \sum_{k=0}^{\infty}\frac{\alpha_k}{k!}x^k$ be an entire function. Let*

$$\varphi(x) = x^2\psi(x) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k,$$

so that $\gamma_0 = \gamma_1 = 0$ and $\gamma_k = k(k-1)\alpha_{k-2}$, for $k = 2, 3, \ldots$. Then

$$\gamma_3^{(3)} = \left(\mathcal{T}_3^{(3)}(\varphi(x))\right)_{x=0} = 768\,\left(3\,\psi'(0)^2 - 2\,\psi(0)\,\psi''(0)\right)E(0), \tag{5.11}$$

where

$$\begin{aligned}E(x) := 729\,\psi'(x)^6 &- 1458\,\psi(x)\,\psi'(x)^4\,\psi''(x) + 324\,\psi(x)^2\,\psi'(x)^2\,\psi''(x)^2\\ &+ 216\,\psi(x)^3\,\psi''(x)^3 + 54x\,\psi(x)^2\,\psi'(x)^3\,\psi^{(3)}(x)\\ &- 360\,\psi(x)^3\,\psi'(x)\,\psi''(x)\,\psi^{(3)}(x) + 100\,\psi(x)^4\,\psi^{(3)}(x)^2\\ &- 90\,\psi(x)^4\,\psi''(x)\,\psi^{(4)}(x).\end{aligned}$$

Preliminaries aside, we are now in a position to prove the principal result of this section.

Theorem 5.5. *Let $\psi(x) := \sum_{k=0}^{\infty}\frac{\alpha_k}{k!}x^k \in \mathcal{L}\text{-}\mathcal{P}^+$. Let*

$$\varphi(x) = x^2\psi(x) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k,$$

so that $\gamma_0 = \gamma_1 = 0$ and $\gamma_k = k(k-1)\alpha_{k-2}$, for $k = 2, 3, \ldots$. Then

$$\gamma_3^{(3)} = \left(\mathcal{T}_3^{(3)}(\varphi(x))\right)_{x=0} \geq 0. \tag{5.12}$$

Proof. In view of (5.11) of Lemma 5.4, since $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, $\left(\mathcal{T}_k^{(1)}(\psi(x))\right)_{x=0} \geq 0$ ($k = 1, 2, 3\ldots$), we only need to establish that $E(0) \geq 0$. Also, since $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, $\psi(x)$ can be uniformly approximated, on compact subsets of $\mathbb{C}$, by polynomials having only real, non-positive zeros. Therefore, it suffices to prove inequality (5.12) when $\psi(x) = \prod_{j=1}^{n}(x + x_j)$, ($x_j \geq 0$), is a polynomial in $\mathcal{L}\text{-}\mathcal{P}^+$. Now if $\psi(0) = 0$, then $E(0) = 729\,\psi'(0)^6 \geq 0$ and so in this case inequality (5.12) is clear. Thus, henceforth we will assume that $\psi(0) \neq 0$ and, for fixed $x \geq 0$, consider $E(x)$ as given in Lemma 5.4. We will prove a stronger result, namely, that for all $x \geq 0$,

$$\begin{aligned}\frac{E(x)}{(\psi(x))^6} &= \frac{729\,\psi'(x)^6}{\psi(x)^6} - \frac{1458\,\psi'(x)^4\,\psi''(x)}{\psi(x)^5} \\ &\quad + \frac{324\,\psi'(x)^2\,\psi''(x)^2}{\psi(x)^4} + \frac{216\,\psi''(x)^3}{\psi(x)^3} + \frac{540\,\psi'(x)^3\,\psi^{(3)}(x)}{\psi(x)^4} \\ &\quad - \frac{360\,\psi'(x)\,\psi''(x)\,\psi^{(3)}(x)}{\psi(x)^3} + \frac{100\,\psi^{(3)}(x)^2}{\psi(x)^2} - \frac{90\,\psi''(x)\,\psi^{(4)}(x)}{\psi(x)^2} \\ &\geq 0.\end{aligned}$$

For fixed $x \geq 0$, by Lemma 5.3 with ψ in place of φ, we obtain

$$\begin{aligned}\frac{E(x)}{(\psi(x))^6} &= 729A^6 - 1458A^4(A^2-B) + 324A^2(A^2-B)^2 \\ &\quad + 216(A^2-B)^3 + 540A^3(A^3-3AB+2C) - 360A(A^2-B)(A^3-3AB+2C) \\ &\quad + 100(A^3-3AB+2C)^2 - 90(A^2-B)(A^4-6A^2B+3B^2+8AC-6D)\end{aligned}$$

or

$$\begin{aligned}\frac{E(x)}{(\psi(x))^6} &= A^6 + 12\,A^4\,B - 18\,A^2\,B^2 + 54\,B^3 + 40\,A^3\,C \\ &\quad + 240\,A\,B\,C + 400\,C^2 + 540\,A^2\,D - 540\,BD \\ &= A^6 + 12\,B\left(\left(A^2 - \frac{3\,B}{4}\right)^2 + \frac{63\,B^2}{16}\right) \\ &\quad + 40\,A^3\,C + 240\,A\,BC + 400\,C^2 + 540\,\left(A^2 - B\right)D.\end{aligned}$$

Since $x_j > 0$ for $j = 1, 2 \ldots, n$, we have $A, B, C, D > 0$ and all the derivatives of $\psi(x)$ are positive for $x \geq 0$. Therefore we also have $(A^2 - B) = \psi''(x)/\psi(x) > 0$, and thus $E(x) > 0$ for $x \geq 0$. $\square$

Remark 5.6. (a) We wish to point out that in Theorem 5.5 we introduced the factor x^2 in order to simplify the ensuing algebra. In the absence of this factor we would have to calculate $\frac{\varphi^{(5)}(x)}{\varphi(x)}$ as well as $\frac{\varphi^{(6)}(x)}{\varphi(x)}$ (see the proof of Lemma 5.3). Then, as in the proof of Theorem 5.5, we would obtain an expression, analogous to $E(x)$, which has 112 terms rather than nine. Nevertheless, it seems that the technique developed above, should yield the desired result (5.12) for an arbitrary multiplier sequence rather than one with the first two terms equal to zero.

(b) We briefly indicate here how the foregoing technique can be used to derive a sufficient condition which guarantees that $A_0^{(4)} = \det H_0^{(4)} < 0$. Let $\varphi(x) := x^2\psi(x)$, where $\psi(x) = \prod_{j=1}^{n}(x + x_j)$, $x_j > 0$, is a polynomial in $\mathcal{L}\text{-}\mathcal{P}^+$. Then, the determinant of the 4^{th} order Hankel matrix $(\varphi^{(i+j-2)}(0))_{i,j=1}^4$ reduces to

$$\begin{aligned}A_0^{(4)} &= W(\varphi(0), \varphi'(0), \varphi''(0), \varphi'''(0)) \\ &= 48\,(27\,\psi'(0)^4 - 54\,\psi(0)\,\psi'(0)^2\,\psi''(0) + 12\,\psi(0)^2\,\psi''(0)^2 \\ &\quad + 20\,\psi(0)^2\,\psi'(0)\,\psi^{(3)}(0) - 5\,\psi(0)^3\,\psi^{(4)}(0)).\end{aligned} \tag{5.13}$$

Guided by (5.13) and the argument used in the proof of Theorem 5.5, we form the expression

$$K(x) = \frac{27\,\psi'(x)^4}{\psi(x)^4} - \frac{54\,\psi'(x)^2\,\psi''(x)}{\psi(x)^3} + \frac{12\,\psi''(x)^2}{\psi(x)^2} + \frac{20\,\psi'(x)\,\psi^{(3)}(x)}{\psi(x)^2} - \frac{5\,\psi^{(4)}(x)}{\psi(x)}$$

and with the aid of Lemma 5.4, for fixed $x \geq 0$, we obtain that

$$K(x) = 3(10\,D - B^2),$$

where the quantities $B = \sum_{j=1}^{n} a_j^2$ and $D = \sum_{j=1}^{n} a_j^4$ have the same meaning as in (5.9). Thus, we readily infer that if the the zeros of the polynomial $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ are distributed such that $10\,D < B^2$ holds at $x = 0$, then $A_0^{(4)} < 0$. By way of illustration, consider $\varphi(x) = x^2\psi(x) = x^2(x+a)^{12}$, where $a > 0$. Then for $x = 0$, we find that $10D = 120/a^4 < 144/a^4 = B^2$, and whence by our criterion, $A_0^{(4)} < 0$. Indeed, direct computation yields that $A_0^{(4)} = -3456a^{44}$.

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